CHAPTER 4

Application of proposed modified upper bound theorem to plane strain problems in homogeneous materials

4.1 Introduction

In the previous chapter the significance of extremum theorems/work principles for a plastic-rigid body was emphasised. Apart from their general applicability to many boundary value problems, these theorems provide a way to a direct construction of solutions, by-passing the integration of the differential equations. In the non-linear problems which constitute the plasticity theory this possibility is extremely important as emphasized by Kachanov (1971). Unfortunately, till now, these extremum theorems have been used only as a crude method of obtaining the limit load, of a plastic-rigid body, using successive approximations by upper and lower bound estimates. On the other hand, slip-line field (SLF) analysis, which also assumes that the material is rigid-plastic, can provide sufficiently accurate estimates of the stresses in the plastic region as well as near the notch tip and the corresponding limit loads. Constructing complete SLF for plane strain, non-hardening plasticity, involves discovering a field that satisfies (i) the Hencky equations for equilibrium and yield condition in the deformed region, (ii) the Geiringer equations for incompressibility and (iii) equilibrium and yield inequality in the rigid regions. As a result constructing a complete SLF is relatively difficult and theoretical solutions exist only for limited geometries and loading conditions. Applications of the SLF theory to fracture
related problems in Charpy and Izod test specimens are discussed in detail by Green (1953, 1956), Green and Hundy (1956), Alexander and Komoly (1962) and Ewing (1968). Till now, these two methods of plastic analysis, that is, the limit theorems and SLF have remained more or less independent apart from the fact that both are upper bounds as they use kinematically admissible velocity fields.

The analytical formulation of a new load bounding technique, Modified Upper Bound (MUB) theorem, was presented in the previous chapter. It was demonstrated that the method (MUB) is actually a new form of already existing extremum/work principle. In this chapter the equivalence of this new form of work principle, that is, the MUB theorem with the classical SLF analysis, for a rigid-plastic body in plane strain, is discussed in detail. Since plastic deformation fields depend on specimen geometry and type of loading, specific cases were considered. Both cracked and uncracked configurations were analysed to establish this equivalence in general. For cracked bodies only deeply cracked configurations were considered so that the plastic deformation remains confined to the uncracked ligament and does not spread to the cracked flanks. At this point it is worth to mention that Kim (2002) has also shown the global equilibrium of least upper bound circular arcs and evaluated fully plastic crack tip stresses. He assumed a plane strain deformation field consisting of rigid-body rotation across a circular arc extending from a crack tip across the remaining ligament. However, such an assumed deformation field has very limited applications.

In the subsequent sections it would be demonstrated that minimization of this new form of general work principle automatically leads to global equilibrium equations, as obtained from SLF analysis. Once this global equilibrium is established, the kinematically
admissible velocity field allows us to use Hencky’s equation for evaluation of stresses at any point in the plastically deformed region and in the vicinity of crack tip. Solutions of crack tip stresses obtained from MUB theorem were compared with detailed SLF solutions. Prandtl (1920) first developed the well-known Prandtl slip-line field for a semi-infinite plane strain mode-I crack. Detailed crack tip SLF solutions have already been provided by Rice & Johnson (1970) for blunted crack tip and Hutchinson (1968) for plane stress condition. Various simplifications resulting from the use of the proposed MUB theorem over SLF method are discussed in this chapter.

As a novel application of the proposed method, single-edge-cracked specimen under combined bending and tensile load was analysed. For plates with deep, single edge cracks, slip line fields are known under pure tension and under opening bending with compression or small tension (Shiratori and Miyoshi, 1980; Shiratori and Dodds, 1980). For such plates under opening bending and large tension, Rice (1972) gave an analytical-graphical formulation for sliding along the circular arc giving the least upper bound to the limit load. He also proposed an approximate elliptical yield locus for all ranges of positive tension and net section moments. Kim et al. (1995) provided a complete analytical formulation for Rice’s least upper bound. They also proposed an improved approximate elliptical yield locus (based on numerical fitting) and compared it with the finite element limit analyses of Lee and Parks (1993). Thus, while SLF solutions are available only for bending with small tensile load, classical upper bound solutions are valid for bending with large tensile load. In this chapter a completely analytical formulation for yield locus for the entire range of tensile and bending load is presented.
4.2 **Equivalence of proposed modified upper bound theorem with slip line field analysis**

As mentioned earlier, the modified upper bound (MUB) theorem is actually a new form of already existing extremum/work principle. In this section equivalence of SLF analysis and MUB theorem would be established in terms of global equilibrium equations for a wide variety of plastic deformation fields. While SLF method requires integration of stress distribution of plastically deformed regions the MUB method minimizes plastic work with respect to unknown parameters of assumed plastic field to arrive at global equilibrium equations. Standard results of stress distribution in plastically deformed zones, that is, Region II (see Fig. 3.1), obtained from SLF analysis would be utilized. It is obvious that if we can obtain stress distribution that satisfies equilibrium conditions in the deforming region by any method (experimental or analytical) we need not to refer to SLF results. Moreover, no information is required regarding the state of stress in region I comprising of rigid blocks of material.

As demonstrated in subsequent sections, the expression of MUB theorem itself provides an equation of global equilibrium for force or bending moment. Minimum work principle is then invoked to evaluate the unknown parameters of the plastic deformation field. This process of minimization automatically leads to other equations of global equilibrium which are identical to those obtained from the SLF analysis. It may be emphasized that this equivalence in terms of global equilibrium equations means that the state of stress in the regions having rigid mode of deformation and hence throughout the body is identical with that obtained from detailed SLF analyses. Thus, the plastic
deformation field assumed in MUB analysis is in fact SLF and Hencky’s theorem can now be used at any point in the plastically deformed regions to evaluate the state of stress. For the sake of simplicity we begin with uncracked configurations.

4.2.1 Bending of cantilever

We start with this classical problem that was first analysed by Green (1954). Let a cantilever beam of rectangular cross-section and length $l$ be bent by a force $P$ (per unit width), applied at the end; the left-hand end of the beam is rigidly clamped. The width $b$ in the horizontal direction is constant and at least six times (Green, 1954) than the height $2h$. In these circumstances plane strain condition can be safely assumed. Depending on $l/2h$ two types of plastic deformation patterns are possible.

4.2.1.1 Short cantilever ($l/2h \leq 13.73$)

In this case the possible plastic deformation field, assumed by Green (1954), is shown in Fig. 4.1(a). Let $d$ be the length of AD, $\varepsilon$ the angle DAC and $2\psi$ the angle subtended by the arc DD’. The right hand part of the cantilever slides along this arc in the limit state. Following the standard results of SLF analysis, (Green, 1954), we have uniaxial tension $+2k$ in ABC and compression $-2k$ in A’B’C’. Adjoining these triangles are the central fields ACD, A’C’D’, which are linked by circular slip line DD’ of radius $R$. The mean pressure on slip line AD is $p = k (1 + 2\varepsilon)$; the tangential stress on AD is clearly equal to $k$. 
This much information regarding the state of stress distribution, in plastically deformed region, is required for MUB analysis.

The scheme used to relate the relative velocity, $v^*$, (with which rigid parts rotate) to the rate of imposed displacement, $\delta^\cdot$, is shown in Fig. 4.1(b). Since the undeformed material is assumed to slide over the circular arc, therefore, its instantaneous centre must lie at the centre of this arc DD’. At instantaneous centre the tangential velocity is zero. As the undeformed portions are moving rigidly, linear variation of velocity (between instantaneous centre and the point of application of imposed load) can be assumed. As a result the Y-component of imposed displacement at D or D’ can be expressed as follows:

$$\delta_R = \frac{\delta R \cos \psi}{(1 + R \cos \psi - d \sin \psi)} \quad (4.1)$$

For kinematic admissibility the Y-component of tangential velocity, $v^*$, at D, must be equal to the Y-component of imposed displacement. Thus, the tangential velocity can be expressed as follows

$$v^* = \frac{\delta R}{(1 + R \cos \psi - d \sin \psi)} \quad (4.2)$$

The angular velocity $\omega$ with which the rigid part of the beam rotate about the instantaneous centre $O$ can be obtained from the following relation
Now invoking work principle, that is, eq. (3.33), limit load can be expressed as,

\[
P_L\delta = 2 \left[ \int_0^\psi \kappa^\ast R d\theta + \int_{DC} \sigma_j n_j v_i dS + \int_{CB} \sigma_j n_j v_i dS \right] \tag{4.4}
\]

The velocity \( v \) of any point \( P \) lying on the circular arc \( DC \) or on the segment \( CB \) can be simply obtained as the product of radial distance \( r \) between the instantaneous centre \( O \) and point \( P \) and the angular velocity \( \omega \). In order to evaluate the work done by the stresses on the elastic-plastic boundary, the velocity \( v \) is resolved into two components: \( v_t \) that is along the slip line and \( v_n \) which is normal to it, as shown in Fig. 4.1(c).

The work done by the stresses on the circular arc \( DC \) (of radius \( d \)) can be expressed as follows

\[
\int_{DC} \sigma_j n_j v_i dS = -\int_0^\epsilon k d\theta + \int_0^\epsilon k (1 + 2\theta) dv_n d\theta \tag{4.5}
\]

\[
\int_{DC} \sigma_j n_j v_i dS = -\int_0^\epsilon k d\theta \sin \lambda d\theta + \int_0^\epsilon k (1 + 2\theta) d\theta \cos \lambda d\theta \tag{4.6}
\]

From simple geometry, Fig. 4.1(c), it can be readily shown that along circular arc \( DC \)
\[ r \sin \lambda = d \left[ 1 - \cos \left( \varepsilon - \theta \right) \right] + R \sin \left( \varepsilon - \theta \right) \]  

(4.7)

\[ r \cos \lambda = R \cos \left( \varepsilon - \theta \right) + d \sin \left( \varepsilon - \theta \right) \]  

(4.8)

On substituting eqs. (4.7 & 4.8) in eq. (4.6), followed by integration, it can be shown that

\[
\int_{BC} \sigma_y n_j v_j dS = -k \omega d \left[ \varepsilon d + R - d \sin \varepsilon - R \cos \varepsilon \right] \\
+ k \omega d \left[ d \left( 1 + 2 \varepsilon \right) + 2R + R \sin \varepsilon - d \cos \varepsilon - 2R \cos \varepsilon - 2d \sin \varepsilon \right]
\]  

(4.9)

Similarly, the work done by the stresses on the segment \( CB \) can be expressed as follows

\[
\int_{CB} \sigma_y n_j v_j dS = \int_0^x k v_i dx + \int_0^x k v_n dx \]  

(4.10)

\[
\int_{CB} \sigma_y n_j v_j dS = \int_0^x k r \omega \sin \lambda dx + \int_0^x k r \omega \cos \lambda dx 
\]  

(4.11)

As per the geometry shown in Fig. 4.1(c), it can be readily shown that along segment \( CB \)

\[ r \sin \lambda = d \left( 1 - \cos \varepsilon \right) + R \sin \varepsilon \]  

(4.12)

\[ r \cos \lambda = R \cos \varepsilon + d \sin \varepsilon + x \]  

(4.13)
Substitution of eqs. (4.12 & 4.13) in eq. (4.11), followed by integration, lead to the following relation

\[
\int_{\mathcal{E}} \sigma_{ij} n_j v_i dS = -k w d \left[ d \left( 1 - \cos \varepsilon \right) + R \sin \varepsilon \right] + k w d \left[ R \cos \varepsilon + d \sin \varepsilon + \frac{d}{2} \right]
\]  

(4.14)

Finally, substitution of eqs. (4.9 & 4.14) in eq. (4.4), using the value of \( \omega \) as given by eq. (4.3), lead to the following relation for the limit load of a short cantilever

\[
P_L = \frac{2k}{\left( 1 - d \sin \psi + R \cos \psi \right)} \left[ R^2 \psi + dR + \frac{(1 + 2\varepsilon) d^2}{2} \right]
\]  

(4.15)

It may be noted that eq. (4.15) is in fact the condition of global moment equilibrium about hinge point \( O \). From geometry it may be easily observed that \( \psi = \frac{\pi}{4} - \varepsilon \) and \( d = \frac{h - R \sin \psi}{\cos \psi} \).

Here \( \varepsilon \) and \( R \) are the two independent unknown parameters that would be evaluated using minimum work principle. Minimizing eq. (4.15) with respect to these two unknown parameters we have

\[
\frac{\partial P_L}{\partial \varepsilon} = \frac{2k}{\left( 1 - d \sin \psi + R \cos \psi \right)} \left[ -R^2 + R \frac{\partial d}{\partial \varepsilon} + (1 + 2\varepsilon) d \frac{\partial d}{\partial \varepsilon} + d^2 \right] + \frac{2k}{\left( 1 - d \sin \psi + R \cos \psi \right)^3} \left[ \frac{d \cos \psi - \sin \psi \frac{\partial d}{\partial \varepsilon} + R \sin \psi \frac{\partial d}{\partial \varepsilon}}{l - d \sin \psi + R \cos \psi} \right] = 0
\]  

(4.16)
\[
\frac{\partial P_L}{\partial R} = \frac{2k}{(l - d \sin \psi + R \cos \psi)} \left[ 2R \psi + d + R \frac{\partial d}{\partial R} + (1 + 2\varepsilon) d \frac{\partial d}{\partial R} \right] + \\
\left[ R^2 \psi + dR + (1 + 2\varepsilon) \frac{d^2}{2} \right] \left[ -2k \left( -\sin \psi \frac{\partial d}{\partial R} + \cos \psi \right) \right] \left( (l - d \sin \psi + R \cos \psi) \right)^{-2} = 0 
\]
(4.17)

After a few algebraic simplifications the resulting expressions can be written as follows

\[
\left[ (1 + 2\varepsilon) R \cos \psi + d \cos \psi - R \sin \psi - (1 + 2\varepsilon) d \sin \psi \right] = -\frac{R^2 \psi + dR + (1 + 2\varepsilon) \frac{d^2}{2}}{(l - d \sin \psi + R \cos \psi)} = \frac{P_L}{2k} 
\]
(4.18)

\[
\left[ 2R \psi \cos \psi + d \cos \psi - R \sin \psi - (1 + 2\varepsilon) d \sin \psi \right] = -\frac{R^2 \psi + dR + (1 + 2\varepsilon) \frac{d^2}{2}}{(l - d \sin \psi + R \cos \psi)} = \frac{P_L}{2k} 
\]
(4.19)

Comparison of eqs. (4.18) and (4.19) provides \(1 + 2\varepsilon = 2\psi\) that can also be obtained from SLF analysis using Hencky’s theorem of constancy of \(\xi\) along the continuous \(\alpha\)-slip-line ADD’A’ (Green, 1954). Thus \(2\varepsilon = \pi/4 - 1/2\). If SLF analysis is performed then unknown radius \(R\) is determined from the condition of force equilibrium. Along the arc DD’, the normal stress is equal to \(2k\chi\) (Kachanov, 1971), where the angle \(\chi\) is measured from the horizontal as shown in Fig. 4.1(a). Thus,

\[
\left[ d \cos \psi - (1 + 2\varepsilon) d \sin \psi + R \int_0^\psi \cos \chi d\chi - 2R \int_0^\psi \chi \sin \chi d\chi \right] = \frac{P_L}{2k} 
\]
(4.20)
After simplification eq. (4.20) can be finally expressed as

\[
\left[ d \cos \psi - (1 + 2\varepsilon) d \sin \psi - R \sin \psi + 2R \psi \cos \psi \right] = \frac{P_L}{2k} \tag{4.21}
\]

Thus, it is proved that the MUB theorem automatically leads to equations of global equilibrium which are identical to those obtained from SLF analyses. Moreover, the method inherently satisfies Hencky’s theorem along a continuous slip line.

### 4.2.1.2 Long cantilever (\(l/2h > 13.73\))

For the case of a long cantilever the possible slip field, assumed by Green (1954), is shown in Fig. 4.2. State of stress in plastically deformed regions ABCD, A’B’C’D’, is identical to that obtained for short cantilever. The shape of assumed plastic field clearly reveals that there is no region in which rigid mode of deformation can be assumed, thus, the first integral term of eq. (3.33) becomes zero. In the limit state a rotation of the rigid part of the cantilever (to the right of BDB’) takes place with respect to point D. Using the stress distribution of plastically deformed region, and evaluating the work done by the stresses on the elastic-plastic boundary DCB, eq. (3.33) can be finally expressed as

\[
P_L = \frac{k (1 + 2\varepsilon) d^2}{l - d \sin \left( \frac{\pi}{4} - \varepsilon \right)} \tag{4.22}
\]
It may be noted that the limit load of a long cantilever expressed by eq. (4.22) can be directly obtained by substituting $R=0$ in eq. (4.15). From geometry it can be readily obtained that $d=\frac{h}{\cos(\pi/4-\epsilon)}$. Here $\epsilon$ is the only unknown parameter that would be evaluated using minimum work principle. Minimizing eq. (4.22) with respect to $\epsilon$ we have

$$\frac{\partial P_L}{\partial \epsilon} = \left( l - d \sin\left(\frac{\pi}{4} - \epsilon\right) \right) \left[ 2d^2k + k(1+2\epsilon)2d \frac{\partial d}{\partial \epsilon} \right] -$$

$$\left[ k(1+2\epsilon)d^2 \right] \left[ -\sin\left(\frac{\pi}{4} - \epsilon\right) \frac{\partial d}{\partial \epsilon} + d \cos\left(\frac{\pi}{4} - \epsilon\right) \right] = 0$$

(4.23)

After a few algebraic rearrangements the following expression can be easily obtained

$$\cos\left(\frac{\pi}{4} - \epsilon\right) - (1+2\epsilon)\sin\left(\frac{\pi}{4} - \epsilon\right) = \frac{\left[(1+2\epsilon)d\right]}{2\left(l-d\sin\left(\frac{\pi}{4} - \epsilon\right)\right)}$$

(4.24)

Using eq. (4.22), eq. (4.24) can be re-written as follows

$$\cos\left(\frac{\pi}{4} - \epsilon\right) - (1+2\epsilon)\sin\left(\frac{\pi}{4} - \epsilon\right) = \frac{P}{2kd}$$

(4.25)

Eq. (4.25) represents the condition of force equilibrium which is identical to that obtained from SLF analysis (Kachanov, 1971). As per SLF procedure, for the case of long cantilever, $2\epsilon<\pi/4-1/2$, the plastic deformation field shown in Fig. 4.2 is valid. When $2\epsilon=\pi/4-1/2$, SLF shown in Fig. 4.1(a) leads to a smaller, and therefore more appropriate, value of the limit load. The first type of field, Fig. 4.1(a), arises with short cantilever ($l/2h$
\( \leq 13.73 \), when \((l/2h = 13.73)\) the two fields coincided, since \(R=0\). All these conditions can be directly obtained from the proposed MUB theorem. Thus, for both short and long cantilever MUB theorem and SLF analysis finally provide identical results. Numerical values of \(\varepsilon\), \(R\), \(d\) and \(P_L\) for both these configurations were already provided by Green (1954).

### 4.2.2 Bending of deeply cracked fracture mechanics specimens

In order to demonstrate equivalence of proposed MUB theorem and SLF analysis for cracked bodies, standard fracture mechanics specimens (plane-strain) viz. pure bending specimen, SE(PB), three-pint bend specimens, SE(B) and compact tension specimen, C(T) were analysed. These cracked bend specimens are nowadays frequently used in fracture mechanics analysis. To ensure that a high crack tip constraint exists in these specimens the testing standards usually recommend deeply cracked geometries subjected to predominant bending load. This recommendation ensures that the fracture toughness so obtained would be a conservative estimate of the fracture toughness of the actual structure under investigation. In low strength metal specimens the remaining ligament is normally fully yielded before crack growth initiation. The plastic deformation, therefore, gets confined to the uncracked ligament and does not spread to the cracked flanks. Under these conditions the proposed MUB theorem, assuming that the material is rigid-plastic, can provide sufficiently accurate estimates of the crack tip stresses and, hence, the crack tip constraint parameters like \(Q\), \(q\) or \(h\). Which of these constraint parameters is more appropriate to characterise ductile fracture process has been the topic of many detailed investigations (e.g.
Roos et al., 1993). In the remaining part of this chapter results of crack tip constraint is expressed in terms of $Q$, though $q$ or $h$ can also be easily obtained. The parameter $Q$ is defined by O’Dowd and Shih (1991) in the form

$$Q = \left. \frac{\sigma_{\theta \theta} - \sigma_{\theta \theta, \text{Prandtl}}}{\sigma_y} \right|_{\theta=0, \frac{ra_y}{J}=2}$$

(4.26)

for a rigid-plastic material; where $\theta$ is the angle in polar co-ordinate system centered at the crack tip and the subscript Prandtl denotes the stress component calculated from the solution of Prandtl crack tip field. In addition, parameters like the limit load, plastic eta factors ($\eta_p$) and plastic rotation factor ($r_p$) were also evaluated. These parameters may serve as an essential preliminary aspect of the subsequent fracture analysis.

In conventional fracture mechanics J-integral has been widely used as a parameter (though it has its own limitations) to characterize stress and strain field in the vicinity of crack tip for a non-linear elastic material. Its experimental evaluation requires a calibration factor ($\eta_{\text{LLD}}$) either based on load-load line displacement ($\eta_{\text{LLD}}$) records or on load-crack mouth opening displacement ($\eta_{\text{CMOD}}$) records. For a given specimen geometry and loading condition, MUB theorem provides the plastic limit load solution, $P_L$, as a function of crack length $a/W$, which then provides $\eta_{\text{LLD}}$ and subsequently $\eta_{\text{CMOD}}$, (see chapter 7 for details).
4.2.2.1 *Single edge cracked specimen in pure bending, SE(PB)*

For a deeply cracked SE(PB) specimen ($a/W > 0.3$), the plastic deformation mechanism, as shown in Fig. 4.3, was assumed. This is exactly the same deformation mode that was assumed by Green (1953) in his slip line field analysis. Instead of SLF method, here MUB theorem (eq. 3.33) was used to evaluate the limit moment and other useful fracture mechanics parameters. In the proposed solution it is assumed that at limit moment, there is a central pivot OPQ that remains rigid, around which the rigid parts of the specimen on either side rotate by shearing over the circular arcs, OPQ. Near free surface, there is a region of constant compressive stress, RQR, due to traction free boundary condition at the edge A-A. Circular arcs, OPQ, merge in this compressive zone tangentially. The line, OPQR', consisting of a straight line segment, QR', and a circular arc, OPQ, has a continuous tangent and therefore the corresponding velocity field is kinematically admissible (Kachanov, 1971). From Hundy’s field (1954), the stress distribution in this compressive zone can be expresses as

$$\sigma_{11} = 0, \sigma_{22} = -2k \quad \text{and} \quad \sigma_{12} = 0$$

(4.27)

Here $k$ is the shear yield strength ($\sigma_y / \sqrt{3}$, according to Von-Mises yield criterion). As far as kinematics is concerned it is assumed that, at limit state, the relative angular velocity, $\varphi^*$, with which rigid parts rotate becomes equal to the rate of imposed rotation i.e. $\varphi^* = \omega$. Using stress distribution of compressive zone and evaluating the work done by the stresses on the
elastic-plastic boundary QR, the resulting expression for limit moment, using eq. (3.33), can be expressed as

\[ M_\ell = k \left[ R^2 \left( \beta + \frac{\pi}{4} \right) + x(R + 0.5x) \right] \]  \hfill (4.28)

Eq. (4.28) represents the condition of global moment equilibrium about the hinge point. From geometry following relation can be easily obtained

\[ R = \frac{l - x}{\sqrt{2} \left( \sin \beta + \frac{1}{\sqrt{2}} \right)} \]  \hfill (4.29)

Here \( x \) and \( \beta \) are the two independent unknown parameters that would be evaluated using minimum work principle. Since the algebra involved is quite standard only important steps/equations are provided. Minimizing eq. (4.28) with respect to these two unknown parameters we have

\[ \frac{\partial M_\ell}{\partial x} = k \left[ \left( \beta + \frac{\pi}{4} \right) 2R \frac{\partial R}{\partial x} + R + x \frac{\partial R}{\partial x} + x \right] = 0 \]  \hfill (4.30)

\[ \frac{\partial M_\ell}{\partial \beta} = k \left[ R^2 + \left( \beta + \frac{\pi}{4} \right) 2R \frac{\partial R}{\partial \beta} + x \frac{\partial R}{\partial \beta} \right] = 0 \]  \hfill (4.31)

On further simplification, these two equations can be expressed as
(\sin \beta + \cos \beta) = 2 \cos \beta \left( \beta + \frac{\pi}{4} \right) \quad (4.32)

x = \frac{R}{\sqrt{2}} \left[ \cos \beta - \sin \beta + 2 \sin \beta \left( \beta + \frac{\pi}{4} \right) - \sqrt{2} \right] \quad (4.33)

Eqs. (4.32) & (4.33) actually represent global equilibrium conditions which can also be obtained from detailed SLF analysis (Chakrabarty, 1987). In addition to limit moment, plastic eta factor, \( \eta_{LLD} \), and plastic rotation factor, \( r_p \), were also evaluated and were found to be in exact agreement with SLF solution. Numerical values of these parameters are presented in Table 4.1. In Table 4.1, the values of the limit moment \( M_L \) were normalised with the limit moment of an uncracked bar \( M_0 \).

It is important to note that if presence of compressive zone is neglected then eq. (4.28) would reduce to the classical upper bound solution proposed by Prager (1955), that is, \( M_L = 0.398 \sigma f^2 \) which is about 10% higher than that obtained from detailed SLF/MUB solution. In addition to limit moment, plastic eta factor, \( \eta_{LLD} \), and plastic rotation factor, \( r_p \), were also compared with the classical SLF solution. MUB theorem provides \( \eta_{LLD} = 2 \) and \( r_p = 0.37 \) which are in exact agreement with Green’s (1953) SLF solution.

### 4.2.2.1.1 Fully plastic crack-tip stress fields for SE(PB) specimen

In the previous section, it was established that, for the assumed plastic deformation field, minimum work principle automatically satisfies the global equilibrium equations. It means that the state of stress in the regions having rigid mode of deformation and hence through
out the body is identical with that obtained from detailed SLF analyses. Thus, the assumed plastic deformation field is in fact SLF and Hencky’s theorem can now be used at any point in the plastically deformed regions to evaluate the state of stress. However, from the stress distribution we cannot determine the constraint parameter \( Q \) at the crack tip directly as the assumed plastic field can only give the stress components along the slip lines which radiate from the crack tip and which are inclined to the horizontal line with an angle larger than 45° (Hao et al., 2000). From this kind of slip line fields the stress field surrounding the crack tip is not uniquely obtainable. Following Hao et al. (2000), possible crack-tip stress field is illustrated in Fig. 4.4. In this figure we assume, asymptotically, that a small segment of straight slip line OO’ exists. It radiates from the crack tip and is connected to the arc OP in the global slip line field. Thus, the stress components on these small lines are constant and equal to the components at the point O’ on the arc O’P. The plastic deformation expand from the line OO’ ahead of the crack tip, as shown in Fig. 4.4 and form a diamond-like plastic zone OBXB like that in Prandtl field. In Fig. 4.3, OPQR’ is a \( \beta \) slip line along which \( \frac{\sigma}{2k} + \theta = \eta \). At point Q, \( \sigma = -k \) and \( \theta = \pi/4 \). Thus, \( \eta_o = \eta_o = \frac{\pi}{4} - \frac{1}{2} \). At point O, \( \theta = -\beta \) and on substituting \( \psi = \pi/4 + \beta \), \( \sigma_o = k(2\psi - 1) \). Also O’BX is \( \alpha \)-slip line along which \( \frac{\sigma}{2k} - \theta = \xi \). Thus, \( \xi_o = \xi_o = \psi - \frac{1}{2} + \beta \). At point B, \( \theta = -\pi/4 \) and the normal pressure (hydrostatic stress) can be expressed in terms of slip angle \( \beta \) as follows.

\[
\sigma_n = \sigma_x = k \left( 2 \psi - 1 \right) + 2k \left( \beta - \frac{\pi}{4} \right)
\]  

(4.34)
In triangle OBX (that is actually a uniform stress zone) lying just below the crack tip, maximum tensile stress is given by the following expression

\[ \sigma_{\theta\theta} = \sigma_B + k \]  

(4.35)

Comparison of these local stresses near the crack tip and, hence, the constraint factor obtained using MUB theorem and those from detailed SLF solution (Green, 1953) is given in Table 4.1.

Table 4.1: Comparison of results of SE(PB) specimen obtained from MUB theorem with SLF analysis (Green, 1953) for a/W = 0.3 - 1.

<table>
<thead>
<tr>
<th>R/l</th>
<th>( \psi^* )</th>
<th>x/l</th>
<th>( M_l/M_o )</th>
<th>( r_p )</th>
<th>( \eta_{LLD} )</th>
<th>( (\sigma_m/\sigma_y) ) at ( \theta=0 )</th>
<th>( (\sigma_{10}/\sigma_y) ) at ( \theta=0 )</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>SLF</td>
<td>0.388</td>
<td>117.04</td>
<td>0.502</td>
<td>1.26</td>
<td>0.369</td>
<td>2.0</td>
<td>2.326</td>
<td>2.903</td>
</tr>
<tr>
<td>MUB</td>
<td>0.388</td>
<td>117.04</td>
<td>0.502</td>
<td>1.26</td>
<td>0.369</td>
<td>2.0</td>
<td>2.326</td>
<td>2.903</td>
</tr>
</tbody>
</table>

4.2.2.2 **Single edge cracked specimen in three-point bending, SE(B)**

For a deeply cracked SE(B) specimen \((a/W > 0.177)\), the plastic deformation mechanism suggested by Green and Hundy (1956), as shown in Fig. 4.5, was used. In constructing this field it was assumed that the crack is sufficiently deep for initial overall yielding not to spread to the surface on the cracked side and that there is a stress singularity at the point R on the flat surface. In practice, the central load is supported over a finite length of the
surface spanning the point R and local deformation would fit the load ‘point’ to this surface. Thus, the singularities shown coinciding at R in Fig. 4.5 should be separated by a small distance; in fact, with them coinciding, the yield criterion is certainly violated in the rigid corner QRQ (Hill, 1954). However, as Ewing (1968) has demonstrated, the effect of neglecting this local disturbance does not lead to much error in the overall pattern or limit load, or in the stress distribution near the crack tip. The effect of finite indenter width would be dealt in more detail in the later part of this sub-section.

In the proposed solution it was assumed that at limit moment, there is a region OPQQPO that remains rigid, around which the rigid parts of the specimen on either side rotate by shearing over the circular arcs, OPQ. Near free surface, we have uniaxial compression in the region RST. Adjacent to it is the central field QRT which merges with the circular slip line of radius R. From Hundy’s field (1954), the stress distribution in this compressive zone can be expressed as \( \sigma_{11} = 0, \ \sigma_{22} = -2k \) and \( \sigma_{12} = 0 \). In the central field QRT, the shear stress along RT is \( k \) and the normal pressure acting on the segment RQ is \( k(1+2\gamma) \). The scheme used to relate the relative velocity, \( \nu^* \), (with which rigid parts rotate) to the rate of imposed displacement, \( \delta^* \), is shown in Fig. 4.6. In an actual SE(B) specimen supports are fixed and load is applied at the center that causes an imposed displacement. However, for a kinematic analysis it can be assumed that load point is fixed and a displacement, \( \delta^* \), is imposed at the supports. From kinematics, it is well known that the instantaneous centre of a body sliding on a curved surface lies at the center of curvature. Since the undeformed material is assumed to slide over the circular arcs, therefore, their instantaneous centre must lie at the centre of these arcs. At instantaneous centre the tangential velocity is zero. As the undeformed portions are moving rigidly, linear variation
of velocity (between instantaneous centre and support) can be assumed. As a result, x-component of imposed displacement (see Fig. 4.6) at crack tip can be expressed as follows:

$$\delta_x = \frac{\delta R \cos \beta}{(S/2 + R \cos \beta)}$$

(4.36)

For kinematic admissibility the x-component of tangential velocity, $v^*$, at crack tip, must be equal to the x-component of imposed displacement i.e.

$$\delta_x = \frac{\delta R \cos \beta}{(S/2 + R \cos \beta)} = v^* \cos \beta$$

(4.37)

Thus, the tangential velocity can be expressed in terms of imposed displacement, $\delta^*$, as given by following equation.

$$v^* = \frac{R \delta}{(S/2 + R \cos \beta)}$$

(4.38)

The angular velocity $\omega$ with which the rigid part of the beam rotate about the hinge H can be obtained from the following relation

$$\omega = \frac{v^*}{R} = \frac{\delta}{(S/2 + R \cos \beta)}$$

(4.39)

Now invoking work principle, that is, eq. (3.33), limit load can be expressed as,
\[ P_L \delta = \int_0^L k v^2 R d\theta + \int_{\gamma} \sigma_y n_y dS + \int_{\beta} \sigma_y n_y dS \]  \hspace{1cm} (4.40)

Since the stress and velocity distribution on the elastic-plastic boundary QTS are quite similar to that of short cantilever, Fig. 4.1 (c), the details are omitted and the resulting expression for the limit load can be expressed as

\[ P_L = \frac{2k}{\left(\frac{S}{2} + R \cos \beta\right)} \left[R^2 (\beta + \pi / 4 - \gamma) + (1 + 2\gamma) \frac{x^2}{2} + xR\right] \]  \hspace{1cm} (4.41)

for

\[ R = \frac{l - x \cos \left(\frac{\pi}{4} - \gamma\right)}{\sin \beta + \sin \left(\frac{\pi}{4} - \gamma\right)} = \frac{x \sin \left(\frac{\pi}{4} - \gamma\right)}{\cos \left(\frac{\pi}{4} - \gamma\right) - \cos \beta} \]  \hspace{1cm} (4.42)

Eq. (4.41) represents the global moment equilibrium, about the hinge point, that needs to be minimized with respect to unknown parameters, that is, \(x, \gamma\) and \(\beta\). These three parameters are not independent but are subjected to a geometrical constraint as specified by eq. (4.42). In the present case, Lagrange’s method of undetermined multiplier was used. Application of this optimization technique requires the geometrical constraint to be re-expressed in following form

\[ \phi(x, \beta, \gamma) = x \sin \left(\frac{\pi}{4} - \gamma\right) \left[\sin \beta + \sin \left(\frac{\pi}{4} - \gamma\right) \right] - \left[\cos \left(\frac{\pi}{4} - \gamma\right) - \cos \beta\right] \left[l - x \cos \left(\frac{\pi}{4} - \gamma\right)\right] \]  \hspace{1cm} (4.43)
Following Lagrange’s multiplier method, we have

\[
\frac{\partial P}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 ; \quad \frac{\partial P}{\partial \beta} + \lambda \frac{\partial \phi}{\partial \beta} = 0 ; \quad \frac{\partial P}{\partial \gamma} + \lambda \frac{\partial \phi}{\partial \gamma} = 0
\]  

(4.44)

On elimination of undetermined multiplier, \(\lambda\), from the above equations we have

\[
\frac{\partial P}{\partial x} \frac{\partial \phi}{\partial \beta} - \frac{\partial P}{\partial \beta} \frac{\partial \phi}{\partial x} = 0
\]  

(4.45)

\[
\frac{\partial P}{\partial \beta} \frac{\partial \phi}{\partial \gamma} - \frac{\partial P}{\partial \gamma} \frac{\partial \phi}{\partial \beta} = 0
\]  

(4.46)

\[
\frac{\partial P}{\partial x} \frac{\partial \phi}{\partial \gamma} - \frac{\partial P}{\partial \gamma} \frac{\partial \phi}{\partial x} = 0
\]  

(4.47)

Minimizing eq. (4.41) with respect to unknown parameters and after a few algebraic simplifications, we have

\[
\frac{\partial P}{\partial x} = \frac{2k}{x} \left[ \frac{2\psi R^2 + (1 + 2\gamma)x^2 + 2Rx}{\left( \frac{S}{2} + R \cos \beta \right)} \right] - \frac{P \cos \beta R}{x \left( \frac{S}{2} + R \cos \beta \right)}
\]  

(4.48)
Here,\[\psi = \beta + \frac{\pi}{4} - \gamma \quad (4.51)\]

Also\[\frac{\partial \phi}{\partial x} = 1 - \cos \psi \quad (4.52)\]

\[
\frac{\partial \phi}{\partial \beta} = x \sin \left( \frac{\pi}{4} - \gamma \right) \cos \beta - R \left[ \sin^2 \beta + \sin \beta \sin \left( \frac{\pi}{4} - \gamma \right) \right] \quad (4.53)
\]

\[
\frac{\partial \phi}{\partial \gamma} = -x \sin \left( \frac{\pi}{4} - \gamma \right) \cos \left( \frac{\pi}{4} - \gamma \right) - x \sin \psi - R \left[ \sin \beta + \sin \left( \frac{\pi}{4} - \gamma \right) \right] \sin \left( \frac{\pi}{4} - \gamma \right) \quad (4.54)
\]
Substitution of eqs. (4.49) & (4.50) and eqs. (4.53) & (4.54) in eq. (4.46) finally leads to the following equation:

\[
\frac{P_L}{2} = k_x \left[ \sin \left( \frac{\pi}{4} + \gamma \right) - (1 + 2\gamma) \cos \left( \frac{\pi}{4} + \gamma \right) \right] + k_R \left[ \left( 2\psi - (1 + 2\gamma) \right) \sin \left( \psi + \frac{\pi}{4} + \gamma \right) + \cos \left( \psi + \frac{\pi}{4} + \gamma \right) + (1 + 2\gamma) \sin \left( \frac{\pi}{4} + \gamma \right) - \cos \left( \frac{\pi}{4} + \gamma \right) \right] \quad (4.55)
\]

Similarly, substitution of eqs. (4.48) & (4.54) and eqs. (4.50) & (4.52) in eq. (4.47) provides the following expression:

\[
k_x \left[ (1 + 2\gamma) \sin \left( \frac{\pi}{4} + \gamma \right) + \cos \left( \frac{\pi}{4} + \gamma \right) \right] + k_R \left[ \left( 2\psi - (1 + 2\gamma) \right) \cos \left( \psi + \frac{\pi}{4} + \gamma \right) - \sin \left( \psi + \frac{\pi}{4} + \gamma \right) + (1 + 2\gamma) \cos \left( \frac{\pi}{4} + \gamma \right) + \sin \left( \frac{\pi}{4} + \gamma \right) \right] = 0 \quad (4.56)
\]

Eqs. (4.55) & (4.56) represent the global force equilibrium equations (identical to those obtained from SLF analysis, Wu et al., 1987). This in turn again establishes equivalence of MUB theorem and SLF analysis. For the case of standard deeply cracked SE(B) specimen, Wu et al. (1987) have presented SLF solution and expressed the results of limit load in the form of plastic constraint factor \( L \) that gives the measure of load enhancement due to presence of notch. Limit load, thus can be expressed in the following form:

\[
P_L = L (a / W) \frac{2\sigma_y l^2}{\sqrt{3} S} \quad (4.57)
\]
In addition to plastic constraint factor, $L(a/W)$, $x$, $\beta$ and $\gamma$, plastic eta factor, $\eta_{LLD}$, and plastic rotation factor, $r_p$, were also compared with the classical SLF solution and their numerical values are given in Table 4.2.

It is worth to note that Joch et al. (1993) have also proposed an upper bound solution for standard SE(B) specimen. They have neglected the constant stress region RST and the fan field QRT (see Fig. 4.5) and assumed a deformation mechanism consisting of circular arcs emanating from the crack tip up to the free surface. The tangential velocity and the load point displacement was related by the following expression

$$v^* = \frac{2R\delta}{S} \quad (4.58)$$

The resulting expression of upper bound limit load (Joch et al., 1993), with Von-Mises plasticity, is given below

$$P_i = \frac{1.593\sigma_l f^2}{S} \quad (4.59)$$

It can be noted that the solution provided by Joch et al. (1993) does not explain the dependence of limit load on $a/W$ ratio. For $a/W=0.5$, the limit load is about 10% higher that that obtained from SLF solution (Wu et al., 1987). Similarly, the plastic constraint factor obtained is about 13.5% higher than that obtained from the SLF solution.
4.2.2.2.1 **Fully plastic crack-tip stress fields for SE(B) specimen**

In Fig. 4.5, assuming $\Delta$ RST to be in a state of compression, ST is $\alpha$-slip line along which $\frac{\sigma}{2k} - \theta = \xi$. At point S, $\sigma = -k$ and $\theta = \pi/4$. Thus, $\xi_s = \xi_0 = -\left(\frac{\pi}{4} + \frac{1}{2}\right)$. At point Q, $\theta = \frac{\pi}{4} - \gamma$ and thus the normal pressure is $\sigma_o = -k(1+2\gamma)$. Now, RQPO is $\beta$-slip line along which $\frac{\sigma}{2k} + \theta = \eta$. Thus, $\eta_o = \eta_0 = \frac{\pi}{4} - 2\gamma - \frac{1}{2}$. At point O, $\theta = -\beta$ and on substituting $\psi = \pi/4 + \beta - \gamma$, $\sigma_o = k(2\psi - 2\gamma - 1)$. Also O’BX is $\alpha$-slip line along which $\frac{\sigma}{2k} - \theta = \xi$. Thus, $\xi_o = \xi_0 = \psi - \gamma - \frac{1}{2} + \beta$. At point B, $\theta = -\pi/4$ and the normal pressure (hydrostatic stress) can be expressed in terms of slip angle $\beta$ as follows.

$$\sigma_\beta = \sigma_x = k(2\psi - 2\gamma - 1) + 2k\left(\beta - \frac{\pi}{4}\right)$$  \hspace{1cm} (4.60)

In triangle OBX (that is actually a uniform stress zone) lying just below the crack tip, maximum tensile stress is given by the following expression

$$\sigma_{\theta\theta} = \sigma_\beta + k$$  \hspace{1cm} (4.61)

Comparison of hydrostatic stress, near the crack tip, obtained using MUB theorem and that from detailed SLF solution (Wu et al., 1987) is given in Table 4.2.
Table 4.2: Comparison of theoretical results of SE(B) specimen obtained from MUB theorem with SLF analysis (Wu et al., 1987).

<table>
<thead>
<tr>
<th>$a/W$</th>
<th>Method</th>
<th>$R/(W-a)$</th>
<th>$\psi$</th>
<th>$\gamma$</th>
<th>$\chi/(W-a)$</th>
<th>$L$</th>
<th>$r_p$</th>
<th>$\eta_{LLD}$</th>
<th>$(\sigma_m/k)$ at $\theta=0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>SLF</td>
<td>0.5</td>
<td>102.38</td>
<td>7.37</td>
<td>0.3</td>
<td>1.215</td>
<td>0.455</td>
<td>1.937</td>
<td>3.006</td>
</tr>
<tr>
<td></td>
<td>MUB</td>
<td>0.5</td>
<td>102.31</td>
<td>7.35</td>
<td>0.3</td>
<td>1.215</td>
<td>0.455</td>
<td>1.921</td>
<td>3.00</td>
</tr>
<tr>
<td>0.3</td>
<td>SLF</td>
<td>0.49</td>
<td>103.99</td>
<td>7.28</td>
<td>0.31</td>
<td>1.227</td>
<td>0.451</td>
<td>1.945</td>
<td>3.118</td>
</tr>
<tr>
<td></td>
<td>MUB</td>
<td>0.49</td>
<td>104.08</td>
<td>7.3</td>
<td>0.31</td>
<td>1.227</td>
<td>0.450</td>
<td>1.935</td>
<td>3.124</td>
</tr>
<tr>
<td>0.4</td>
<td>SLF</td>
<td>0.48</td>
<td>105.63</td>
<td>7.18</td>
<td>0.32</td>
<td>1.238</td>
<td>0.447</td>
<td>1.953</td>
<td>3.233</td>
</tr>
<tr>
<td></td>
<td>MUB</td>
<td>0.48</td>
<td>105.66</td>
<td>7.17</td>
<td>0.32</td>
<td>1.238</td>
<td>0.447</td>
<td>1.948</td>
<td>3.235</td>
</tr>
<tr>
<td>0.5</td>
<td>SLF</td>
<td>0.47</td>
<td>107.29</td>
<td>7.07</td>
<td>0.34</td>
<td>1.248</td>
<td>0.443</td>
<td>1.961</td>
<td>3.349</td>
</tr>
<tr>
<td></td>
<td>MUB</td>
<td>0.47</td>
<td>107.29</td>
<td>7.07</td>
<td>0.34</td>
<td>1.248</td>
<td>0.443</td>
<td>1.959</td>
<td>3.346</td>
</tr>
<tr>
<td>0.6</td>
<td>SLF</td>
<td>0.47</td>
<td>108.96</td>
<td>6.96</td>
<td>0.35</td>
<td>1.258</td>
<td>0.439</td>
<td>1.969</td>
<td>3.466</td>
</tr>
<tr>
<td></td>
<td>MUB</td>
<td>0.47</td>
<td>108.96</td>
<td>6.94</td>
<td>0.35</td>
<td>1.258</td>
<td>0.439</td>
<td>1.97</td>
<td>3.462</td>
</tr>
<tr>
<td>0.7</td>
<td>SLF</td>
<td>0.46</td>
<td>110.65</td>
<td>6.84</td>
<td>0.36</td>
<td>1.267</td>
<td>0.435</td>
<td>1.977</td>
<td>3.584</td>
</tr>
<tr>
<td></td>
<td>MUB</td>
<td>0.46</td>
<td>110.64</td>
<td>6.81</td>
<td>0.36</td>
<td>1.267</td>
<td>0.435</td>
<td>1.979</td>
<td>3.582</td>
</tr>
<tr>
<td>0.8</td>
<td>SLF</td>
<td>0.45</td>
<td>112.35</td>
<td>6.71</td>
<td>0.37</td>
<td>1.275</td>
<td>0.431</td>
<td>1.985</td>
<td>3.702</td>
</tr>
<tr>
<td></td>
<td>MUB</td>
<td>0.45</td>
<td>112.34</td>
<td>6.71</td>
<td>0.37</td>
<td>1.275</td>
<td>0.431</td>
<td>1.988</td>
<td>3.701</td>
</tr>
<tr>
<td>0.9</td>
<td>SLF</td>
<td>0.44</td>
<td>114.05</td>
<td>6.58</td>
<td>0.38</td>
<td>1.282</td>
<td>0.427</td>
<td>1.987</td>
<td>3.821</td>
</tr>
<tr>
<td></td>
<td>MUB</td>
<td>0.44</td>
<td>114.05</td>
<td>6.53</td>
<td>0.38</td>
<td>1.282</td>
<td>0.427</td>
<td>1.994</td>
<td>3.821</td>
</tr>
</tbody>
</table>
4.2.2.2 Effect of indenter width on the limit load and local stresses near the crack tip for SE(B) specimen

In actual practice, SE(B) specimen is loaded in three-point bending by an indenter of finite width. This requires evaluation of correction to be expected due to finite indenter width. The indenter surface in practice is a circular arc of about 2.5 mm radius but following the suggestions of Alexander and Komoly (1962), Ewing (1968) replaced it by a flat punch of width $2b$ and analyse the effects on the SLF by varying this small dimension $b$. The plastic deformation field (as suggested by Ewing, 1968) is shown in Fig. 4.7. The stress distribution in compressive zone and central field remain same as obtained earlier. Application of MUB theorem, eq. (3.33), yields the following expression for limit load (same as eq. (4.41)).

$$P_L = \frac{2k}{S + R \cos \beta} \left[ R^2 \left( \beta + \pi \sqrt{4 - \gamma} \right) + \left( \frac{x^2}{2} + xR \right) \right]$$

(4.62)

for

$$R = \frac{b + x \sin \left( \frac{\pi}{4} - \gamma \right)}{\cos \left( \frac{\pi}{4} - \gamma \right) - \cos \beta}$$

(4.63)

$$\left( b + x \sin \left( \frac{\pi}{4} - \gamma \right) \right) \left( \sin \beta + \sin \left( \frac{\pi}{4} - \gamma \right) \right) = \left( \cos \left( \frac{\pi}{4} - \gamma \right) - \cos \beta \right) \left( l - x \cos \left( \frac{\pi}{4} - \gamma \right) \right)$$

(4.64)
Rest of the analysis is exactly same as discussed under sub-section 4.2.2.2 and we proceed directly to results. Effect of finite indenter width on plastic constraint factor, $L(a/W)$, $R$, $x$, $\beta$ and $\psi$, and on local stresses near the crack tip is quantified in Table 4.3. Again, both MUB theorem and SLF analysis provides identical results.

Table 4.3: Effect of indenter width on plastic field parameters and crack tip stresses obtained from MUB theorem and SLF analysis (Ewing, 1968) for $a/W=0.2$, $W=10$ mm.

<table>
<thead>
<tr>
<th>Span $S$ (mm)</th>
<th>$b$ (mm)</th>
<th>Method</th>
<th>$R$ (mm)</th>
<th>$\psi^*$</th>
<th>$\gamma^*$</th>
<th>$x$ (mm)</th>
<th>$\beta$</th>
<th>$(\sigma_{00}/2k)_{\theta=0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S=44$</td>
<td>0</td>
<td>SLF</td>
<td>3.9629</td>
<td>103.548</td>
<td>7.309</td>
<td>2.4779</td>
<td>1.224</td>
<td>2.044</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MUB</td>
<td>3.9627</td>
<td>103.538</td>
<td>7.312</td>
<td>2.4786</td>
<td>1.224</td>
<td>2.043</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>SLF</td>
<td>4.0881</td>
<td>103.755</td>
<td>9.431</td>
<td>2.2456</td>
<td>1.251</td>
<td>2.051</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MUB</td>
<td>4.0879</td>
<td>103.769</td>
<td>9.419</td>
<td>2.2453</td>
<td>1.251</td>
<td>2.051</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>SLF</td>
<td>4.2173</td>
<td>104.119</td>
<td>11.532</td>
<td>2.0322</td>
<td>1.287</td>
<td>2.064</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MUB</td>
<td>4.2163</td>
<td>104.128</td>
<td>11.531</td>
<td>2.0335</td>
<td>1.287</td>
<td>2.064</td>
</tr>
<tr>
<td>$S=40$</td>
<td>0</td>
<td>SLF</td>
<td>4.0219</td>
<td>102.375</td>
<td>7.376</td>
<td>2.4078</td>
<td>1.215</td>
<td>2.003</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MUB</td>
<td>4.0221</td>
<td>102.364</td>
<td>7.375</td>
<td>2.4077</td>
<td>1.215</td>
<td>2.002</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>SLF</td>
<td>4.1475</td>
<td>102.590</td>
<td>9.510</td>
<td>2.1757</td>
<td>1.243</td>
<td>2.01</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MUB</td>
<td>4.1480</td>
<td>102.591</td>
<td>9.499</td>
<td>2.1746</td>
<td>1.243</td>
<td>2.01</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>SLF</td>
<td>4.2771</td>
<td>102.957</td>
<td>11.625</td>
<td>1.9624</td>
<td>1.279</td>
<td>2.023</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MUB</td>
<td>4.2766</td>
<td>102.960</td>
<td>11.622</td>
<td>1.9630</td>
<td>1.279</td>
<td>2.023</td>
</tr>
</tbody>
</table>
4.2.3 **Compact tension C(T) specimen**

Theoretical solution proposed here is valid for a deeply cracked C(T) specimen \((a/W \geq 0.1)\). The same plastic deformation mechanism that was suggested by Green (1953) for SE(PB) specimen was used by Ewing and Richards (1974) in their SLF analysis of C(T) specimen. The stress distribution in compressive zone is assumed to be same as that in case of SE(B) specimen i.e. \(\sigma_{11} = 0\), \(\sigma_{22} = -2k\) and \(\sigma_{12} = 0\).

The scheme used to relate the relative velocity, \(\nu^*\), (with which rigid parts rotate) to the rate of imposed displacement, \(\delta^*\), is similar to that used for SE(B) specimen. Since the undeformed material is assumed to slide over the circular arcs, therefore, their instantaneous centre must lie at the centre of these arcs. At instantaneous centre the tangential velocity is zero. As the undeformed portions are moving rigidly, linear variation of velocity (between instantaneous centre and pin) can be assumed. As a result, Y-component of imposed displacement (see Fig. 4.6) at the crack tip can be expressed as follows

\[
\delta_y = \frac{\delta R \sin \beta}{2(a + R \sin \beta)}
\]

(4.65)

For kinematic admissibility, Y-component of the tangential velocity, \(\nu^*\), at the crack tip, must be equal to Y-component of imposed displacement i.e.

\[
\delta_y = \frac{\delta R \sin \beta}{2(a + R \sin \beta)} = \nu^* \sin \beta
\]

(4.66)
Thus, the tangential velocity can be expressed in terms of imposed displacement, $\delta^*$, as given by the following equation.

$$v^* = \frac{R\delta}{2(a + R\sin \beta)} \quad (4.67)$$

Using the stress distribution of compressive zone and the proposed velocity field in the MUB theorem, eq. (3.33), the resulting expression for limit load can be expressed as

$$P_L = \frac{k}{(a + R\sin \beta)} \left[ R^2(\beta + \pi/4) + x(R + 0.5x) \right] \quad (4.68)$$

Eq. (4.68) represents the condition of global moment equilibrium about the hinge point. From geometry following relation can be easily obtained

$$R = \frac{l - \frac{x}{\sqrt{2}}}{(\sin \beta + \frac{1}{\sqrt{2}})} \quad (4.69)$$

Similar to the case of SE(PB) specimen, here $x$ and $\beta$ are the two independent unknown parameters that would be evaluated using minimum work principle. Minimizing eq. (4.68) with respect to these two unknown parameters and after a few algebraic re-arrangements we have
\[ \sqrt{2} \left( \beta + \frac{\pi}{4} \right) R = \left[ (l - \sqrt{2}x) + x \left( \sin \beta + \frac{1}{\sqrt{2}} \right) \right] + \frac{P_1 \sin \beta}{k \sqrt{2}} \tag{4.70} \]

\[ 1 - 2 \cos \left( \beta + \frac{\pi}{4} \right) \frac{P_1 \cos \beta}{\sin \beta + \frac{1}{\sqrt{2}}} k \sqrt{2} R \left( \sin \beta + \frac{1}{\sqrt{2}} \right) = x \cos \beta \frac{R}{\sin \beta + \frac{1}{\sqrt{2}}} \tag{4.71} \]

On further simplification, these two equations can be rearranged to give

\[ (\sin \beta + \cos \beta) = 2 \cos \left( \beta + \frac{\pi}{4} \right) \tag{4.72} \]

\[ x = R \left( \frac{1 - \sqrt{2} \cos \beta}{\sqrt{2} \cos \beta} \right) - \frac{P_1}{k \sqrt{2}} \tag{4.73} \]

Solution of eq. (4.72) provides \( \beta = 72.04^\circ \), that is, the angle subtended by the circular arc \( OQ \) to its centre (see Fig. 4.8) is always \( 117.04^\circ \) for all \( a/W \geq 0.1 \). This leads to considerable simplification and a closed-form expression for the limit load can be expressed as follows

\[ \frac{l}{2W} = 1.26 \sqrt{m^2 + m} - m \tag{4.74} \]

Here, \( m = \frac{P_1}{4kW} \), thus results obtained from MUB theorem are in exact agreement with SLF solution (Ewing and Richards, 1974). In addition to factor \( m \), plastic eta factor, \( \eta_{LLD} \), and
plastic rotation factor, $r_p$, were also compared with the classical SLF solution and their comparison is given in Table 4.4. It is worth to note that for deep notches as $a/W \to 1$, results obtained from eq. (4.60) reduces to the case of pure bending specimen, SE(PB), as discussed by Ewing and Richards (1974).

Following the procedure used to evaluate the fully plastic crack tip stresses, for a pure bending specimen SE(PB), the pressure (hydrostatic stress) in the diamond shaped plastic zone OBXB (see Fig. 4.4) can be expressed in terms of slip angle $\beta$ as follows:

\[ \sigma_b = \sigma_x = k(2\psi - 1) + 2k \left( \beta - \frac{\pi}{4} \right) \]  
\[ \text{(4.75)} \]

In triangle OBX (that is actually a uniform stress zone) lying just below the crack tip, maximum tensile stress is given by the following expression

\[ \sigma_{\theta\theta} = \sigma_b + k \]  
\[ \text{(4.76)} \]

For a deeply cracked C(T) specimen ($a/W \geq 0.1$), the angle subtended by the circular arc at its centre is $117.04^\circ$ and, thus, the hydrostatic stress near the crack tip, that is, $\sigma_{\mu}/2k$ is 2.014 and the tensile stress at the tip of crack ($\sigma_{\theta\theta}/\sigma_o$) is 2.903 (the same as found for a deeply cracked pure bending specimen).
Table 4.4: Comparison of results obtained from MUB theorem with SLF solutions (Ewing and Richards, 1974) for $W=10$ mm.

<table>
<thead>
<tr>
<th>$a/W$</th>
<th>$m$ (SLF)</th>
<th>$m$ (MUB)</th>
<th>$R$ (mm) (SLF)</th>
<th>$R$ (mm) (MUB)</th>
<th>$\eta_p$ (MUB)</th>
<th>$\eta_p$ (SLF)</th>
<th>$r_p$ (MUB)</th>
<th>$r_p$ (SLF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.244</td>
<td>0.244</td>
<td>5.3959</td>
<td>5.3955</td>
<td>2.63</td>
<td>2.63</td>
<td>0.57</td>
<td>0.57</td>
</tr>
<tr>
<td>0.2</td>
<td>0.179</td>
<td>0.179</td>
<td>4.5050</td>
<td>4.5033</td>
<td>2.591</td>
<td>2.59</td>
<td>0.53</td>
<td>0.53</td>
</tr>
<tr>
<td>0.3</td>
<td>0.127</td>
<td>0.127</td>
<td>3.7124</td>
<td>3.7105</td>
<td>2.531</td>
<td>2.53</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>0.4</td>
<td>0.087</td>
<td>0.087</td>
<td>3.0075</td>
<td>3.0061</td>
<td>2.458</td>
<td>2.453</td>
<td>0.47</td>
<td>0.47</td>
</tr>
<tr>
<td>0.5</td>
<td>0.056</td>
<td>0.056</td>
<td>2.3785</td>
<td>2.3776</td>
<td>2.377</td>
<td>2.376</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>0.6</td>
<td>0.033</td>
<td>0.033</td>
<td>1.8136</td>
<td>1.8126</td>
<td>2.295</td>
<td>2.293</td>
<td>0.43</td>
<td>0.43</td>
</tr>
<tr>
<td>0.7</td>
<td>0.017</td>
<td>0.017</td>
<td>1.3019</td>
<td>1.3008</td>
<td>2.214</td>
<td>2.212</td>
<td>0.41</td>
<td>0.41</td>
</tr>
<tr>
<td>0.8</td>
<td>0.007</td>
<td>0.007</td>
<td>0.8344</td>
<td>0.8334</td>
<td>2.137</td>
<td>2.133</td>
<td>0.39</td>
<td>0.39</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0017</td>
<td>0.0017</td>
<td>0.4038</td>
<td>0.4017</td>
<td>2.06</td>
<td>2.045</td>
<td>0.38</td>
<td>0.38</td>
</tr>
</tbody>
</table>

4.2.4 **Single-edge-cracked specimen under combined bending and tension**

Shiratori and Dodds (1980) discussed that the plastic deformation mechanism that was suggested by Green (1953) for SE(PB) specimen can also be used for single-edge-cracked specimen under combined bending and tension but with modification to account for the tensile load. Thus, in the proposed solution, for opening bending with small tension, it is assumed that at limit state, there is a region OPQQPO that remains rigid, around which the rigid parts of the specimen on either side rotate by shearing over the circular arcs, OPQ.
Near free surface, we have uni-axial compression in the region RQR’. Thus, plastic deformation mechanism (as shown in Fig. 4.9) consists of a circular arc that merges into the constant stress region tangentially. The stress distribution in this constant region is already known from Hundy’s field (1954).

Before proceeding further let us introduce two non-dimensional parameters $\hat{N}_n$ and $\hat{M}_n$ representing net section tension and bending moment normalized in terms of an uncracked plate with the shear strength $k$ and remaining ligament $l$:

\[
\hat{N}_n = \frac{N_n}{2kl} \quad (4.77)
\]

\[
\hat{M}_n = \frac{M_n}{0.5kl^2} \quad (4.78)
\]

The condition $-1 \leq \hat{N}_n < 0.5512$ is generally referred as small tension for which compressive zone near the free surface exists and SLFs are well known. $\hat{N}_n > 0.5512$ is referred as large tension for which compressive zone vanishes and SLFs are unknown.

The scheme used to relate the relative angular velocity, $\omega$, (with which rigid parts rotate) to the rate of imposed displacement, $\delta^\prime$, was proposed by Kim et al. (1995). Thus, the relative velocity, $\omega$, can be expressed in terms of imposed displacement, $\delta^\prime$, as given by following equation.
Using the stress distribution of compressive zone and the proposed velocity field in the MUB theorem, eq. (3.33), the resulting expression for limit moment can be expressed as

\[
M_L = k \left[ R^2 \left( \beta + \frac{\pi}{4} \right) + x(R + 0.5x) \right] - N_L \left( R \sin \beta - \frac{l}{2} \right)
\]  

(4.80)

Eq. (4.80) represents the condition of global moment equilibrium about the hinge point. From geometry following relation can be easily obtained

\[
R = \frac{l - x}{\sqrt{2}} \frac{1}{\sin \beta + \frac{1}{\sqrt{2}}}
\]  

(4.81)

Here \(x\) and \(\beta\) are the two independent unknown parameters that would be evaluated using minimum work principle. Minimizing eq. (4.80) with respect to these two unknown parameters we have

\[
\frac{\partial M_L}{\partial x} = k \left[ \left( \beta + \frac{\pi}{4} \right) R^2 \frac{\partial R}{\partial x} + x \frac{\partial R}{\partial x} \right] - N_L \sin \beta \frac{\partial R}{\partial x} = 0
\]  

(4.82)

\[
\frac{\partial M_L}{\partial \beta} = k \left[ R^2 \left( \beta + \frac{\pi}{4} \right) 2 R \frac{\partial R}{\partial \beta} + x R \frac{\partial R}{\partial \beta} \right] - N_L \left( R \cos \beta + \sin \beta \frac{\partial R}{\partial \beta} \right) = 0
\]  

(4.83)
On further simplification, these two equations can be expressed as

\[(\sin \beta + \cos \beta) = 2\cos \beta \left(\beta + \frac{\pi}{4}\right)\]  \hspace{1cm} (4.84)

\[x = R \left(\frac{1 - \sqrt{2} \cos \beta}{\sqrt{2} \cos \beta}\right) - \frac{N_i}{k\sqrt{2}}\]  \hspace{1cm} (4.85)

Eqs. (4.84) & (4.85) actually represent global equilibrium conditions which can also be obtained from detailed SLF analysis (Shiratori and Dodds, 1980). Solution of eq. (4.84) provides \(\beta = 72.04^\circ\), that is, the angle subtended by the circular arc \(OQ\) to its centre (see Fig. 4.9) is always \(117.04^\circ\) for all \(a/W > 0.35\) and small tension. This leads to considerable simplification and a closed-form expression for resulting yield locus that is in exact agreement with SLF solution (Shiratori and Dodds, 1980) can be expressed as follows

\[\Phi_{MUB} = \hat{M}_n + 0.7394 \hat{N}_n^2 - 0.5212 \hat{N}_n - 1.2606 = 0\]  \hspace{1cm} (4.86)

for \(-1 \leq \hat{N}_n < 0.5512\)

For deeply cracked specimens subjected to opening bending with large tensile load compressive zone region i.e. \(x\) becomes zero and hence MUB theorem would reduce to the classical upper bound theorem of limit analyses. The plastic field for this case can be approximated by a circular arc emanating from the crack tip and extending up to the free
surface (Rice, 1972). Using this deformation mechanism, as suggested by Rice (1972),
Kim et al. (1995) and Kim (2002) have already provided a complete analytical formulation
for Rice’s least upper bound yield locus that can be obtained directly by substituting $x=0$ in
eq (4.78). It may be observed that now there is no way to impose traction free boundary
condition and, therefore, the total angle subtended by circular arc is simply $\beta + \gamma$. The
resulting expression for limit moment can then be expressed as

$$M_L = kR^2(\beta + \gamma) - N_L \left( R \sin \beta - \frac{l}{2} \right)$$

(4.87)

for

$$R = \frac{l}{(\sin \beta + \sin \gamma)}$$

(4.88)

Minimizing eq. (4.87) with respect to the two unknown parameters, that is, $\beta$ and $\gamma$ we
have

$$\frac{\partial M_L}{\partial \beta} = k \left[ R^2 + (\beta + \gamma)2R \frac{\partial R}{\partial \beta} \right] - N_L \left[ R \cos \beta + \sin \beta \frac{\partial R}{\partial \beta} \right] = 0$$

(4.89)

$$\frac{\partial M_L}{\partial \gamma} = k \left[ R^2 + (\beta + \gamma)2R \frac{\partial R}{\partial \gamma} \right] - N_L \left[ \sin \beta \frac{\partial R}{\partial \gamma} \right] = 0$$

(4.90)

On further simplification, these two equations can be expressed as

$$(\beta + \gamma) \cos \beta - \frac{1}{2}(\sin \beta + \sin \gamma) + \frac{N_L \cos \beta \sin \gamma}{2kR} = 0$$

(4.91)
As expected, eqs. (4.91) and (4.92) are in exact agreement with the classical upper bound solution proposed by Kim et al. (1995). Thus, for the case of deeply cracked specimens subjected to opening bending with small tensile load MUB theorem has provided yield locus that is in exact agreement with detailed SLF solutions where as for opening bending with large tensile load MUB theorem reduces to the classical upper bound solution.

The procedure used to evaluate the fully plastic crack tip stresses for a pure bending specimen SE(PB) can be used for the case of opening bending with small tensile load (for which SLF exists). As the angle subtended by the circular arc at its centre is $117.04^\circ$ thus the hydrostatic stress near the crack tip, that is, $\sigma_\text{m}/2k$ is 2.014 and the tensile stress at the tip of crack ($\sigma_\theta/\sigma_\text{o}$) is 2.903 (same as found for a deeply cracked pure bending specimen). For the case of opening bending with large tensile load SLF breakdown and thus the Hencky’s equations cannot be used directly to evaluate stress distribution in plastically deformed region. For this case Kim (2002) has suggested an approximate procedure that can be used to estimate fully plastic crack tip stresses based on equilibrium condition of the least upper bound for plane strain deformation fields consisting of rigid-body rotation.

### 4.3 Discussion

In this chapter an analytical formulation of MUB theorem is presented and it is demonstrated that MUB theorem is actually a new form of already existing general work
principle. The most important consideration in the proposed method (as well as in SLF analysis) is the choice of assumed plastic deformation field which is subjected to restrictions imposed by kinematic admissibility and boundary conditions. Once this plastic field is chosen then either the concept of global static equilibrium (the SLF method) or minimum work principle (the MUB theorem) can be invoked to evaluate the dimensions of this plastic field. Unfortunately, till now these extremum/work principles have normally been utilized as a crude method of load bounding mainly in metal forming operations. Thus, whenever it was required to analytically evaluate the stress distribution near the tip of crack or any crack tip constraint parameter, SLF analysis was the only choice.

In the present context it is worth to discuss the work performed by Kim (2002). He has presented a simple method to estimate fully plastic crack tip stresses based on equilibrium condition of the least upper bound for plane strain deformation fields consisting of rigid-body rotation across a circular arc extending from the crack tip across the remaining ligament. However, such an assumed plastic field has very limited application and no attempt was made to establish the general equivalence of work principle and SLF analysis. In the present work it was established, for a wide variety of cases, that consideration of minimum work principle automatically leads to global equilibrium and, thus, the two methods, that is, MUB theorem and SLF analysis would give identical results. A wide variety of plastic deformation fields were analysed to establish this equivalence in general. The proposed MUB theorem was used to obtain theoretical solutions of the limit load, plastic eta factor ($\eta_p$), plastic rotation factor ($r_p$), and crack tip constraint parameter $Q$ for standard deeply cracked SE(PB), SE(B) and C(T) specimens, under plane strain condition. In addition, standard problem of bending of cantilever was
also analysed. Results of these standard homogeneous specimens were found to be in exact agreement with those obtained by detailed SLF analyses. The case of single-edge-cracked specimen under combined bending and tensile load was also analysed. A complete analytical formulation of yield locus for the entire range of tensile and bending load was obtained using the proposed MUB theorem. These findings have demonstrated that the proposed MUB theorem is a promising technique to solve a class of plane strain plasticity problems in rigid-plastic materials.

In SLF analysis, in addition to equilibrium considerations, Hencky’s theorem is invoked to set up additional equations which need to be solved simultaneously to evaluate the unknown parameters. As these equations are generally transcendental considerable mathematics is involved in SLF analysis. No such calculations are involved in the proposed method. Standard optimization algorithms can be readily used to minimize the plastic work done and thus the present method becomes very amenable to computational analysis. It is worth to note that MUB theorem automatically satisfies Hencky’s theorem. This work has shown one successful application of the proposed theorem and it is expected that similar other cases, particularly in metal forming processes, where in one region rigid plastic flow of the material is occurring and in other region statically governed stress field exists, may also be treated.
Fig. 4.1 (a): Assumed plastic deformation mechanism for a short cantilever under transverse load, Green (1954).
Fig. 4.1 (b): Schematic describing the relationship between tangential velocity $v^*$ and imposed displacement $\delta^\cdot$. 
Fig. 4.1 (c): Stress and velocity distribution on the elastic-plastic boundary of a short cantilever.
Fig. 4.2: Assumed plastic deformation mechanism for a long cantilever under transverse load, Green (1954).
Fig. 4.3: Assumed plastic deformation mechanism for SE(PB) specimen, Green (1953).
Fig. 4.4: Asymptotic fully plastic crack-tip stress fields for a deeply cracked pure bending SE(PB) and three-point bend SE(B) specimen.
Fig. 4.5: Assumed plastic deformation mechanism for a three-point bend SE(B) specimen, Green and Hundy (1956).
Fig. 4.6: Schematic describing the relationship between the tangential velocity and imposed displacement for a three-point SE(B) specimen.
Fig. 4.7: Effect of indenter width on plastic deformation mechanism for SE(B) specimen, Ewing (1968).
Fig. 4.8: Assumed plastic deformation mechanism for a compact tension C(T) specimen, Ewing and Richards (1974).
Fig. 4.9: Assumed plastic field for a single-edge-cracked specimen under bending with small tensile load, Shiratori and Dodds (1980).