Chapter 4

Kramers’ fission width for variable inertia

4.1 Introduction

As described in the first chapter, a dynamical model for fission of a hot compound nucleus was first proposed by Kramers [63] based on its analogy to the motion of a Brownian particle in a heat bath. In this model, the collective fission degrees of freedom represent the Brownian particle while the rest of the intrinsic degrees of freedom of the compound nucleus (CN) correspond to the heat bath. The dynamics of such a system is governed by the appropriate Langevin equations or equivalently by the corresponding Fokker-Planck equation. Kramers analytically solved the Fokker-Planck equation with a few simplifying assumptions and obtained the stationary width of fission. The detail derivation is given in the first chapter where, following the work of Kramers, parabolic shapes are considered for the nuclear potential at the ground state and at the saddle region and the inertia of the fissioning system is assumed to be shape independent and constant. The stationary width predicted by Kramers was found to be in reasonable agreement with the asymptotic fission width obtained from numerical solutions of the Fokker-Planck [45, 156, 157, 162, 163, 164, 165] and Langevin [105, 166, 167, 168, 169] equations in which harmonic oscillator potentials and constant inertia were used.
In the present chapter, we examine the applicability of Kramers’ expression for stationary fission width for more realistic systems. Specifically, we use the finite range liquid drop model (FRLDM) potential [128, 129] and shape-dependent inertia. To this end, we first approximate the FRLDM potential with suitably defined harmonic oscillator potentials, as we have done in the previous chapter also, in order to make use of Kramers’ expression for fission width. Since the centrifugal barrier changes the potential profile as the nuclear spin increases, the frequencies of the harmonic oscillator potentials approximating the FRLDM potential also develop a spin dependence [107, 108, 170]. Though the oscillator potentials are fitted to closely resemble the FRLDM potential, it is instructive to compare Kramers’ fission width with that obtained from the numerical solution of Langevin equations where the full FRLDM potential is employed. Considering the width from the Langevin equations to represent the true fission width, this comparison enables us to confirm the validity of Kramers’ expression for systems described by realistic potentials over the entire range of compound nuclear spin populated in a heavy ion induced fusion reaction. We next extend Kramers’ formulation of stationary width in order to include the slow variation of the collective inertia with deformation. The Kramers’ formula was generalized earlier [171] for variable inertia where a factor \(\sqrt{m_g/m_s}\) was introduced in the expression for the fission width. The inertias at the ground state and at the saddle point are denoted respectively by \(m_g\) and \(m_s\) here. In a Langevin calculation with variable inertia, Karpov et al. [167], however, reported that Kramers’ width (without the above mentioned factor) predicts the asymptotic fission width very accurately. We therefore address this issue here in some detail and show that the difference lies in different matching conditions. We draw our conclusions by comparing Kramers’ predicted widths with the widths calculated from the Langevin equations.

Further, in the present chapter, we examine the applicability of Kramers’ fission width when the variation of collective inertia is made very sharp but continuous. In the stochastic dynamical models of nuclear fission described by Langevin or Fokker-Planck equations, the collective kinetic energy of the CN expressed in terms of the speeds of the relevant collective coordinates contains an inertia term, which also depends on the collective coordinates. This collective inertia can be evaluated under suitable assumptions regarding the intrinsic nuclear motion. By considering the intrinsic nuclear motion as that of a classical irrotational and incompressible
fluid, the inertia can be calculated in the Werner-Wheeler approximation [71, 172] as described in Chapter 2. The inertia against slow shape distortion can also be obtained from the cranking model where nuclear single-particle states are considered [137]. Both these approaches predict a substantial increase of the collective inertia of a fissioning heavy nucleus as its shape evolves from the ground state to the saddle configuration. Inertia parameters in a dinuclear system have also been evaluated using linear response theory [69, 173]. In general, the inertia associated with a collective coordinate depends on the choice of the collective coordinate and the underlying microscopic motion.

The dissipation coefficient \( \eta \) is usually obtained by considering one- or two-body mechanisms of dissipation [67, 68, 70, 71, 138]. The shape dependencies of inertia and the dissipation coefficient from different models are found to be similar. As an illustration, the shape dependencies of the one-body dissipation coefficient (\( \eta \)) and the irrotational fluid inertia (\( m \)) are given in Chapter 2. Similar plots of \( \eta \) and \( m \) are shown in Fig. 4.1 (top and middle panels, respectively) for the \(^{224}\)Th nucleus; the bottom panel shows the reduced dissipation coefficient \( \beta = \eta/m \). Evidently, both the dissipation coefficient and the inertia have strong shape dependencies of similar nature whereas their ratio, the reduced dissipation coefficient \( \beta \), has a weaker shape dependence. A similar observation is also made for the reduced two-body dissipation coefficient (see Fig. 4 of Ref. [72]) where the ratio of two-body viscosity divided by the hydrodynamical inertia is found to be almost shape independent. As a first approximation, we therefore consider in the subsequent discussion the reduced dissipation coefficient \( \beta \) to be shape independent while allowing both the collective inertia \( m \) and the dissipation coefficient \( \eta \) to assume similar shape-dependent forms.

In the next section, we present the necessary steps taken to include the effect of slowly varying inertia in Kramers’ expression for the stationary fission width. A comparison between the results from the Langevin calculation and Kramers’ prediction is made in Sec. 4.3. Subsequently in Sec. 4.5, the necessary steps to include the effects of steeply varying inertia in the stationary fission width are given. The comparison between the Kramers’ predicted widths thus obtained and the corresponding stationary widths from the Langevin simulations is given in Sec. 4.6. A summary of all the results is presented in the last section.
Figure 4.1: One-body dissipation coefficient $\eta$ (top), irrotational fluid inertia $m$ (middle), and reduced dissipation coefficient $\beta = \eta/m$ (bottom) as a function of dimensionless elongation parameter $c$ for $^{224}$Th. The spherical shape and the scission configuration with zero neck radius correspond to $c = 1$ and $c = 2.09$, respectively [109].

4.2 Kramers’ width for slowly varying inertia

To introduce a shape-dependent collective inertia in the analytical formulation of stationary fission width, we follow the work of Kramers [63] (discussed in Chapter 1) very closely. The Liouville equation describing the fission dynamics in one-dimensional classical phase space is

$$\frac{\partial \rho}{\partial t} + \frac{p}{m} \frac{\partial \rho}{\partial c} + \left\{ K - \frac{p^2}{2} \frac{\partial}{\partial c} \left( \frac{1}{m} \right) \right\} \frac{\partial \rho}{\partial p} = \beta p \frac{\partial \rho}{\partial p} + \beta \rho + m \beta T \frac{\partial^2 \rho}{\partial p^2}, \tag{4.1}$$
where $\rho$ denotes the phase space density, $c$ is the collective coordinate with $p$ as its conjugate momentum and $m$ is the collective inertia. The conservative and dissipative forces are given as, $K = -\partial V/\partial c$ and $-\beta p$ respectively where $V$ is the collective potential and $\beta$ is the dissipation coefficient. $T$ represents the temperature of the CN. In what follows, we consider fission as a slow diffusion of Brownian particles across the fission barrier. When quasi-equilibrium is reached and a steady diffusion rate across the fission barrier has been established, Eq. (4.1) becomes

$$\frac{p}{m} \frac{\partial \rho}{\partial c} + \left\{ K - \frac{p^2}{2} \frac{\partial}{\partial c} \left( \frac{1}{m} \right) \right\} \frac{\partial \rho}{\partial p} = \beta p \frac{\partial \rho}{\partial p} + \beta \rho + m \beta T \frac{\partial^2 \rho}{\partial p^2}. \tag{4.2}$$

The calculations of potential and inertia are explained and plotted for $^{224}$Th in the previous chapter. We make the Werner-Wheeler approximation [71] for incompressible and irrotational flow to calculate the collective inertia (Fig. 2.4). The FRLDM potential is obtained by double folding a Yukawa-plus-exponential potential with the nuclear density distribution using the parameters given by Sierk [128]. For convenience, the potential and inertia are shown again in Fig. 4.2.

In nuclear fission, a CN that is at a temperature significantly less than the height of the fission barrier mostly stays close to its ground-state configuration except for occasional excursions toward the saddle region when it has picked up sufficient kinetic energy from the thermal motion and which may eventually result in fission. Evidently, we do not consider transients that are fast nonequilibrium processes and happen for nuclei with vanishing fission barriers. Therefore, in the present picture, the Brownian particles are initially confined in the potential pocket at the ground-state configuration with a fission barrier $V_B$ and for $V_B \gg T$, they can be assumed to be in a state of thermal equilibrium described by the Maxwell-Boltzmann distribution,

$$\rho = Ae^{-\left(\frac{p^2}{2m} + V\right)/T}, \tag{4.3}$$

where $A$ is a normalization constant. We next seek a stationary solution of the Liouville equation which corresponds to a steady flow of the Brownian particles across the fission barrier. The desired solution should be of the form

$$\rho = AF(c, p)e^{-\left(\frac{p^2}{2m} + V\right)/T} \tag{4.4}$$
Figure 4.2: FRLDM potential (gray-colored line) and the collective inertia (black line) of $^{224}$Th. The dotted line is obtained by fitting the FRLDM potential with two harmonic oscillator potentials (see section 3.1). The ground-state ($c_g$) and saddle ($c_s$) configurations are also marked [27].

such that $F(c, p)$ satisfies the boundary conditions

$$
F(c, p) \simeq 1 \quad \text{at} \quad c = c_g, \\
\simeq 0 \quad \text{at} \quad c \gg c_s, 
$$

where $c_g$ and $c_s$ define the ground-state and the saddle deformations. The first boundary condition corresponds to a continuous change of both the potential and the inertia values with deformation. In this context, it may be pointed out that Hofmann et al. [171] considered discrete values of inertia for the saddle and ground-state configurations which resulted in the factor $\sqrt{m_g/m_s}$ in the stationary fission width expression. This factor, however, does not ap-
pear in the present work, since we consider a continuous variation of the inertia value.

Substituting Eq. (4.4) in the stationary Liouville equation we obtain

$$m\beta T \frac{\partial^2 F}{\partial p^2} = \frac{p}{m} \frac{\partial F}{\partial c} + \frac{\partial F}{\partial p} \left\{ -\frac{\partial V}{\partial c} + \beta p - \frac{p^2}{2} \frac{\partial}{\partial c} \left( \frac{1}{m} \right) \right\}.$$

To assess the importance of the inertia derivative term in this equation, we estimate the magnitude of the term $\frac{p^2}{2} \frac{\partial}{\partial c} \left( \frac{1}{m} \right)$ with respect to $\beta p$ in the neighborhood of the fission barrier. Considering the inertia values as given in Fig. 4.2 for $^{224}$Th and a temperature of 2MeV, which gives the most probable momentum values, we find $\beta p > \frac{p^2}{2} \frac{\partial}{\partial c} \left( \frac{1}{m} \right)$ for $\beta > 0.1$MeV/h. Since a conservative estimate of the magnitude of nuclear dissipation $\beta$ is about 2MeV/h [105], we can neglect the inertia derivative term in Eq. (4.6). It may be pointed out here that though we neglect the inertia derivative term in Eq. (4.6) for $F$, the Boltzmann factor $\exp \left[ -(p^2/2m + V)/T \right]$ of the density in Eq. (4.4) fully satisfies Eq. (4.2). This is the reason for not neglecting the inertia derivative term earlier in Eq. (4.2) for the full density function $\rho(c,p)$. In fact, we also retain the inertia derivative term in the Langevin equations, which we discuss in the next section.

Since we require the solution for $F$ in the vicinity of the saddle point, we approximate the FRLDM potential in this region with a harmonic oscillator potential

$$V = V_B - \frac{1}{2} m_s \omega_s^2 (c - c_s)^2,$$

where the frequency $\omega_s$ is obtained by fitting the FRLDM potential. Introduction of $X = c - c_s$ further reduces Eq. (4.6) to

$$m_s \beta T \frac{\partial^2 F}{\partial p^2} = \frac{p}{m_s} \frac{\partial F}{\partial X} + \frac{\partial F}{\partial p} \left( m_s \omega_s^2 X + \beta p \right).$$

Following Kramers [63], we next assume for $F$ the form

$$F(X,p) = F(\zeta),$$

where $\zeta = p - aX$ and $a$ is a constant. The value of $a$ is subsequently fixed as follows. Substituting Eq. (4.9) for $F$ in Eq. (4.8), we obtain

$$m_s \beta T \frac{d^2 F}{d\zeta^2} = - \left( \frac{a}{m_s} - \beta \right) \left\{ p - \frac{m_s \omega_s^2}{a} X \right\} \frac{dF}{d\zeta}. $$
To have consistency between Eq. (4.10) and Eq. (4.9), we require

\[ \frac{m_s \omega_s^2}{a} = a, \quad (4.11) \]

which leads to

\[ \frac{a}{m_s} - \beta = -\frac{\beta}{2} + \sqrt{\omega_s^2 + \frac{\beta^2}{4}}, \quad (4.12) \]

where the positive root of \( a \) is chosen in order to satisfy the following boundary conditions: \( F(X, p) \to 1 \) for \( X \to -\infty \) (assuming the ground state to be far on the left of the saddle point) and \( F(X, p) \to 0 \) for \( X \to +\infty \). Equation (4.10) then becomes

\[ m_s \beta T \frac{d^2 F}{d\zeta^2} = -\left( \frac{a}{m_s} - \beta \right) \frac{dF}{d\zeta}. \quad (4.13) \]

The solution of Eq. (4.13) satisfying the above boundary conditions is

\[ F(\zeta) = \frac{1}{m_s} \sqrt{\frac{(a - m_s \beta)}{2\pi \beta T}} \int_{-\infty}^{\zeta} e^{-\left( \frac{\beta}{m_s} \right) (\zeta - \beta)^2 / 2m_s \beta T} d\zeta. \quad (4.14) \]

Substituting for \( F \) according to this equation in Eq. (4.4), the stationary density in the saddle region is finally obtained.

We next obtain the net flux or current across the saddle as

\[ j = \int_{-\infty}^{+\infty} \rho(X = 0, p) \frac{p}{m_s} dp = AT e^{-V_B/T} \sqrt{\frac{a - m_s \beta}{a}} \left\{ \sqrt{1 + \left( \frac{\beta}{2 \omega_s} \right)^2} - \frac{\beta}{2 \omega_s} \right\}. \quad (4.15) \]

The total number of particles in the potential pocket at the ground-state deformation is

\[ n_g = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho dc dp = \frac{2\pi AT}{\omega_g}, \quad (4.16) \]

where we have approximated the FRLDM potential with the following harmonic oscillator potential near ground state,

\[ V = \frac{1}{2} m_g \omega_g^2 (c - c_g)^2 \quad (4.17) \]
in which the frequency $\omega_g$ is again obtained by fitting the FRLDM potential. The probability $P$ of a Brownian particle crossing the fission barrier per unit time is then

$$P = \frac{j}{n_g} = \frac{\omega_g}{2\pi} e^{-V_B/T} \left\{ \sqrt{1 + \left( \frac{\beta^{2}}{2\omega_s} \right)^2} - \frac{\beta}{2\omega_s} \right\}.$$  \hspace{1cm} (4.18)

It is immediately noticed that this expression is exactly the same as the one obtained by Kramers using a shape-independent collective inertia. Eq. (4.18), however, is obtained with different inertia values at the ground-state and saddle configurations, which consequently define the frequencies ($\omega_g$ and $\omega_s$) in this equation. The fission width from Eq. (4.18) is

$$\Gamma = \hbar P = \frac{\hbar \omega_g}{2\pi} e^{-V_B/T} \left\{ \sqrt{1 + \left( \frac{\beta^{2}}{2\omega_s} \right)^2} - \frac{\beta}{2\omega_s} \right\},$$  \hspace{1cm} (4.19)

which we compare with the stationary width from Langevin equations in the following subsection.

### 4.3 Comparison with Langevin width for slowly varying inertia

Before we proceed to compare the fission widths from Langevin dynamics and Kramers’ formula, we point out that the net flux leaving the potential pocket is calculated at different points in the two approaches, though both of them represent the time rate of fission. In a stochastic process such as nuclear fission, a fission trajectory can return to a more compact shape even after it crosses the saddle configuration due to the presence of the random force in the equations of motion. This back streaming is typical of Brownian motion and has been noted earlier by several authors [78, 97, 174]. The back streaming is described by the phase-space density for negative momentum values at the saddle point in Kramers’ solution [Eq. (4.15)]. If one considers outward trajectories passing a larger coordinate value, the probability of returning approaches zero as the potential becomes steeper beyond the saddle point. In numerical simulations of the Langevin dynamics, the scission point is usually so chosen such that the strong Coulomb repulsion beyond the scission point makes the return of a trajectory highly unlikely after it crosses the scission point. The calculated outgoing flux of the Langevin trajectories
at the scission point then represents the net flux and hence corresponds to the net flux as defined in Kramers’ approach. This feature is illustrated in Fig. 4.3, where fission trajectories crossing the saddle and the scission points are considered separately in order to obtain the time-dependent fission rates from the Langevin equations. Clearly, the stationary width calculated at the saddle point is higher than that obtained at the scission point, since the former does not include the back-streaming effects. In what follows, we therefore compare Kramers’ width with the stationary widths from Langevin equations obtained at the scission configuration.

Figure 4.3: Time-dependent fission widths from Langevin equations. The thick black and thin gray lines represent the fission rates obtained at the scission point and at the saddle point, respectively [27].

The fitted values of $\omega_g$ and $\omega_s$ are obtained both for a constant value of the collective inertia [108] (shown earlier in Fig. 3.4) as well as for its different values at $c_g$ and $c_s$. Fig. 4.4
Figure 4.4: Compound nuclear spin ($\ell$) dependence of the frequencies of the harmonic oscillator potentials approximating the rotating FRLDM potential at the ground state ($\omega_g$) and at the saddle point ($\omega_s$). In the upper panel, the values of inertia at the ground state and at the saddle are taken to be the same, while the Werner-Wheeler approximation to the inertia is used in the lower panel [27].

shows the compound nuclear spin dependence of the frequencies thus obtained. The Langevin equations [Eq. (2.37)] are next solved with a constant value of the inertia at all deformations. A constant value of $\beta = 5\text{MeV}/\hbar$ is used in all the calculations. The time-dependent fission widths from the Langevin dynamics are displayed in Fig. 4.5 for different values of spin of the CN $^{224}$Th. The corresponding Kramers’ widths are also shown in this figure. A close agreement between the stationary widths from Langevin dynamics and those from the Kramers’ formula is observed for compound nuclear spin ($\ell$) of 0 and 25\hbar while for $\ell = 50\hbar$, the Kramer’ limit
underestimates the fission width by about 20%. The last discrepancy possibly reflects the fact that the condition $V_B \gg T$ required for validity of Kramers’ limit is not met in this case, since the fission barrier is 1.64 MeV for $\ell = 50h$, while the temperature of the CN is 2 MeV.
The Langevin equations are subsequently solved using shape-dependent values of the collective inertia, and the calculated time-dependent fission widths are shown in Fig. 4.6. Kramers’ widths are calculated using the frequencies $\omega_g$ and $\omega_s$, which are obtained using the local values $\ell \neq 0$, and $V_B = 5.82$ MeV for $T = 1.0$ MeV, $\ell = 40\hbar$, and $V_B = 2.98$ MeV for $T = 1.5$ MeV.

![Figure 4.6](image)

Figure 4.6: Time-dependent fission widths from Langevin equations with shape-dependent collective inertia. Results for different values of compound nuclear spin ($\ell$) and temperatures ($T$) are shown. The shape dependence is continuous for the histograms in thick black lines. The corresponding values of Kramers’ width [Eq. (4.19)] are indicated by the horizontal thick black lines. The histograms in thin gray lines are obtained with discrete values of inertia in the ground-state and saddle regions (see text) and the horizontal thin gray lines represent the corresponding stationary limits [Eq. (4.20)] [27].
of the collective inertia at \( c_g \) and \( c_s \), respectively. Kramers’ widths, also shown in Fig. 4.6, are found to be in excellent agreement with the stationary widths from the Langevin equations. It is thus demonstrated that Kramers’ formula [Eq. (4.19)] gives the correct stationary fission width even when the collective inertia of the system has a shape dependence.

We now study the stationary fission rate with a different type of shape dependence of collective inertia. We assume that the value of the inertia remains constant at \( m_g \) for all deformations around the ground state and at an intermediate deformation between the ground state and the saddle point, its value abruptly increases to \( m_s \) and remains so for all deformations in the saddle region. Such a system was considered by Hofmann et al. [171] and a modified version of the Kramers’ fission width was obtained as

\[
\Gamma = \sqrt{\frac{m_g \hbar \omega_g}{m_s}} e^{-V_B/T} \left\{ \sqrt{1 + \left( \frac{\beta}{2 \omega_s} \right)^2} - \frac{\beta}{2 \omega_s} \right\}.
\]  

(4.20)

We have solved the Langevin equations with the inertia defined as in the above and the calculated fission widths are also plotted in Fig. 4.6. The modified Kramers’ width from Eq. (4.20) is also shown for each case. The modified Kramers’ width is found to predict satisfactorily the stationary fission width from dynamic calculations. This result shows that the Kramers’ width and the stationary width from Langevin dynamical calculation remain in close agreement even under very distinct prescriptions of shape dependence of inertia. We, however, consider the Kramers’ width as given by Eq. (4.19) to be more appropriate for nuclear fission since it is obtained for realistic and smooth dependence of inertia on deformation. Therefore, a system with a deformation dependent slowly-varying collective inertia, the stationary fission width retains the form as was originally obtained by Kramers for constant inertia.

### 4.4 Connection between Kramers’ and Bohr-Wheeler fission widths

In this section, we discuss the following expression for Kramers’ width

\[
\Gamma = \frac{\hbar \omega_g}{T} \Gamma_{BW} \left\{ \sqrt{1 + \left( \frac{\beta}{2 \omega_s} \right)^2} - \frac{\beta}{2 \omega_s} \right\}.
\]  

(4.21)
which is often used in the literature [60, 108, 166, 168]. \( \Gamma_{BW} \) in this equation is the transition-state fission width due to Bohr and Wheeler [8], and it is introduced in Eq. 4.21 in the following manner. According to Bohr and Wheeler, the transition-state fission width is given as (the detailed derivation can be found in Chapter 1)

\[
\Gamma_{BW} = \frac{1}{2\pi \rho_g(E_i)} \int_{0}^{E_i - V_B} \rho_s(E_i - V_B - \epsilon) d\epsilon,
\]

where \( \rho_g \) is the level density at the initial state \((E_i, \ell_i)\) and \( \rho_s \) is the level density at the saddle point. Under the condition \( V_B/E_i << 1 \) and assuming the level density parameter for the ground state and at the saddle point to be the same and further assuming a simplified form of the level density as \( \rho(E) \sim \exp(2\sqrt{aE}) \), the Bohr-Wheeler width reduces to

\[
\Gamma_{BW} = \frac{T}{2\pi} e^{-V_B/T}.
\]

Substituting for \( \Gamma_{BW} \) from Eq. (4.23) in Eq. (4.21), the Kramers’ width as given in Eq. (4.20) is obtained. In other words, Eq. (4.21) represents the fission width which was originally obtained by Kramers only when the approximate expression for \( \Gamma_{BW} \) is used. Consequently, it is not appropriate to obtain Kramers’ width from Eq. (4.21) where the transition-state fission width \( \Gamma_{BW} \) is calculated from Eq. (4.22) using a shape-dependent level-density parameter. This observation follows from the fact that while the density of quantum mechanical microscopic states are explicitly taken into account in the work of Bohr and Wheeler [8], Kramers’ work [63] essentially concerns the classical phase space of the collective motion. Since there is no scope of introducing any detailed information regarding density of states apart from the nuclear temperature in dissipative dynamical models of nuclear fission, Kramers’ fission width cannot be connected to the Bohr-Wheeler expression where detailed density of states are employed. In fact, as shown in Fig. 1.3, the magnitude of the fission width obtained from the simplified version of the Bohr-Wheeler expression [Eq. (4.23)] differs substantially from that calculated using Eq. (4.22), particularly at high excitation energies where the dissipative effects are important, when the standard form of the shape-dependent level-density formula [see Eq. (1.5)] is used [27]. Therefore, the use of the Bohr-Wheeler fission width obtained with a shape-dependent level density in Eq. (4.21) does not correspond to dissipative dynamics, as envisaged in Kramers’ formula. Further, it can introduce an energy dependence in the dissipation coefficient when Eq. (4.21) is employed to fit experimental data. This can be one of the contributing factors
leading to the inference of very large values of nuclear dissipation [105].

It may be worthwhile to discuss at this point the distinguishing features of the transition-state fission width $\Gamma_{K}^{tr}$ which was obtained by Kramers (see Chapter 1) and is given as

$$\Gamma_{K}^{tr} = \frac{\hbar \omega_g}{2\pi} e^{-V_B/T}. \quad (4.24)$$

This width differs by a factor of $\hbar \omega_g/T$ from the approximate form of the Bohr-Wheeler fission width as given by Eq. (4.23). As described earlier, this difference arises because the accessible phase spaces are considered differently in the two approaches as we pointed out earlier. Strutinsky [85] introduced a phase-space factor in the Bohr-Wheeler transition-state fission width to account for the collective vibrations around the ground-state shape and obtained the same width as given in Eq. (4.24). It is important to recognize here that while the Bohr-Wheeler expressions [Eqs. (4.22) and (4.23)] represent the low-temperature limit ($T \ll \hbar \omega_g$) of fission width, Kramers’ width [Eq. (4.20)] corresponds to fission at higher temperatures ($T \gg \hbar \omega_g$). At low temperatures of a CN, quantal treatment of the collective motion is required, since the energy available to the collective motion is also very small [172]. Consequently, in the low-temperature limit, the collective motion is restricted to one state, namely, the zero-point vibration. Therefore, the Bohr-Wheeler width based upon density of quantum mechanical intrinsic nuclear states alone represents the low temperature limit of nuclear fission width. On the other hand, the phase space for collective vibrations increases with increasing temperature, and the Strutinsky-corrected width thus becomes the high-temperature limit of transition-state fission width. At higher temperatures, however, the nuclear collective motion also turns out to be dissipative in nature. Thus Kramers’ expression [Eq. (4.20)] should be considered as the high-temperature limit of the width of nuclear fission.

4.5 Kramers’ fission width for sharply varying inertia

As discussed in the introduction of this chapter, the collective inertia associated with fission dynamics depends on the collective coordinates and different different microscopic models for inertia suggest a strong coordinate dependence. It is also shown in Sec. 4.3 that the Kramers’ width remains valid when a slow variation of collective inertia is considered. The frequencies $\omega_g$ and $\omega_s$ in the Kramers’ width for this case are defined in terms of $m_g$ and $m_s$, the inertia values
at ground-state and saddle configurations respectively. The question therefore arises as to how
the Kramers’ expression for fission width is affected when the inertia value changes steeply
but continuously from $m_g$ to $m_s$ and when the inertia derivative term in the Fokker-Planck
equation is retained. We address this issue in some detail in the present subsection and investi-
gate the limiting factors in extending the Kramers’ approach to systems with a sharp increase
of collective inertia between the ground state and the saddle. We draw our conclusions by
comparing Kramers’ predicted widths with the widths obtained from the Langevin equations,
which represent the true fission width.

To study the effect of a sharp variation of inertia with deformation in the solution of the
Liouville equation [Eq. (4.2)], we consider a model shape-dependent inertia where its value
rises steeply at the deformation $c_t$, as illustrated in Fig. 4.7 (bottom panel) [109]. The inertia
has a constant value $m_g$ in the ground-state region up to a deformation of $c_1$. Beyond $c_1$, the
inertia rises fast and attains a value $m_s$ at $c_2$, which remains constant in the saddle region.
The variation of inertia over the entire range is considered to be smooth and continuous. The
potential $V$ is given by two harmonic oscillator potentials defined by Eq. (4.17) and Eq. (4.7)
near $c_g$ and $c_s$, respectively. Now we proceed in a similar way as described for slowly varying
inertia and seek a solution of Eq. (4.2) in the form

$$\rho = AF(c, p) e^{-\left(\frac{p^2}{2m(c)} + V\right)/T}$$

such that $F(c, p)$ satisfies the boundary conditions

$$F(c, p) \approx 1 \quad \text{at} \quad c = c_g,$$
$$F(c, p) \approx 0 \quad \text{at} \quad c \gg c_s.$$  

For a shape-dependent inertia of the type shown in Fig. 4.7 where the inertia increases fast
at a deformation $c_t$, the change of inertia value takes place over a limited region of deformation
space between $c_1$ and $c_2$. Consequently the inertia values are constant at the ground-state and
at the saddle regions. The inertia derivative term in Eq. (4.6) is thus zero beyond $c_2$ and the
solution for $F(c, p)$ in the saddle region is given by

$$F(c, p) = \sqrt{\frac{B}{\pi}} \int_{-\infty}^{\zeta_{\text{max}}} e^{-B\zeta^2} d\zeta,$$
Figure 4.7: Collective potential (top) for $^{224}$Th and the sharply varying model inertia (bottom) where the sharp variation takes place at $c_t$ between $c_1$ and $c_2$. $\Delta V_1$ is the potential difference between $c_1$ and $c_g$ and $\Delta V_2$ is the same between $c_2$ and $c_s$ [109].

where

$$B = \frac{1}{2m_s\beta T}\left(\frac{a}{m_s} - \beta\right)$$

with

$$\frac{a}{m_s} = \frac{\beta}{2} + \sqrt{\omega_s^2 + \frac{\beta^2}{4}}$$

and

$$\zeta_{max}(c, p) = p - a(c - c_s).$$

We first notice that for $c \gg c_s$, $F \to 0$ and the second boundary condition in Eq. (4.26) is satisfied. It can be further seen that the upper limit $\zeta_{max}$ of the integral in Eq. (4.27) evaluated
at \((c_2, \tilde{p})\), where \(\tilde{p}\) is the magnitude of the most probable momentum \((\sqrt{2m_sT})\), is given as

\[
\zeta_{\text{max}}(c_2, \tilde{p}) = \sqrt{2m_sT} \left\{ \sqrt{\frac{\Delta V_2}{T}} \left( \gamma + \sqrt{1 + \gamma^2} \right) - 1 \right\},
\]

(4.31)

where \(\Delta V_2\) is the potential difference between \(c_2\) and \(c_s\) (Fig. 4.7). Consequently, the leading term in \(B\zeta_{\text{max}}^2\) becomes \(\frac{\Delta V_2}{2T} (1 + \sqrt{1 + 1/\gamma^2})\). Hence for \(\Delta V_2/T > 1\), the integrand becomes much smaller than unity. This implies \(F(c, p) \simeq 1\) even at \(c = c_2\). Since the Maxwell-Boltzmann distribution

\[
\rho = Ae^{-\left(\frac{\beta^2}{2m(c)} + V\right)/T}
\]

(4.32)
satisfies the full Liouville equation Eq. (4.2) including the inertia derivative term and it represents the density of particles confined in the potential pocket at the ground-state configuration, we find in the above that the Maxwell-Boltzmann distribution also remains a solution at \(c = c_2\). The required solution for \(\rho\) thus takes the form,

\[
\rho = \begin{cases} 
Ae^{-\left(\frac{\beta^2}{2m(c)} + V\right)/T} & \text{up to } c_2, \\
AF(c, p) e^{-\left(\frac{\beta^2}{2m(c)} + V\right)/T} & \text{beyond } c_2.
\end{cases}
\]

(4.33)

The above scenario essentially implies that the diffusion process affects the density distribution only beyond \(c_2\). As obtained previously, the net flux or current across the saddle [Eq. (4.15)] and the total number of particles in the potential pocket at the ground-state deformation [Eq. (4.16)] are given respectively by

\[
j = AT e^{-V_B/T} \left\{ \sqrt{1 + \left( \frac{\beta}{2\omega_s} \right)^2} - \frac{\beta}{2\omega_s} \right\}
\]

(4.34)

and

\[
n_g = \frac{2\pi AT}{\omega_g},
\]

(4.35)

where we make use of the fact that the inertia has a constant value of \(n_g\) near the ground-state deformations and it is also assumed that \(\frac{\Delta V_1}{T} > 1\) where \(\Delta V_1\) is the potential difference between \(c_g\) and \(c_1\) (see Fig. 4.7). The fission width is obtained subsequently from the probability of a Brownian particle crossing the fission barrier per unit time and is given as,

\[
\Gamma = \frac{\hbar \omega_g}{2\pi} e^{-V_B/T} \left\{ \sqrt{1 + \left( \frac{\beta}{2\omega_s} \right)^2} - \frac{\beta}{2\omega_s} \right\}.
\]

(4.36)
It is immediately noticed that this expression is similar to the one that was obtained by Kramers using a shape-independent collective inertia. This equation however is obtained [109] with a strong shape dependence in the collective inertia resulting in different inertia values at the ground-state and saddle configurations and which, in turn, define the frequencies ($\omega_g$ and $\omega_s$) in this equation.

4.6 Comparison with Langevin width for sharply varying inertia

We now compare the Kramers' fission width obtained for sharply varying inertia with the corresponding stationary fission width from Langevin dynamical calculations. $^{224}$Th is considered as the CN to perform the calculations for both underdamped ($\beta/2\omega_s = 0.38 < 1$) and overdamped ($\beta/2\omega_s = 7.55 > 1$) motions with $\beta = 0.4\text{MeV}/\hbar$ and $8\text{MeV}/\hbar$, respectively. The $\beta$ values thus chosen cover the range of dissipation strengths obtained from fitting experimental data [72]. We first compare the dynamical fission widths from Langevin equations with the Kramers’ width as given by Eq. (4.36). Figure 4.8 shows the comparison where results for both underdamped and overdamped motions are plotted. The transition point $c_t$ is chosen as the mid-point between $c_g$ and $c_s$ for the Langevin dynamical calculations. $\Gamma_{gs}$ represents the Kramers' fission width when the inertia values at $c_g$ and $c_s$ are different as considered in Eq. (4.36). A close agreement between the stationary widths from Langevin equations and $\Gamma_{gs}$ is observed in Fig. 4.8 for both the underdamped and overdamped fission. Since the assumptions regarding potential variation ($\Delta V_{1,2} > T$) are reasonably met for the cases considered here, the agreement demonstrates that Eq. (4.36) gives the fission width correctly when extended for a steep shape dependence of inertia.

It is of interest to note here that, in the limit of strong dissipation when $\beta/2\omega_s \gg 1$, Kramers’ fission width [Eq. (4.36)] for systems with shape-dependent inertia and dissipation coefficient becomes

$$
\Gamma = \frac{\hbar}{2\pi} \frac{1}{\sqrt{\beta_g/\beta_s}} \sqrt{\left| \frac{\partial^2 V}{\partial c^2} \right|_g} \sqrt{\left| \frac{\partial^2 V}{\partial c^2} \right|_s} e^{-V_B/T},
$$

(4.37)
Figure 4.8: Time-dependent Langevin width (solid line) for sharply varying inertia with $c_t = 1.3$. $\Gamma_{gs}$ (dashed line) represents the fission width with different values of inertia at $c_g$ and $c_s$ as considered in Eq. (4.36). The top and bottom panels, respectively, show results for underdamped and overdamped fission [109].

where $(\frac{\partial^2 V}{\partial c^2})_g$ and $(\frac{\partial^2 V}{\partial c^2})_s$ are the potential curvatures at the ground-state and at the saddle configurations, respectively. The dissipation coefficients at the ground state and at the saddle are denoted by $\beta_g$ and $\beta_s$, respectively. Equation 4.37 is similar to the expression for fission width [72] that one obtains in the strong friction limit from the Smoluchowski equation. The shape-independent dissipation coefficient $\beta$ that appears in the fission width from the Smoluchowski equation is however replaced by $\sqrt{\beta_g \beta_s}$, the geometric mean of $\beta_g$ and $\beta_s$, in Eq. (4.36).

We next perform Langevin dynamical calculations with shape-dependent inertias in which the steep rise in the inertia value takes place at different points. Figure 4.9 shows the dependence of the stationary fission width ($\Gamma_L$) from Langevin equations on the location of the transition
Figure 4.9: Sharply varying inertia for different values of $c_t$ (top panel). Middle and bottom panels, respectively, show results for underdamped and overdamped fission. In the lower two panels, solid circles represent stationary Langevin width ($\Gamma_L$) plotted as a function of $c_t$. $\Gamma_{gs}$ represents the fission width with different values of inertia at $c_g$ and $c_s$ as considered in Eq. (4.36). $\Gamma_g$ and $\Gamma_s$ denote the widths obtained from Eq. (3.1) with shape-independent constant values of inertia, $m_g$ and $m_s$, respectively [109].

point $c_t$. Results for both underdamped and overdamped fission widths are shown in this figure. The Kramers’ widths obtained with shape-independent inertia values are also shown in this figure. Here $\Gamma_g$ is obtained with a shape-independent constant value of $m_g$ while $\Gamma_s$ is similarly defined with $m_s$. We first observe in Fig. 4.9 that $\Gamma_L$ is very close to $\Gamma_{gs}$ for
$c_t$ values near the mid-point between ground-state and saddle configurations, confirming the applicability of Eq. (4.36) in this region. However, as the transition point moves closer to the ground-state deformation, Eq. (4.16) for $n_g$ increasingly starts losing its validity. Similarly, when $c_t$ is shifted toward $c_s$, the solution for $\rho$ as given by Eq. (4.33) does not remain accurate. This essentially reflects the fact that most part of the Langevin dynamics takes place with inertia value at the saddle when $c_t < c_g$ and therefore $\Gamma_L$ approaches $\Gamma_s$ here. Similarly, the Brownian particles move mostly with ground-state inertia for $c_t > c_s$ and $\Gamma_L$ is close to $\Gamma_g$ in this region. Therefore, we have expanded the domain of validity of Kramers’ fission width formula by including a steep variation of collective inertia with deformation.

4.7 Summary

In the preceding sections, we considered the applicability of Kramers’ formula to the stationary fission width of a CN that is described by a realistic collective potential and a shape-dependent collective inertia. It is shown that for a system with a deformation-dependent collective inertia, the stationary fission width retains the form as originally obtained by Kramers for constant inertia. The accuracy of the various approximations in deriving the above fission width is tested by comparing its values with the stationary fission widths obtained by solving the Langevin equations. Both approaches are found to be in excellent agreement with each other. The present work thus extends the applicability of Kramers’ formula for stationary fission width to more realistic systems.

Further, we have expanded the domain of validity of Kramers’ fission width formula by including a steep variation of collective inertia with deformation in the Brownian motion. Comparison with numerical simulations from the corresponding Langevin equations confirms the adequacy of the extended formula and also demonstrates its region of validity and the consequences of the limiting conditions.

We also compare the strength of the statistical-model fission width obtained under different simplifying assumptions and point out the constraints in interpreting Kramers’ width in terms of the statistical-model fission width of Bohr and Wheeler.