CHAPTER 6

A study of the limit load and crack tip constraint of middle tension M(T) specimen having a weld centre crack

6.1 Introduction

The problem of middle tension M(T) specimen having a weld centre crack has been studied extensively. It was first systematically analysed by Varias et al. (1991). They numerically (finite element) examined the case where a crack was postulated at the centre of ductile metal foil sandwiched between two rigid ceramic blocks. The focus of this study was to understand the ductile failure mechanisms that are likely to occur in the metal foil under such a high constraint state. It was demonstrated that for such an extreme mismatch case, under small-scale yielding condition, under uni-axial tensile load a high tri-axial stress exists ahead of crack tip at a distance several times the foil thickness. A formula for evaluating the stress intensity factor was also suggested.

On analytical front, initial study on M(T) specimen having a weld centre crack, under large-scale plasticity, was carried out by Joch et al. (1993). Main objective of this study was to quantify the influence of weld strength mismatch on the limit load and plastic $\eta$-factor. Classical upper bound theorem of limit analysis was used to derive analytical solutions. However, more detailed description of this problem was presented by Hao et al. (1997). Using classical approach of Slip-line theory, they provided sufficiently detailed analytical solutions of the limit load and crack tip stresses for the case where plasticity was
confined only in the weld material. In case where the yield strength of base and weld material is comparable plastic deformation occurs in both the materials. Constructing SLF solutions for such cases is not straightforward as the stress connections conditions at the base-weld interface are unknown.

Hao et al. (1997) made an attempt to solve this problem analytically by assuming continuity of normal and shear stress along the slip line passing through the interface of base and weld material. However, this assumption was not well supported by the results obtained from detailed elastic-plastic finite element analysis. Based on comparison of their analytical results with finite element (FE) studies they indicated the possibility of jump in tractions at the interface of the two materials. This problem was then re-examined numerically by Kim and Schwalbe (2001, 2004). Based on extensive FE analyses, these authors proposed closed-form expressions of the limit load of M(T) specimen for weld centre crack, interfacial crack, and asymmetrically located crack in the weld region. They also performed detailed numerical studies to examine the strength mismatch effect on crack tip constraint parameter $h$ under fully plastic condition (2004). The problem of M(T) specimen having an asymmetric crack in the weld region was also analysed by Lei et al. (1999). The classical upper bound theorem of limit analyses was used to obtain analytical solutions of the limit load. Using the concept of equivalent stress-strain relation proposed by Lei and Ainsworth (1997) these authors also provided an estimation of J-integral. Analytical solutions of the limit load of an overmatched M(T) specimen were also provided by Alexandrov et al. (1999). Based on kinematically admissible velocity fields and statically admissible stress fields corresponding upper and lower bound estimates of limit load were arrived. Their numerical results for upper bound limit load were identical.
to those provided by Joch et al. (1993), however, Alexandrov et al. (1999) demonstrated that the upper bound limit load depends on a single parameter that can account for the effects of weld strength mismatch $M$ as well as weld slenderness ratio $\psi$.

In this chapter, a discontinuous stress solution is proposed to analyse M(T) specimen having a weld centre crack. Discontinuity is incorporated in the proposed solution by assuming an unknown value of the mean (hydrostatic) stress at the base-weld interface. Modified upper bound (MUB) theorem along with global equilibrium equations was utilised to obtain this unknown mean stress and hence the whole stress field. The results obtained were found to be in excellent agreement with the known FE solutions available in literature. In addition to the limit load, effect of weld strength mismatch on crack tip constraint parameter $h$ was quantified.

### 6.2 Analysis of overmatched middle tension M(T) specimen having a weld centre crack

Consider the case of a M(T) specimen having a weld centre crack as shown in Fig. 6.1. Assumption of plane strain was made and analysis was carried out on an idealised weld without any heat affected zone (HAZ). The two materials (base and weld) were considered as rigid-plastic having mismatch in their yield strength. The effects of weld geometry were modeled by changing the weld slenderness ratio $\psi$ while the strength mismatch effects were incorporated by changing the mismatch ratio $M$. These two parameters are defined as follows
Material strength mismatch ratio $M$ and weld slenderness ratio $\psi$ were systematically varied to account all practical cases. $M>1$ corresponds to an overmatch weld while $M<1$ refers to an undermatch weld.

For a $M(T)$ specimen, having a weld centre crack, the assumed plastic field for an overmatch weld is shown in Fig. 6.2. In this field it was assumed that a straight slip line APB emanated from the crack tip and crossed the base-weld interface. It was then merged into the fan field BEC of angular extent $\gamma$ whose centre $E$ lied at the base-weld interface. Near the free surface there was a region of constant stress (uniform tension) CDE which merged with the fan field tangentially. It may be mentioned that this type of field was first suggested by Hao et al. (1997), however, the authors did not provide any details of its analysis. The stress distribution in the constant stress region, CDE, can be expressed as

$$\sigma_{11} = 0, \quad \sigma_{22} = -2k \quad \text{and} \quad \sigma_{12} = 0 \quad (6.2)$$

In the central field BEC, the shear stress along BE was $k$ and the pressure acting on it, as per Hencky's relation, was $k(1+2\gamma)$. Thus, the stress distribution in the plastic region of the base material up to point P was readily known. At point P, that is, at the interface of two materials we propose that the continuity of tractions is violated. Thus, both the in-plane shear stress $\sigma_{12}$ as well as the mean stress undergoes a sudden jump at the interface. As a result, the stress distribution cannot be obtained directly for the weld region. As discussed in chapter 5, this problem was solved by the use of MUB theorem. Unlike SLF analysis.
this technique does not require any information about the mean stress along the slip line separating the two rigid regions. Since the tangential velocity and the shear stress (by virtue of yield criterion) were known along the slip line APB, plastic dissipation of energy can be easily computed. As the mean stress acting on the slip line APB and, hence, the unknown mean stress at the interface, does not enter in MUB analysis problem becomes amenable to a fully analytical treatment. We now proceed to analyse this field.

From kinematics the relation between the rate of imposed displacement $\delta$ and the tangential velocity $v^*$ along the slip line APB can be easily established, that is,

$$v^* = \frac{\delta}{\sin\left(\frac{\pi}{4} + \gamma\right)}$$  \hspace{1cm} (6.3)

Now invoking work principle, that is, eq. (3.33), limit load can be expressed as,

$$P_L \delta = 2 \left[ \int_{AP} k_x v^* dS + \int_{PB} k_y v^* dS + \int_{BC} \sigma_y n_j v_j dS + \int_{CD} \sigma_y n_j v_j dS \right]$$  \hspace{1cm} (6.4)

The work done by the stresses on the circular arc BC (of radius $x$), Fig. 6.2 (b), can be expressed as follows

$$\int_{BC} \sigma_y n_j v_j dS = \int_0^\gamma k x v_x d\theta + \int_0^\gamma k (1 + 2\theta) x v_x d\theta$$  \hspace{1cm} (6.5)
\[
\int_{BC} \sigma_y n_j v_j dS = \int_0^\gamma kx\delta \sin\left(\frac{\pi}{4} + \theta\right) d\theta + \int_0^\gamma k(1 + 2\theta)x\delta \cos\left(\frac{\pi}{4} + \theta\right) d\theta \quad (6.6)
\]

\[
\int_{BC} \sigma_j n_j v_j dS = kx\delta \cos\left(\frac{\pi}{4} + \gamma\right) + k(1 + 2\gamma)x\delta \sin\left(\frac{\pi}{4} + \gamma\right) - \sqrt{2}kx\delta \quad (6.7)
\]

Similarly, work done by the stresses on the segment CD can be expressed as follows

\[
\int_{CD} \sigma_y n_j v_j dS = \int_{CD} kv_j dS + \int_{CD} kv_j dS = \sqrt{2}kx\delta \quad (6.8)
\]

Finally, substitution of eqs. (6.7) & (6.8) in eq. (6.4), using the value of \(v^*\) as given by eq. (6.3), lead to the following relation for the limit load

\[
F_y = \frac{2\sigma_{\text{cr}}l}{\sqrt{3}} \left[\frac{(M - 1)H/1}{\sin^2(\pi/4 + \gamma)} + \frac{1 - x}{l} \cos\left(\frac{\pi}{4} - \gamma\right)\right] + \frac{0.5 \cos 2\gamma}{l} + \frac{x}{l}\left[(1 + 2\gamma)\sin\left(\frac{\pi}{4} + \gamma\right) + \sin\left(\frac{\pi}{4} - \gamma\right)\right] \quad (6.9)
\]

Since \(\gamma\) is the only independent variable, as per MUB theorem

\[
\frac{dF_y}{d\gamma} = 0 \quad (6.10)
\]
Thus, the limit load $F_\gamma$ and all the unknown parameters of the assumed plastic field can be easily evaluated. When $\gamma=0$ the field reduces to that of homogeneous M(T) specimen (McClintock, 1971). As far as evaluation of the limit load is concerned, the unknown value of the mean stress at the base-weld interface does not enter in the analysis. However, for evaluation of crack tip stress field the jump in the value of mean stress that occurs at the base-weld interface must be quantified.

For a wide variety of specimen geometry and loading conditions it has been established in chapter 4 that the MUB theorem provides results that are identical to those obtained from SLF analysis. Moreover, it was demonstrated in chapter 5 that MUB theorem when applied to SE(PB) and C(T) specimen having a weld centre crack satisfy global equilibrium equations. We now proceed to establish this for a M(T) specimen also. If it is assumed that $\sigma_w^*$ is the unknown value of hydrostatic stress that occurs at point P, in the weld material, then the equations of global equilibrium can be expressed as follows

$$\frac{F_x}{2} = 2k_x \cos\left(\frac{\pi}{4} + \gamma\right) + k_y \left(\frac{1}{\sin(\pi/4+\gamma)} + k_w \H + \frac{\sigma_w^*H}{\tan(\pi/4+\gamma)}\right)$$  \hspace{1cm} (6.11)

$$\frac{F_x}{2} = -k_x \sin(\pi/4+\gamma) + k_y \H \sin(\pi/4+\gamma) - \sigma_w^*H = 0$$  \hspace{1cm} (6.12)

Thus, if the equivalence of MUB theorem and SLF analysis is assumed to hold good for a M(T) specimen also then the value of assumed plastic field parameters ($x$ and $\gamma$) obtained from the MUB theorem may be used to obtain $\sigma_w^*$ from eq. (6.12). As a cross check, the value of $\sigma_w^*$ was substituted in eq. (6.11) to confirm that the limit load so obtained is quite
close to that obtained directly from the MUB theorem. This again has validated our assumption that for the case of a strength mismatch weld also (having two different material interfaces) the parameters of plastic field as obtained from MUB theorem also satisfies the global equilibrium equations.

As discussed in chapter 5, from the stress distribution so obtained the state of stress ahead of crack tip can not be directly evaluated. A construction similar to that shown in Fig. 5.4 was used to evaluate the crack tip stress distribution. Thus, the mean (hydrostatic) stress, directly ahead of crack tip, can be expressed as follows.

\[
\sigma^Y = \sigma^B = \sigma_w^* + 2k_w\gamma
\]  

(6.13)

The crack tip constraint parameter \( h \) (eq. 5.15) was used to describe the effect of weld strength mismatch on the local stress tri-axiality.

6.3 **Analysis of undermatched middle tension M(T) specimen having a weld centre crack**

For a M(T) specimen having a crack at the centre of an undermatched weld, in general, plasticity passes through both base and weld material. Detailed FE analyses performed by the author revealed that in comparison to an overmatch case the global stress fields for an undermatch case are more complex. No detailed global stress field could be developed for such a case. Limit load solution for such cases were obtained using the simple kinematically admissible velocity field that was first proposed by Joch et al. (1993).
However, it is important to mention that such simplified velocity field can not be used for evaluation of crack tip constraint as it has not been demonstrated that the global equilibrium equations are satisfied.

When the yield strength of weld material is sufficiently low (with respect to base metal), the entire plastic deformation gets confined in the weaker weld material and, thus, the weld slenderness ratio $\psi$ is the only important parameter affecting the plastic deformation. For such cases Hao et al. (1997) proposed slip line fields for various values of $\psi$ and obtained analytical solution of the limit load and crack tip constraint parameter $h$. The case of extreme undermatch was also analysed by Kim and Schwalbe (2001a). Based on SLF analyses a different expression of the limit load was proposed, however, no details of solution were provided. It was discussed that their solutions provide slightly different value of the limit load than that suggested by Hao et al. (1997).

In the following sections detailed analyses of the proposed slip line fields for the case of a M(T) specimen having a crack at the centre of an extreme undermatch weld are presented. The proposed fields were used to obtain analytical solutions of the limit load and crack tip constraint parameter $h$.

6.3.1 Slip Line Field-1 ($1 \leq \psi \leq 3.6$)

The complete structure of the proposed field is shown in Fig. 6.3. At point C lying at the base-weld interface, near the free surface, it was assumed that a singularity exist and the stress distribution in the plastic sector BCD is described by a fan field of radial extent $x$ and angular extent $\gamma$. Adjacent to the fan field is the constant stress region CED in which a
uniform tensile stress of magnitude $2k$ exists. It is assumed, asymptotically, that a small segment of straight slip line $AA'$ exists. It radiates from the crack tip at an angle of $\pi/4$ with the horizontal axis. The fan field $BCD$ is connected to the straight slip line $AA'$ by a circular arc $A'B$ of radius $y$ and angular extent $\gamma$. The stress components on the small line $AA'$ are constant and equal to the components at the point $A'$ on the arc $A'B$. The global stress field is completely described by the three unknown parameters, that is, $x$, $y$, and $\gamma$.

It can be easily established that $ABC$ is an $\alpha$-slip line and the mean (hydrostatic) stress at point $B$ is $k(1+2\gamma)$. From Hencky's relation, the hydrostatic stress at point $A'$ can be expressed by the following equation

$$\sigma''_w = k_w \left[ (1+2\gamma) - \frac{\pi}{2} + 2\left( \frac{\pi}{4} + \gamma \right) \right] \quad (6.14)$$

From geometry the following two relations can be easily obtained

$$y\left[ \cos\left( \frac{\pi}{4} - \gamma \right) - \frac{1}{\sqrt{2}} \right] + x\sin\left( \frac{\pi}{4} - \gamma \right) = H \quad (6.15)$$

$$y\left[ \frac{1}{\sqrt{2}} - \sin\left( \frac{\pi}{4} - \gamma \right) \right] + x\cos\left( \frac{\pi}{4} - \gamma \right) = l \quad (6.16)$$

The remaining third equation can be obtained from the equilibrium consideration, that is
Eqs. 6.15-6.17 may be used to obtain the values of the three unknown parameters of the plastic field and the stress distribution in the plastic sectors can be easily evaluated. The resulting limit load can be obtained from the following expression

\[ \sum F_x = \int_{ABC} \sigma_i n_i dS = 0 \]  

(6.17)

Once the hydrostatic stress at point A' is known, the crack opening stress directly ahead of crack tip can be obtained from the following relation

\[ \sigma_{\theta \theta} \bigg|_{\theta=0} = \sigma_{w}^{d} + k_w \]  

(6.19)

It is worth to mention that the proposed slip line field is applicable for \(1 \leq \psi \leq 3.6\). As the weld slenderness ratio \(\psi\) increases the angle \(\gamma\) describing the angular extent of fan field BCD increases. This results in a increase of hydrostatic stress at point A near the crack tip. When \(\psi=3.6\), the angle \(\gamma\) becomes equal to \(\pi/4\) and the hydrostatic stress ahead of crack tip is high enough to cause complete yielding of the crack tip and the Prandtl's field develops.
6.3.2 **Slip Line Field-2** $(3.6 \leq \psi \leq 5)$

The construction of this field is guided by the consideration that for $\psi \geq 3.6$, the stress tri-axiality ahead of crack tip is sufficient enough to cause complete yielding of the crack tip. As a result the asymptotic distribution of stresses near the crack tip can be completely described by the Prandtl field which extends up to a distance $z$ from the crack tip in the radial direction as shown in Fig. 6.4. At point J lying at the base-weld interface, near the free surface, it is assumed that a singularity exist and the stress distribution in the plastic sector EFJ is described by a fan field of radial extent $x$ and angular extent of $\pi/4$. Adjacent to the fan field is the constant stress region JFG in which a uniform tensile stress of magnitude $2k$ exists. The fan field EFJ is connected to the Prandtl field by a circular arc DE of radius $y$ that intersects the horizontal axis at an angle of $\pi/4$. The radial extent of the Prandtl field is described by $z$. The global stress field is, thus, completely described by the three unknown parameters, that is, $x$, $y$, and $z$.

From Hencky’s relation, the hydrostatic stress at point D can be expressed by the following equation

$$\sigma_{ww}^D = k_n (1 + \pi)$$  \hspace{1cm} (6.20)

Eq. (6.20) indicates that the stress distribution at point D is the same as that obtained from the Prandtl field. The crack opening stress directly ahead of crack tip in the Prandtl field can be expressed as
\[ \sigma_{\theta\theta}|_{\theta=0} = \sigma_w^D + k_w = k_w (2 + \pi) \]  \hfill (6.21)

Now from geometry the following two relations can be easily obtained

\[ y \left[ 1 - \frac{1}{\sqrt{2}} \right] = H \]  \hfill (6.22)

\[ \frac{2z}{\sqrt{2}} + \frac{y}{\sqrt{2}} + x = l \]  \hfill (6.23)

The equilibrium equation in horizontal direction remain same as expressed by eq. (6.12) as the presence of Prandtl field near the crack tip does not effect the force equilibrium in X-direction. Thus, eqs. (6.17), (6.22) and (6.23) may be used to evaluate the unknown parameters of the plastic field and, hence, the stress distribution in the plastic regions. The resulting limit load can be obtained from the following expression.

\[ \sum F_y = \int_{DEJ} \sigma_{z\gamma} n_1 dS + \sqrt{2} (2 + \pi) k_w z \]  \hfill (6.24)

As the weld slenderness ratio \( \psi \) increases the size of the Prandtl field near the crack tip, as measured by the distance \( z \), increases monotonically and at \( \psi=5 \) the Prandtl field just
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touches the base-weld interface. For \( \psi \geq 5 \) the stress fields of the weld zone become complicated and numerical construction of slip line fields is required. Slip line fields for such cases were constructed by Hao et al. (1997), and Kim and Schwalbe (2001a).

6.4 Results

For the case of M(T) specimen having a weld centre crack the limit load solutions have been provided by various authors. However, the solutions provided by Kim and Schwalbe (2001a) which are based on detailed FE analysis are most accurate and are widely accepted. Thus, in the present study analytical solutions of the limit load obtained from MUB theorem were compared with detailed FE analyses performed by the author and with the solutions provided by Kim and Schwalbe (2001a) as shown in Fig. 6.5. As mentioned earlier, analytical solutions of the limit load based on continuity of tractions at the base-weld interface were proposed by Hao et al. (1997). A comparison of their limit load solutions with FE results is shown in Fig. 6.6. In Figs. 6.5 and 6.6, the normalised limit load represent the ratio of limit load of M(T) specimen having a weld centre crack to that of homogeneous M(T) specimen. It may be observed from Fig. 6.6 that as the weld strength mismatch ratio \( M \) increases the difference between the limit load solutions of Hao et al. (1997) and FE results becomes higher. This actually suggests that the assumption of continuous stress solution is not valid particularly for higher mismatch ratios. On the other hand the limit load obtained from MUB theorem is in very good agreement with FE results (see Fig. 6.5) for all mismatch ratios. In Fig. 6.7 analytical solution of crack tip constraint parameter \( h \) obtained from MUB theorem is compared with FE results of Kim and
Schwalbe (2004). For an extremely undermatched M(T) specimen having a weld centre crack analytical solutions of crack tip constraint parameter $h$ obtained from the proposed slip line fields were compared with the solutions provided by Kim and Schwalbe (2004) in Fig. 6.8. Very good agreement was obtained between the two solutions.

6.5 Discussion

In this chapter analytical solutions of the limit load and crack tip constraint parameter $h$ for a rigid-plastic material under mode-I loading were described. For standard homogeneous fracture specimens MUB theorem provides results that are in exact agreement with SLF solutions. Classical methods like SLF analysis are, however, applicable to macroscopically homogeneous/single material. Welded structures have an abrupt material discontinuity at the base weld interface. Constructing SLF solutions for such problems is not straightforward as the stress connections conditions at the interface are unknown. Detailed FE analysis performed by Hao et al. (1997) and Kim and Schwalbe (2001a) have revealed that SLF solutions based on continuity of stress at base-weld interface are not in good agreement with FE results. In this chapter MUB theorem was successfully used to obtain analytical solution of the limit load and crack tip constraint parameter $h$ for M(T) specimen having a weld centre crack. At this point it is worth to discuss that while the analytical solutions of the limit load obtained from MUB theorem are in excellent agreement with the widely accepted solutions of Kim and Schwalbe (2001a) and detailed FE results performed by the author, however, analytical results of crack tip constraint parameter $h$ do not show such a good match with FE solutions. Such kind of differences in the crack tip stresses
obtained from SLF analysis and FE results have also been observed for a homogeneous M(T) specimen (Zhu and Chao, 2000). These authors performed detailed FE studies and observed that there exist tensile and compressive stresses along the vertical centerline of M(T) specimen which result in a bending moment $M_V$. The difference between $M_V$ and the moment generated by the applied far-field load makes the crack opening stress non-uniform along the remaining ligament. However, the slip line field for M(T) specimen (McClintock, 1971) comprises of a constant stress sector creating uniform opening stress along the ligament. These authors, thus, concluded that at the limit load the crack tip stress fields obtained from FE analysis can only approach to, but cannot attain to, the slip-line fields of homogeneous M(T) specimen. A closer look of Fig. 6.7 reveals that for higher weld overmatch ratios ($M \to 2$), FE results were close to analytical solutions, however, as $M$ approaches unity (homogeneous case) difference between FE results and analytical predictions increases as noted by Zhu and Chao (2000).
Fig. 6.1: Middle tension M(T) specimen having a weld centre crack.
Fig. 6.2 (a): Proposed stress field for an overmatched M(T) specimen having a weld centre crack.
Fig. 6.2 (b): Stress and velocity distribution on the elastic-plastic boundary of an overmatched M(T) specimen having a weld centre crack.
Fig. 6.3: Proposed slip line field of extremely undermatched M(T) specimen having a weld centre crack ($1 \leq \psi \leq 3.6$).
Fig. 6.4: Proposed slip line field of extremely undermatched M(T) specimen having a weld centre crack \((3.6 \leq \psi \leq 5)\).
Fig. 6.5: Comparison of normalised limit load of M(T) specimen, having a weld centre crack, obtained from MUB theorem with FE results.
Fig. 6.6: Comparison of normalised limit load of M(T) specimen, having a weld centre crack, provided by Hao et al. (1997) with author's FE results.
Fig. 6.7: Comparison of crack tip constraint parameter $h$ of overmatched M(T) specimen obtained from MUB theorem with FE results.
Fig. 6.8: Comparison of crack tip constraint parameter $h$ of extremely undermatched M(T) specimen obtained from proposed SLF with FE results.