Chapter 3
Chapter 3

ANALYSIS OF IMPATIENCE IN A MULTISERVER MARKOVIAN QUEUING MODEL

3.1. Introduction:

In this chapter, we discuss the analysis of a multi server Markovian queuing system i.e M/M/k model. While this system received wide attention, we revisit it under the assumption that customers arriving into the queuing system may balk as well as renege. This is because closed form useable results for this model in case customers are of impatient type are still not available. In the analysis of many queuing system, it is assumed that customers who arrive stay on till they receive service. In real life, this does not always happen. In our fast-paced world, we have to admit that customer impatience is there all around us. Every customers wait for service but only for a limited time. They have a patience time beyond which they are not willing to wait. If service is not offered by that time, the customer leaves and is therefore loss to the system. We have carried out the analysis considering both the reneging rules–R_BOS and R_EOS. A work of similar nature has been carried out by Haghighi et al. (1986). However they have considered only reneging of type R_BOS. We extend their work by providing closed form expressions of various performance measures including a few newly designed ones. Further, we extend the analysis to the case where reneging is of the type R_EOS.

3.2. Assumptions.

We begin by first describing the model. The assumptions of this model are
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1) The arrivals occur in accordance with a Poisson process with parameter $\lambda$.

2) The distribution of service time is assumed as exponential with parameter $\mu$.

3) Each customer joining the system is assumed to have random patience time following $\text{exp}(v)$ which is independent of the position of the customer in the system.

4) There are $k$ servers in parallel that are independent of each other. Further, their efficiencies are same and the service time distribution of each of these servers is $\text{exp}(\mu)$.

5) Service discipline is considered as FIFO.

6) The size of calling population is infinite.

Let $\mu_n$ denotes the service rate of the system when the system is in state 'n'. Then because of model assumption we have, 

$$\mu_n = \begin{cases} n\mu & \text{if } n \leq k \\ k\mu & \text{if } n > k \end{cases}$$

3.3. Analysis of $M/M/k$ model with position independent reneging (PIR) and state independent balking (SIB).

In this section, we assume that each arriving customer has probability $(1-p)$ of balking from a system with no idle servers. Further each customer joining the system has random patience time following $\text{exp}(v)$. The reneging rate is assumed to remain the same irrespective of the position of the customer.
3.3.1. The system state probabilities.

In this section, the steady state probabilities are derived by the Markov process method. We first analyze the case where customers renege only from the queue. The rate-in rate-out diagram under R_BOS is given below:

Under R_BOS, let $p_n$ denote the probability that there are ‘n’ customers in the system. The steady state probabilities under R_BOS were derived by Haghighi et al (1986). We reproduce the equations below.

$$p_0 = \mu \rho_1,$$  \hspace{1cm} (3.3.1.1)

$$\lambda p_{n+1} + (n+1)\mu p_{n+1} = \lambda p_n + n \mu p_n; \ n=1,2,...,k-1,$$ \hspace{1cm} (3.3.1.2)

$$\lambda p_{k+1} + (k\mu + \nu) p_{k+1} = \lambda p_k + k \mu p_k$$ \hspace{1cm} (3.3.1.3)

$$\lambda p_{n+1} + \{k\mu + (n-k+1)\nu\} p_{n+1} = \lambda p_n + \{k\mu + (n-k)\nu\} p_n; \ n=k+1,.....$$ \hspace{1cm} (3.3.1.4)

Solving recursively, we get (under R_BOS)

$$p_n = \left\{ \frac{\lambda^n}{(n!\mu^n)} \right\} p_0; \ \ \ \ n=1,2,...,k$$ \hspace{1cm} (3.3.1.5)

$$p_n = \left\{ \frac{\lambda^n}{k!\mu^k \prod_{r=k+1}^{n}(k\mu + r-\nu)} \right\} p_0; \ n=k+1,.....$$ \hspace{1cm} (3.3.1.6)
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where $p_0$ is obtained from the normalizing condition $\sum_{n=0}^{\infty} p_n = 1$ and is given as

$$p_0 = \left[ \sum_{n=0}^{\infty} \frac{\lambda^n}{(n! \mu^n)} + \sum_{n=k+1}^{\infty} \frac{\lambda^n}{(n! \mu^n)} \prod_{r=k+1}^{\infty} (k\mu + r - k\nu) \right]^{-1}. \quad (3.3.1.7)$$

The steady state probabilities satisfy the recurrence relation, under R_BOS

$$p_n = \frac{\lambda/(n\mu)}{p_{n-1}}; \quad n=1,2,...k.$$

$$p_n = \left[ \frac{\lambda/(k\mu + n - k\nu)}{p_{n-1}} \right]; \quad n=k+1, k+2,.....$$

We shall denote by $K_{R\_BOS}$ the probability that an arriving unit has to wait on arrival (under R_BOS). Then

$$K_{R\_BOS} = Pr (N \geq k)$$

$$= \sum_{n=k}^{\infty} p_n. \quad (3.3.1.8)$$

We may call $K_{R\_BOS}$ as 'Erlang’s second (Erlang’s delay probability) formula for balking (SIB) and reneging (R_BOS)' in line with similar nomenclature in Medhi (2003, page 87).

The rate-in rate-out diagram under R_EOS is given as

Under R_EOS where customers may renege from queue as well as while being served, let $q_n$ denote the probability that there are $n$ customers in the
system. Applying again the Markov process method, we obtain the following set of steady state equations.

\[ \lambda q_0 = (\mu + \nu) q_1, \quad (3.3.1.9) \]

\[ \lambda q_{n-1} + (n + 1)(\mu + \nu) q_{n+1} = \lambda q_n + n(\mu + \nu) q_n; n = 1, 2, \ldots, k-1, \quad (3.3.1.10) \]

\[ \lambda q_{k-1} + (k\mu + (k + 1)\nu) q_{k+1} = \lambda p q_k + k(\mu + \nu) q_k, \quad (3.3.1.11) \]

\[ \lambda p q_{n-1} + (k\mu + (n + 1)\nu) q_{n+1} = \lambda p q_n + (k\mu + n\nu) q_n; n = k+1, \ldots \quad (3.3.1.12) \]

Solving recursively, we get (under R_EOS)

\[ q_n = \left[ \frac{(\mu + \nu)^n}{n!(\mu + \nu)} \right] \frac{q_0}{n + 1}; n = 1, 2, \ldots, k \quad (3.3.1.13) \]

\[ q_n = \left[ \frac{\lambda^n}{(\mu + \nu) + (\mu + \nu)^k \prod_{r=k+1}^n (k\mu + r\nu)} \right] q_0; n = k+1, \ldots \quad (3.3.1.14) \]

where \( q_0 \) is obtained from the normalizing condition \( \sum_{n=0}^{\infty} q_n = 1 \) and is given as

\[ q_0 = \left[ \sum_{n=0}^{k} \frac{\lambda^n}{(n!(\mu + \nu)^n)} + \sum_{n=k+1}^{\infty} \frac{\lambda^n}{(n!(\mu + \nu)^n) \prod_{r=k+1}^n (k\mu + r\nu)} \right]^{-1} \quad (3.3.1.15) \]

The steady state probabilities satisfy the recurrence relation, under R_EOS

\[ q_n = \left[ \frac{\lambda}{n!(\mu + \nu)} \right] q_{n-1}; \quad n = 1, 2, \ldots, k. \]

\[ q_n = \left[ \frac{\lambda p}{(k\mu + n\nu)} \right] q_{n-1}; \quad n = k+1, k+2, \ldots. \]

We shall denote by \( K_{R_EOS} \) the probability that an arriving unit has to wait on arrival (under R_EOS). Then

\[ K_{R_EOS} = \Pr (N \geq k) = \sum_{n=k}^{\infty} q_n. \quad (3.3.1.16) \]
which may be called 'Erlang's second (Erlang's delay probability) formula for 
balking (SIB) and reneging (R_EOS)' in line with similar nomenclature in Medhi 
(2003, page 87).

3.3.2. Performance Measures.

An important measure is 'L', which denotes the mean number of 
customers in the system. To obtain an expression for the same, we note that 
$L = P'(1)$

where 

$$P'(1) = \frac{d}{ds} P(s) |_{s=1}.$$ 

Here $P(S)$ is the p.g.f. of the steady state probabilities and is given by 

$$P(s) = \sum_{n=0}^{\infty} P_n s^n.$$ 

To obtain $P'(1)$ under R_BOS we proceed as follows:

From equation (3.3.1.2) we have 

$$\lambda p_{n-1} + (n + 1) \mu p_{n+1} = \lambda p_{n} + n \mu p_{n}; \quad n=1,2,\ldots k-1$$

Multiplying both sides of the equation by $s^n$ and summing over $n$

$$\lambda s \sum_{n=0}^{k-1} p_{n-1} s^{n-1} - \lambda s \sum_{n=0}^{k-1} p_{n} s^{n} = \mu s \sum_{n=0}^{k-1} p_{n} s^{n} - \frac{1}{s} \mu \sum_{n=0}^{k-1} (n+1) p_{n+1} s^{n+1} \quad (3.3.2.1)$$

From (3.3.1.3) we have 

$$\lambda p_{k-1} + (k \mu + \nu) p_{k+1} = \lambda p_{k} + k \mu p_{k}$$

Again multiplying both sides of the equation by $s^k$

$$\lambda s p_{k-1} s^{k-1} - \lambda p_{k} s^k = \mu p_{k} s^k - \frac{1}{s} (k \mu + \nu) p_{k+1} s^{k+1} \quad (3.3.2.2)$$
From (3.3.1.4) we have
\[ \lambda p_{n+1} + \{k \mu + (n-k+1)\nu\} p_{n+1} = \lambda p_n + \{k \mu + (n-k)\nu\} p_n \quad ; \quad n=k+1, k+2, \ldots \]

Similarly multiplying both sides of the equation by $s^n$ and summing over $n$

\[ \lambda \sum_{n=k+1}^{\infty} p_{n+1} s^{n+1} - \lambda \sum_{n=k+1}^{\infty} p_n s^n = \sum_{n=k+1}^{\infty} (k \mu + (n-k)\nu) p_n s^n - \frac{1}{s} \sum_{n=k+1}^{\infty} (k \mu + (n-k+1)\nu) p_{n+1} s^{n+1} \]

(3.3.2.3)

Adding (3.3.2.1), (3.3.2.2) and (3.3.2.3) we get

\[ \Rightarrow \lambda \left[ \sum_{n=0}^{k} p_{n+1} s^{n+1} + p_k s^{k+1} + p \sum_{n=k+1}^{\infty} p_{n+1} s^{n+1} \right] - \lambda \left[ \sum_{n=0}^{k} p_n s^n + p \sum_{n=k+1}^{\infty} p_n s^n \right] = \left[ \sum_{n=0}^{k} \frac{\mu}{s} (n+1) p_n s^n + (k \mu + \nu) p_k s^k \right] + \left[ \sum_{n=k+1}^{\infty} (k \mu + (n-k)\nu) p_{n+1} s^{n+1} \right] \]

\[ \Rightarrow \lambda \left[ \sum_{n=0}^{k} \left\{ p_{n+1} s^{n+1} + p_k s^{k+1} + p \sum_{n=k+1}^{\infty} p_{n+1} s^{n+1} \right\} \right] - \lambda \left[ \sum_{n=0}^{k} \left\{ p_n s^n + p \sum_{n=k+1}^{\infty} p_n s^n \right\} \right] = \left[ \sum_{n=0}^{k} \left\{ \frac{\mu}{s} (n+1) p_n s^n + (k \mu + \nu) p_k s^k \right\} \right] + \left[ \sum_{n=k+1}^{\infty} \left\{ (k \mu + (n-k)\nu) p_{n+1} s^{n+1} \right\} \right] \]

Adding (3.3.2.1), (3.3.2.2) and (3.3.2.3) we get

\[ \Rightarrow \lambda \left[ \sum_{n=0}^{k} p_{n+1} s^{n+1} + p_k s^{k+1} + p \sum_{n=k+1}^{\infty} p_{n+1} s^{n+1} \right] - \lambda \left[ \sum_{n=0}^{k} p_n s^n + p \sum_{n=k+1}^{\infty} p_n s^n \right] = \left[ \sum_{n=0}^{k} \frac{\mu}{s} (n+1) p_n s^n + (k \mu + \nu) p_k s^k \right] + \left[ \sum_{n=k+1}^{\infty} (k \mu + (n-k)\nu) p_{n+1} s^{n+1} \right] \]

Adding (3.3.2.1), (3.3.2.2) and (3.3.2.3) we get

\[ \Rightarrow \lambda \left[ \sum_{n=0}^{k} p_{n+1} s^{n+1} + p_k s^{k+1} + p \sum_{n=k+1}^{\infty} p_{n+1} s^{n+1} \right] - \lambda \left[ \sum_{n=0}^{k} p_n s^n + p \sum_{n=k+1}^{\infty} p_n s^n \right] = \left[ \sum_{n=0}^{k} \frac{\mu}{s} (n+1) p_n s^n + (k \mu + \nu) p_k s^k \right] + \left[ \sum_{n=k+1}^{\infty} (k \mu + (n-k)\nu) p_{n+1} s^{n+1} \right] \]

Adding (3.3.2.1), (3.3.2.2) and (3.3.2.3) we get

\[ \Rightarrow \lambda \left[ \sum_{n=0}^{k} p_{n+1} s^{n+1} + p_k s^{k+1} + p \sum_{n=k+1}^{\infty} p_{n+1} s^{n+1} \right] - \lambda \left[ \sum_{n=0}^{k} p_n s^n + p \sum_{n=k+1}^{\infty} p_n s^n \right] = \left[ \sum_{n=0}^{k} \frac{\mu}{s} (n+1) p_n s^n + (k \mu + \nu) p_k s^k \right] + \left[ \sum_{n=k+1}^{\infty} (k \mu + (n-k)\nu) p_{n+1} s^{n+1} \right] \]
\begin{align*}
\Rightarrow \lambda P(s) - \lambda s & \sum_{n=1}^{\infty} p_n s^n + \lambda \psi p(s) - \lambda \psi s - \lambda P(s) + \lambda p_0 + \lambda s \sum_{n=1}^{\infty} p_n s^n - \lambda P(s) + \lambda p_0 + \lambda s \sum_{n=1}^{\infty} p_n s^n \\
& = \mu P(s) - \mu s \sum_{n=1}^{\infty} \eta p_n s^{n-1} + k \mu P(s) - k \mu s \sum_{n=1}^{\infty} p_n s^n + v \psi P(s) - v \psi s \sum_{n=1}^{\infty} p_n s^n - k \psi P(s) + k \psi s \sum_{n=1}^{\infty} p_n s^n \\
& - \mu P'(s) + \mu \frac{\lambda}{\mu} p_0 + \mu s \sum_{n=1}^{\infty} \eta p_n s^{n-1} - \mu \sum_{n=1}^{\infty} p_n s^n - v P'(s) + \\
& v \sum_{n=1}^{\infty} \eta p_n s^{n-1} + \frac{kv}{s} P(s) - \frac{kv}{s} \sum_{n=1}^{\infty} p_n s^n \\
\Rightarrow P(s)(\mu + v) = & \lambda P(s) - \lambda s \sum_{n=1}^{\infty} p_n s^n + \lambda \psi p(s) - \lambda \psi s - \lambda P(s) + \lambda p_0 + \lambda s \sum_{n=1}^{\infty} p_n s^n - \lambda P(s) + \lambda p_0 + \lambda s \sum_{n=1}^{\infty} p_n s^n \\
& + v \sum_{n=1}^{\infty} \eta p_n s^{n-1} + \frac{kv}{s} P(s) - \frac{kv}{s} \sum_{n=1}^{\infty} p_n s^n \\
\text{Now}
\lim P'(s) = \lim \frac{1}{(\mu + v)} \left[ \lambda P(s) - \lambda s \sum_{n=1}^{\infty} p_n s^n + \lambda \psi p(s) - \lambda \psi s - \lambda P(s) + \lambda p_0 + \lambda s \sum_{n=1}^{\infty} p_n s^n \\
& + \mu \frac{\lambda}{\mu} p_0 + \mu s \sum_{n=1}^{\infty} \eta p_n s^{n-1} - \mu \sum_{n=1}^{\infty} p_n s^n - v P'(s) + \\
& v \sum_{n=1}^{\infty} \eta p_n s^{n-1} + \frac{kv}{s} P(s) - \frac{kv}{s} \sum_{n=1}^{\infty} p_n s^n \\
\Rightarrow P(0) = & \frac{1}{(\mu + v)} \left[ \lambda (1 - \sum_{n=1}^{\infty} p_n) + \lambda \psi (1 - \sum_{n=1}^{\infty} p_n) + \mu P'(0) - \mu \sum_{n=1}^{\infty} \eta p_n - k \mu (1 - \sum_{n=1}^{\infty} p_n) + v \sum_{n=1}^{\infty} \eta p_n + k \psi (1 - \sum_{n=1}^{\infty} p_n) \\
\Rightarrow P(0) = & \left[ \lambda (1 - K_{\text{R-BOS}}) + \lambda \psi (1 - K_{\text{R-BOS}}) - (\mu - \nu) \sum_{n=1}^{\infty} \eta p_n - k(K_{\text{R-BOS}} - p_k)(\mu - \nu) \\
\right] \text{\{using (3.3.1.8)\}}
\end{align*}

Therefore the mean system size under R_BOS is given by

\[ L_{\text{R-BOS}} = (1/v) \left[ \lambda (1 - K_{\text{R-BOS}}) + \lambda \psi (1 - K_{\text{R-BOS}}) - (\mu - \nu) \sum_{n=1}^{\infty} \eta p_n - k(K_{\text{R-BOS}} - p_k)(\mu - \nu) \right] \]

Mean number of customers in the queue is denoted by 'Lq' and under R_BOS it is given by
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\[ L_{q(R_{-EOS})} = \sum_{n=k+1}^{\infty} (n-k) p_n \]
\[ = L_{R_{-EOS}} - \sum_{n=1}^{k} np_n - k(K_{R_{-EOS}} - p_k) \]
\[ = \frac{1}{\nu} \{ \lambda (1 - K_{R_{-EOS}}) + \lambda p K_{R_{-EOS}} - \mu \sum_{n=1}^{k} np_n - k\mu (K_{R_{-EOS}} - p_k) \} \]

To calculate the mean system size and mean queue size under R_EOS we proceed as follows

Let \( Q(s) \) denote the probability generating function, defined by \( Q(s) = \sum_{n=0}^{\infty} q_n s^n \).

From equation (3.3.1.10) we have,
\[ \lambda q_{n+1} + (n+1)(\mu + \nu)q_{n+1} = \lambda q_n + n(\mu + \nu)q_n; \quad n=1,2,\ldots,k-1 \]

Multiplying both sides of this equation by \( s^n \) and summing over \( n \) from we get
\[ \lambda s \sum_{n=1}^{k-1} q_{n-1} s^{n-1} - \lambda \sum_{n=1}^{k-1} q_n s^n = (\mu + \nu) \sum_{n=1}^{k-1} nq_n s^n - \frac{1}{s} (\mu + \nu) \sum_{n=1}^{k-1} (n+1)q_{n+1} s^{n+1} \quad (3.3.2.5) \]

From equation (3.3.1.11) we have
\[ \lambda q_{k+1} + (k\mu + (k+1)\nu)q_{k+1} = \lambda p q_k + k(\mu + \nu)q_k \]

Multiplying both sides of this equation by \( s^k \) we get
\[ \lambda s q_{k-1} s^{k-1} - \lambda p q_k s^k = k(\mu + \nu) q_k s^k - \frac{1}{s} [k\mu + (k+1)\nu] q_{k+1} s^{k+1} \quad (3.3.2.6) \]

From equation (3.3.1.12)
\[ \lambda pq_{n+1} + \{k\mu + (n+1)\nu\}q_{n+1} = \lambda pq_n + \{k\mu + n\nu\}q_n; \quad n=k+1, k+2,\ldots \]

Multiplying both sides of this equation by \( s^n \) and summing over \( n \) from we get
\[ \lambda ps \sum_{n=k+1}^{\infty} q_{n-1} s^{n-1} - \lambda p \sum_{n=k+1}^{\infty} q_n s^n = \sum_{n=k+1}^{\infty} \{k\mu + n\nu\} q_n s^n - \frac{1}{s} \sum_{n=k+1}^{\infty} \{k\mu + (n+1)\nu\} q_{n+1} s^{n+1} \quad (3.3.2.7) \]
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Adding (3.3.2.5), (3.3.2.6) and (3.3.2.7) and proceeding in a manner similar to mean system size under $R_BOS$, we obtain,

$$Q'(l) = \frac{1}{\nu} \left[ \lambda (1 - K_{R\_EOS}) + \lambda p K_{R\_EOS} - \mu \sum_{n=1}^{k} nq_n - k \mu (K_{R\_EOS} - q_k) \right]$$

Thus, the mean system size under $R\_EOS$ is given by

$$L_{R\_EOS} = (1/\nu) \{ \lambda (1 - K_{R\_EOS}) + \lambda p K_{R\_EOS} - \mu \sum_{n=1}^{k} nq_n - k \mu (K_{R\_EOS} - q_k) \}.$$ 

and the mean queue size under $R\_EOS$ is given by

$$L_{q(R\_EOS)} = L_{R\_EOS} - \sum_{n=1}^{k} nq_n - k (K_{R\_EOS} - q_k)$$

$$= (1/\nu) \{ \lambda (1 - K_{R\_EOS}) + \lambda p K_{R\_EOS} - (\mu + \nu) \sum_{n=1}^{k} nq_n - k (\mu + \nu) (K_{R\_EOS} - q_k) \}.$$

Using Little's formula, we can calculate the average waiting time in the system and average waiting time in queue from the above mean lengths both under $R\_BOS$ and $R\_EOS$. Under $R\_BOS$ mean waiting time in system and queue are given by

$$W_{R\_BOS} = L_{R\_BOS} / \lambda$$

$$= (1/\lambda \nu) \{ \lambda (1 - K_{R\_BOS}) + \lambda p K_{R\_BOS} - (\mu - \nu) \sum_{n=1}^{k} np_n - k (K_{R\_BOS} - p_k) (\mu - \nu) \}.$$ 

$$W_{q(R\_BOS)} = L_{q(R\_BOS)} / \lambda$$

$$= (1/\lambda \nu) \{ \lambda (1 - K_{R\_BOS}) + \lambda p K_{R\_BOS} - \mu \sum_{n=1}^{k} np_n - k \mu (K_{R\_BOS} - p_k) \}.$$
and under $R_{EOS}$ these are given by

$$W_{R_{EOS}} = \frac{L_{R_{EOS}}}{\lambda}$$

$$= (1/\lambda \nu) \left\{ \lambda (1 - K_{R_{EOS}}) + \lambda p K_{R_{EOS}} - \mu \sum_{n=1}^{k} q_n - k \mu (K_{R_{EOS}} - q_k) \right\}$$

$$W_q(R_{EOS}) = L_q(R_{EOS}) / \lambda$$

$$= (1/\lambda \nu) \left\{ \lambda (1 - K_{R_{EOS}}) + \lambda p K_{R_{EOS}} - (\mu + \nu) \sum_{n=1}^{k} q_n - k (\mu + \nu) (K_{R_{EOS}} - q_k) \right\}$$

Customers arrive into the system at rate $\lambda$. However all the customers who arrive do not join the system because of balking. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$\lambda^*(R_{EOS}) = \lambda \sum_{n=0}^{k-1} p_n + \lambda p \sum_{n=k}^{m} p_n$$

$$= \lambda (1 - K_{R_{EOS}}) + \lambda p K_{R_{EOS}}$$

Similarly in case of $R_{EOS}$

$$\lambda^*(R_{EOS}) = \lambda (1 - K_{R_{EOS}}) + \lambda p K_{R_{EOS}}$$

We have assumed that each customer has a random patience time following $exp(\nu)$. Clearly then, the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rates (avgrr) under $R_{BOS}$ and $R_{EOS}$ are given by
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\[ \text{Avgrr}_{(R\_BOS)} = \sum_{n=1}^{\infty} (n-k)p_n \]
= \nu \left\{ \lambda(1-K_{R\_BOS}) + \lambda p K_{R\_BOS} - \mu \sum_{n=1}^{k} n p_n - k \mu (K_{R\_BOS} - p_k) \right\} 
\]
\[ \text{Avgrr}_{(R\_EOS)} = \sum_{n=1}^{\infty} n q_n \]
= \nu' Q'(1) 
= \lambda(1-K_{R\_EOS}) + \lambda p K_{R\_EOS} - \mu \sum_{n=1}^{k} n q_n - k \mu (K_{R\_EOS} - q_k) 

Average balking rates under R\_BOS and R\_EOS are given by

\[ \text{Avgbr}_{(R\_BOS)} = \lambda(1-p) \sum_{n=k}^{\infty} p_n \]
= \lambda(1-p)K_{R\_BOS} 
\[ \text{Avgbr}_{(R\_EOS)} = \lambda(1-p) \sum_{n=k}^{\infty} q_n \]
= \lambda(1-p)K_{R\_EOS} 

In a real life situation, customers who balk or renege represent the business lost. It is therefore of interest to determine the proportion of customers lost, both out of those joining the system as well as out of those arriving into the system.

These performance measures are given below

Proportion of customer lost due to reneging out of those joining the system (under R\_BOS) is

\[ = \text{Avgrr}_{(R\_BOS)} / \lambda'_{(R\_BOS)} \]
= 1 - \left\{ \frac{1}{\lambda(1-K_{R\_BOS}) + \lambda p K_{R\_BOS}} \left\{ \mu \sum_{n=1}^{k} n p_n + k \mu (K_{R\_EOS} - p_k) \right\} \right\} 

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Proportion of customer lost due to reneging out of those joining the system (under R\_EOS) is
\[
= \frac{\text{Avgrr}(R\_EOS)}{X_{R\_EOS}}
\]
\[
= 1 - \left\{ \frac{1}{\lambda (1 - K_{R\_EOS}) + \lambda p K_{R\_EOS}} \right\} \left\{ \mu \sum_{n=1}^{k} nq_n + k\mu(K_{R\_EOS} - q_k) \right\}
\]

Proportion of customer lost due to reneging out of total customers arriving in the system (under R\_BOS) is
\[
= \frac{\text{Avgrr}(R\_BOS)}{\lambda}
\]
\[
= (1 - K_{R\_BOS}) + p K_{R\_BOS} - (1/\lambda) \left[ \mu \sum_{n=1}^{k} nq_n + k\mu(K_{R\_BOS} - q_k) \right]
\]

Proportion of customer lost due to reneging out of total customers arriving in the system (under R\_EOS)
\[
= \frac{\text{Avgrr}(R\_EOS)}{\lambda}
\]
\[
= (1 - K_{R\_EOS}) + p K_{R\_EOS} - (1/\lambda) \left[ \mu \sum_{n=1}^{k} nq_n + k\mu(K_{R\_EOS} - q_k) \right]
\]

Customers are lost to the system in two ways, due to balking and due to reneging. Management would like to know the proportion of total customers lost in order to have an idea of total business lost.

Hence the mean rate at which customers are lost (under R\_BOS) is
\[
\lambda - X_{R\_BOS} + \text{Avgrr}(R\_BOS)
\]
\[
= \lambda - \mu \sum_{n=1}^{k} np_n - k\mu(K_{R\_BOS} - p_k)
\]
and the mean rate at which customers are lost (under R_EOS) is

\[ \lambda - X_{EOS} + \text{Avgrr}_{R_{EOS}} \]

\[ = \lambda - \mu \sum_{n=1}^{k} nq_n - k\mu(K_{R_{EOS}} - q_k) \]

These rates help in the determination of proportion of customers lost which is of interest to the system manager as also an important measure of business lost. This proportion (under R_BOS) is given by

\[ \frac{\{ \lambda - X_{BOS} + \text{Avgrr}_{R_{BOS}} \}}{\lambda} \]

\[ = 1 - \frac{1}{\lambda} \left[ \mu \sum_{n=1}^{k} nq_n + k\mu(K_{R_{BOS}} - p_k) \right] \]

and the proportion (under R_EOS) is given by

\[ \frac{\{ \lambda - X_{EOS} + \text{Avgrr}_{R_{EOS}} \}}{\lambda} \]

\[ = 1 - \frac{1}{\lambda} \left[ \mu \sum_{n=1}^{k} nq_n + k\mu(K_{R_{EOS}} - q_k) \right] \]

The proportion of customer completing receipt of service can now be easily determined from the above proportion.

The customers who leave the system from the queue do not receive service. Consequently, only those customers who reach the service station constitute the actual load of the server. From the server’s point of view, this provides a measure of the amount of work the server has to do. Let us call the rate at which customers reach the service station as \( \lambda^s \). Then under R_BOS

\[ \lambda^s_{R_{BOS}} = \lambda^s_{R_{BOS}}(1 - \text{proportion of customers lost due to reneging out of those joining the system}) \]
In case of R_EOS, one needs to recall that customers may renege even while being served and only those customers who renege from the queue will not constitute any work for the server. Thus

\[ \lambda^*_{(R\_EOS)} = \lambda^e_{(R\_EOS)} (1 - \text{proportion of customers lost due to reneging from the queue out of those joining the system}) \]

\[ = \lambda^*_{(R\_EOS)} \left\{ 1 - \sum_{n=k+1}^{\infty} (n-k)q_n / \lambda^*_{(R\_EOS)} \right\} \]

\[ = \lambda^*_{(R\_EOS)} - \nu\left( Q'(1) - \sum_{n=1}^{\infty} nq_n \right) + k\nu \left( 1 - \sum_{n=0}^{k} q_n \right) \]

\[ = \lambda^*_{(R\_EOS)} - \nu Q'(1) + \nu \sum_{n=1}^{\infty} nq_n + k\nu (K_{R\_EOS} - q_k) \]

\[ = (\mu + \nu) \sum_{n=1}^{k} nq_n + k(\mu + \nu) (K_{R\_EOS} - q_k) \]

In order to ensure that the system is in steady state, it is necessary for the rate of customers reaching the service station to be less than the system capacity. This translates to

\[ (\lambda^i / k\mu) < 1. \]

3.3.3. Sensitivity Analysis:

We have assumed that there are essentially three parameters viz: \( \lambda, \mu \) and \( \nu \) relating to the stochastic nature of arrival, service and reneging.
patterns. We shall follow the following notational convention in the rest of this section.

\[ p_n(\lambda, \mu, \nu) \text{ and } q_n(\lambda, \mu, \nu) \]

will denote the probability that there are ‘n’ customers in a system with parameters \( \lambda, \mu, \nu \) in steady state under R_BOS and R_EOS respectively.

i) Let \( \lambda > \lambda_0 \), then

\[
\frac{p_0(\lambda_1, \mu, \nu)}{p_0(\lambda_0, \mu, \nu)} < 1
\]

\[
\Rightarrow \frac{(\lambda_0 - \lambda)}{\mu} + \frac{(k+1) - \lambda}{k! \mu^k} p(\lambda_0^{k+1} - \lambda_0^{k+1}) \frac{1}{k! \mu^k(k\mu + \nu)} + ... < 0
\]

which is true. Hence \( p_0 \downarrow \) as \( \lambda \uparrow \).

ii) Let \( \mu > \mu_0 \), then

\[
\frac{p_0(\lambda, \mu_1, \nu)}{p_0(\lambda, \mu_0, \nu)} > 1
\]

\[
\Rightarrow \lambda \left( \frac{1}{\mu_0} - \frac{1}{\mu_1} \right) + \frac{\lambda}{k! \mu_0^k} \left( \frac{1}{\mu_0^k} - \frac{1}{\mu_1^k} \right) + \frac{\lambda^2 p}{k! \mu_0^k (k\mu_0 + \nu)} \left( \frac{1}{\mu_0^k(k\mu_0 + \nu)} - \frac{1}{\mu_1^k(k\mu_1 + \nu)} \right) + ... > 0
\]

which is true. Hence \( p_0 \uparrow \) as \( \mu \uparrow \).

iii) Let \( \nu > \nu_0 \), then

\[
\frac{p_0(\lambda, \mu, \nu_1)}{p_0(\lambda, \mu, \nu_0)} > 1
\]

\[
\Rightarrow \frac{\lambda^2 p}{k! \mu_0^k} \left( \frac{1}{(k\mu + \nu_0)} - \frac{1}{(k\mu + \nu_1)} \right) + \frac{\lambda^2 p}{k! \mu_0^k} \left( \frac{1}{(k\mu + \nu_0)} \left( \frac{1}{(k\mu + 2\nu_0)} \right) \right) + ... > 0
\]

which is true. Hence \( p_0 \uparrow \) as \( \nu \uparrow \).
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The following can similarly be shown.

iv) $q_0 \downarrow$ as $\lambda \uparrow$

v) $q_0 \uparrow$ as $\mu \uparrow$

vi) $q_0 \uparrow$ as $v \uparrow$

These results state that an increase in arrival rate would result in lowering of the fraction of time the server is idle. An increase in service rate would mean the server is able to work efficiently so that it can process same amount of work quickly. This translates to higher server idle time. An increase in reneging rate would mean the server has fewer work to do and hence higher fraction of idle time.

3.3.4. Numerical Example:

To illustrate the use of our results, we apply them to a queuing problem. We quote below an example from Allen (2005, page 278).

'KAMAKAZY AIRLINES is planning a new telephone reservation center. Each agent will have a reservation terminal and can serve a typical caller in 5 minutes, the service time being exponentially distributed. Calls arrive randomly and the system has a large message buffering system to hold calls that arrive when no agent is free. An average of 36 calls per hour is expected during the peak period of the day. The design criteria for the new facility is

The probability a caller will find all agents busy must not exceed 0.1 (10%).

How many agents (and terminals) should be provided?'

This is a design problem. Here $\lambda = 36/\text{hr}$ and $\mu = 12/\text{hr}$. As required by the telephone reservation centre, we examine the minimum number of agents so as to meet the design criterion. Though not explicitly mentioned, it is necessary to assume
reneging and balking. Balking because in telecommunication based service delivery systems, it is known that an incoming call that gets a busy tone may withdraw from the system. Even if the call joins the queue, it is natural to assume that callers have a patience time. This implies reneging.

We assume a possible Markovian reneging rate of $v=10$/hr. We further assume that balking probability is state independent and is 0.01.

Various performance measures of interest computed are given in Table 3.1. These measures were arrived at using a FORTRAN 77 program coded by the authors. Different choices of $k$ were considered. Results relevant with regard to the number of agents to ensure design requirement of the problem ($\sum_{n=k+1}^{\infty} P_n < 0.1$) are presented in Table 3.1. (All rates in the table are per hour rates).

**Table 3.1: Table of Performance Measures:**

(Assuming $\lambda=36$, $\mu=12$, $v=10$ and balking probability 0.01)

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Number of agents</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k=5$</td>
</tr>
<tr>
<td>$\sum_{n=k+1}^{\infty} P_n$</td>
<td>0.18801</td>
</tr>
<tr>
<td>$\lambda^x$ (i.e. arrival rate of customers reaching service station)</td>
<td>34.48748</td>
</tr>
<tr>
<td>Effective mean arrival rate($\lambda^x$)</td>
<td>35.93232</td>
</tr>
<tr>
<td>Fraction of time server is idle ($p_0$)</td>
<td>0.04959</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th></th>
<th>k=5</th>
<th>k=6</th>
<th>k=7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average length of queue</td>
<td>0.14448</td>
<td>0.05349</td>
<td>0.00179</td>
</tr>
<tr>
<td>Average length of system</td>
<td>3.01844</td>
<td>3.00636</td>
<td>3.00197</td>
</tr>
<tr>
<td>Mean reneging rate</td>
<td>1.44484</td>
<td>0.53496</td>
<td>0.01791</td>
</tr>
<tr>
<td>Average balking rate</td>
<td>0.06768</td>
<td>0.03059</td>
<td>0.01217</td>
</tr>
<tr>
<td>Mean rate of customers lost</td>
<td>1.51252</td>
<td>0.56556</td>
<td>0.19129</td>
</tr>
<tr>
<td>Proportion of customers lost due to reneging, and balking.</td>
<td>0.04201</td>
<td>0.01571</td>
<td>0.00534</td>
</tr>
</tbody>
</table>

The above problem required identification of ideal number of agents with respect to the design criteria that the probability a caller will find all agents busy must not exceed 0.1 (10%). In the table we have analyzed three alternatives with 5, 6 and 7 agents (k=5, 6, 7). In case the number of agents is 5, the probability that all the agents busy is 0.18801. This violates the design criteria. In case the number of agents is 6 or 7, the design criteria is met (with probabilities 0.08499 and 0.03381 respectively). If we consider the case of k=7, that would necessitate an additional server compared to k=6. Considering cost implications, the idea would be to attain the design criteria with minimal number of servers and it appears from the above table that an ideal choice of number of agents is 6. The airlines may therefore design the reservation centre so as to hold six agents. One may note here that with six agents, the steady state condition would also be satisfied (as $\lambda^* / (k*\mu) = 35.43444/6*12=0.49215<1$).
3.4. Analysis of M/M/k model with position independent reneging (PIR) and state dependent balking (SDB).

In this section, we assume that balking is state dependent. Such an assumption implies that higher the queue size, higher is the probability that a customer may balk. It is not difficult to find many queuing systems where such customer behavior can be observed. Further, we assume that each arriving customer has probability \((1-p^{n-k+1})\) of balking from a system with no idle servers where ‘n’ is the state of the system and ‘k’ is the number of servers. The reneging distribution remain the same as in the previous section.

3.4.1. The system state probabilities.

In this section, the steady state probabilities are derived by the Markov process method. We first analyze the case of R_BOS where customers renge only from the queue. The rate-in rate-out diagram under R_BOS is given below:

Under R_BOS, let \(p_n\) denote the probability that there are ‘n’ customers in the system. The steady state probabilities under R_BOS are
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\[ \dot{\lambda}p_0 = \mu p_1, \quad (3.4.1.1) \]

\[ \dot{\lambda}p_{n+1} + (n+1)\mu p_{n+1} = \lambda p_n + n\mu p_n; n = 1, 2, \ldots, k-1 \] \hspace{1cm} (3.4.1.2)

\[ \dot{\lambda}p_{n+1} + [k\mu + (n-k+1)\nu]p_{n+1} = \lambda p_n + [k\mu + (n-k)\nu]p_n; n = 1, 2, \ldots, k \] \hspace{1cm} (3.4.1.3)

Solving recursively, we get (under R_BOS)

\[ p_n = \left( \frac{\lambda^n}{n!\mu^n} \right) p_0 \quad ; n = 1, 2, \ldots, k \] \hspace{1cm} (3.4.1.4)

\[ p_n = \left[ \lambda^n p^{[(n-k)(n-k+1)]/2} \left\{ k! \mu^k \prod_{r=k+1}^{n} (k\mu + r-k\nu) \right\} \right] p_0; n = k+1, \ldots \] \hspace{1cm} (3.4.1.5)

where \( p_0 \) is obtained from the normalizing condition \( \sum_{n=0}^{\infty} p_n = 1 \) and is given as

\[ p_0 = \left[ \sum_{n=0}^{k} \frac{\lambda^n}{n!\mu^n} + \sum_{n=k+1}^{\infty} \frac{\lambda^n p^{[(n-k)(n-k+1)]/2}}{k! \mu^k \prod_{r=k+1}^{n} (k\mu + r-k\nu)} \right]^{-1}. \] \hspace{1cm} (3.4.1.6)

The steady state probabilities satisfy the recurrence relation. Under R_BOS

\[ p_n = \left\{ \frac{\lambda}{n\mu} \right\} p_{n-1} \quad ; n = 1, 2, \ldots, k, \]

and \[ p_n = \left\{ \frac{2\lambda p^{-(n-k)}}{(k\mu + n-k\nu)} \right\} p_{n-1} \quad ; n = k+1, k+2, \ldots. \]

We shall denote by \( K_{R_BOS} \) the probability that an arriving unit has to wait on arrival (under R_BOS). Then

\[ K_{R_BOS} = \Pr (N \geq k) \]

\[ = \sum_{n=k}^{\infty} p_n. \] \hspace{1cm} (3.4.1.7)

We may call \( K_{R_BOS} \) as ‘Erlang’s second (Erlang’s delay probability) formula for balking (SDB) and reneging (R_BOS)’ in line with similar nomenclature in Medhi (2003, page 87).
The rate-in rate-out diagram under R_EOS is given as

\[ \begin{array}{cccccccc}
\lambda & \lambda & \lambda & \lambda & \lambda p & \lambda p' \\
0 & 1 & 2 & \cdots & K-1 & K & k+1 & \cdots \\
\mu + \nu & 2(\mu + \nu) & 3(\mu + \nu) & (k-1)(\mu + \nu) & k(\mu + \nu) & k\mu + (k+1)\nu & k\mu + (k+2)\nu
\end{array} \]

Under R_EOS customer may renege from the queue as well as while receiving service, let \( q_n \) denote the probability that there are \( n \) customers in the system. Applying the Markov process method, we obtain the following set of steady state equations.

\[
\begin{align*}
\lambda q_0 &= (\mu + \nu)q_1, \quad (3.4.1.8) \\
\lambda q_{n+1} + (n+1)(\mu + \nu)q_{n+1} &= \lambda q_n + n(\mu + \nu)q_n; \quad n = 1, 2, \ldots, k-1 \quad (3.4.1.9) \\
\lambda p^{n-k}q_{n-1} + [k\mu + (n+1)\nu]q_{n-1} &= \lambda p^{n-k+1}q_n + [k\mu + n\nu]q_n; \quad n = k+1, \ldots \quad (3.4.1.10)
\end{align*}
\]

Solving recursively, we get (under R_EOS)

\[
q_n = \left[ \lambda^n / \left[n(\mu + \nu)^n\right] \right] q_0; \quad n = 1, 2, \ldots, k \quad (3.4.1.11)
\]

\[
q_n = \left[ \lambda^n p^{\left[(n-k)(n-k+1)\right]/2} \left( \prod_{r=k+1}^{n} (k\mu + r\nu) \right) \right] q_0; \quad n = k+1, \ldots \quad (3.4.1.12)
\]

where \( q_0 \) is obtained from the normalizing condition \( \sum_{n=0}^{\infty} q_n = 1 \) and is given as

\[
q_0 = \left[ \sum_{n=0}^{k} \lambda^n / \left[n(\mu + \nu)^n\right] + \sum_{n=k+1}^{\infty} \lambda^n p^{\left[(n-k)(n-k+1)\right]/2} \left( \prod_{r=k+1}^{n} (k\mu + r\nu) \right) \right]^{-1} \quad (3.4.1.13)
\]
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The recurrence relations under R_EOS are

\[ q_n = \left[ \frac{\lambda}{n(\mu + \nu)} \right] q_{n-1} \quad ; \quad n = 1, 2, \ldots k, \]

and \[ q_n = \left\{ \frac{\lambda P^{(n+1)}}{(k\mu + n\nu)} \right\} q_{n-1} \quad ; \quad n = k+1, k+2, \ldots. \]

We shall denote by \( K_{R_{_{EOS}}} \) the probability that an arriving unit has to wait on arrival (under R_EOS). Then

\[ K_{R_{_{EOS}}} = \Pr (N \geq k) \]

\[ = \sum_{n=k}^{\infty} q_n. \]  

(3.4.1.14)

which may be called 'Erlang's second (Erlang's delay probability) formula for balking (SDB) and reneging (R_EOS)'.

3.4.2. Performance Measures.

We first obtain an expression for 'L', which denotes the mean number of customers in the system. To obtain the expression for the same, we note again that

\[ L = P'(1) \]

where

\[ P'(1) = \frac{d}{ds} P(s) \Big|_{s=1}. \]

Here \( P(S) \) is the p.g.f. of the steady state probabilities and is given by

\[ P(s) = \sum_{n=0}^{\infty} p_n s^n. \]

To obtain \( P'(1) \) under R_BOS we proceed as follows:

From equation (3.4.1.2) we have

\[ \lambda p_{n-1} + (n+1) \mu p_{n+1} = \lambda p_n + n \mu p_n, \quad n=1, 2, \ldots k-1. \]

Multiplying both sides of the equation by \( s^n \) and summing over \( n \)
\[
\lambda \sum_{n=1}^{k-1} p_{n-1}s^{n-1} - \lambda \sum_{n=1}^{k-1} p_n s^n = \sum_{n=1}^{k-1} \mu_p s^n - \frac{1}{s} \sum_{n=1}^{k-1} (n+1)p_n s^{n+1} \tag{3.4.2.1}
\]

From (3.4.1.3) we have

\[
\lambda p^{s-k} p_{n+1} \{k\mu + (n-k+1)\nu\} p_{n+k+1} = \lambda p^{s-k} p_n \{k\mu + (n-k)\nu\} p_n \quad ; \quad n = k, k+1, \ldots
\]

Similarly multiplying both sides of the equation by \(s^n\) and summing over \(n\)

\[
\lambda s \sum_{n=1}^{k-1} p_{n-1}s^{n-1} - \lambda \sum_{n=1}^{k-1} p^{s-k+1} p_n s^n = \sum_{n=1}^{k-1} \{k\mu + (n-k)\nu\} p_n s^n - \frac{1}{s} \sum_{n=1}^{k-1} \{k\mu + (n-k+1)\nu\} p_n s^{n+1} \tag{3.4.2.2}
\]

Adding (3.4.2.1) and (3.4.2.2)

\[
\Rightarrow \lambda \left[ \sum_{n=1}^{k-1} p_{n-1}s^{n-1} + \sum_{n=1}^{k-1} p^{s-k+1} p_n s^n \right] - \lambda \left[ \sum_{n=1}^{k-1} \mu_p s^n + \sum_{n=1}^{k-1} p^{s-k+1} p_n s^n \right]
\]

\[
= \mu \sum_{n=1}^{k-1} p_{n-1}s^{n-1} + \sum_{n=1}^{k-1} \{k\mu + (n-k)\nu\} p_n s^n - \frac{1}{s} \sum_{n=1}^{k-1} \{k\mu + (n-k)\nu\} p_n s^{n+1} \tag{3.4.2.2}
\]

Adding (3.4.2.1) and (3.4.2.2)
\[ \Rightarrow \lambda \left( P(s) - \sum_{n=k}^{\infty} p_n s^n \right) + \left( \lambda / p^{\lambda^k} \right) \left( P(p) - \sum_{n=k}^{\infty} p_n p^n \right) - \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) - (\lambda / p^{\lambda^k}) \left( P(p) - \sum_{n=0}^{k-1} p_n p^n \right) \]

\[ = \lambda k \left( P(s) - \sum_{n=0}^{k-1} p_n s^n \right) + k \mu \left( P(s) - \sum_{n=0}^{k-1} p_n s^n \right) + \lambda \lambda \left( P(s) - \sum_{n=0}^{k-1} p_n s^n \right) - k \lambda \left( P(s) - \sum_{n=0}^{k-1} p_n s^n \right) \]

\[ - k \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) \]

\[ \Rightarrow \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) + \lambda \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) - k \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) \]

\[ \Rightarrow \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) + \lambda \lambda \left( P(p) - p_0 - \sum_{n=0}^{k-1} p_n p^n \right) - k \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) \]

\[ + \lambda \lambda \left( P(p) - p_0 - \sum_{n=0}^{k-1} p_n p^n \right) - k \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) \]

\[ \Rightarrow P(s)(\mu+v) = \lambda \left( P(s) - \sum_{n=0}^{k-1} p_n s^n \right) + \lambda \lambda \left( P(p) - p_0 - \sum_{n=0}^{k-1} p_n p^n \right) - k \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) \]

\[ + \lambda \lambda \left( P(p) - p_0 - \sum_{n=0}^{k-1} p_n p^n \right) - k \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) \]

Now

\[ \lim_{s \to 1^-} P(s) = \frac{1}{(\mu+v)} \left[ \frac{k \mu}{s} P(s) + \frac{k \mu}{s} \sum_{n=0}^{k-1} p_n s^n + \lambda \lambda \left( P(s) - p_0 - \sum_{n=0}^{k-1} p_n s^n \right) \right] \]

\[ \Rightarrow P'(1) = \frac{1}{(\mu+v)} \left[ \lambda \left( 1 - \sum_{n=0}^{k-1} p_n \right) + (\lambda / p^{\lambda^k}) \left( P(p) - \sum_{n=0}^{k-1} p_n p^n \right) + \mu P'(1) - k \mu (1 - \sum_{n=0}^{k-1} p_n) \right] \]

\[ + \frac{1}{(\mu-v)} \left[ \lambda (1 - \sum_{n=0}^{k-1} p_n) + (\lambda / p^{\lambda^k}) \left( P(p) - \sum_{n=0}^{k-1} p_n p^n \right) - (\mu - v) \sum_{n=0}^{k-1} p_n \right] \] (3.4.2.3)
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Here \( P(p) = \sum_{n=0}^{\infty} p_n(\lambda, \mu, \nu) p^n \) where the symbol \( p_n(\lambda, \mu, \nu) \) will denote the probability that there are \( n \) customers in a system with parameters \( p\lambda, \mu, \nu \) in steady state under \( R\_BOS \) is as described in section 5. We use \( p_n \) and \( p_n(\lambda, \mu, \nu) \) interchangeably. However should any of the parameters \( \lambda, \mu, \nu \) change, it is explicitly stated. To obtain a closed form expression for \( P(p) \), let us for the time being, consider another queuing system with parameter and assumptions similar to the queuing system we are presently considering except that the arrival rate is \( p\lambda' \). For this new system, the steady state equations are same as (3.4.1.1), (3.4.1.2) and (3.4.1.3) with \( \lambda \) is replaced by \( p\lambda' \). Denoting the steady state probabilities of this new system by \( p_n(p\lambda, \mu, \nu) \), we can obtain

\[
p_n(p\lambda, \mu, \nu) = \left[ (p\lambda)^n / n! \mu^n \right] p_0(p\lambda, \mu, \nu) ; n = 1,2,...,k,
\]

\[
p_n(p\lambda, \mu, \nu) = \left[ (p\lambda)^n p^{(n-k)(n-k+1)}/k! \prod_{r=k+1}^{n} \left\{ \mu + (r-k)\nu \right\} \right] p_0(p\lambda, \mu, \nu) ; n = k+1, ...
\]

(3.4.2.4)

where

\[
p_0(p\lambda, \mu, \nu) = \left[ \sum_{s=0}^{k} (p\lambda)^s / s! \mu^s + \sum_{n=k+1}^{\infty} (p\lambda)^n p^{(n-k)(n-k+1)}/k! \prod_{r=k+1}^{n} \left\{ \mu + (r-k)\nu \right\} \right]^{-1}.
\]

(3.4.2.5)

Let \( P(S; p\lambda, \mu, \nu) \) denotes the probability generating function of this new queuing system so that
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\[ P(S; p\lambda, \mu, \nu) = \sum_{n=0}^{\infty} p_n(p\lambda, \mu, \nu) s^n. \]

Now

\[ P(p) = \sum_{n=0}^{\infty} p_n(p\lambda, \mu, \nu) p^n \]
\[ = p_0 + \sum_{n=1}^{\infty} (p\lambda)^n / n! \mu^n + \sum_{n=k+1}^{\infty} \left( (p\lambda)^n / n! \mu^n + \sum_{m=k+1}^{n} \frac{(p\lambda)^m}{m!} \prod_{r=k+1}^{m} \{\mu + (r-k)\nu} \right) p_0 \]
\[ \Rightarrow \left( P(p) - p_0 \right) / p_0 = \sum_{n=1}^{\infty} (p\lambda)^n / n! \mu^n + \sum_{n=k+1}^{\infty} \left[ (p\lambda)^n p^{(n-k)(n-k+1)/2} / k! \mu^k \prod_{r=k+1}^{n} \{\mu + (r-k)\nu} \right] p_0 (p\lambda, \mu, \nu) \]
\[ \Rightarrow 1 = p_0 (p\lambda, \mu, \nu) + \left\{ (P(p) - p_0) / p_0 \right\} p_0 (p\lambda, \mu, \nu) \]
\[ \Rightarrow P(p) = p_0 / p_0 (p\lambda, \mu, \nu). \]

(3.4.2.6)

Now putting \( S=1 \) in \( P(S; p\lambda, \mu, \nu) \) we get

\[ P(I; p\lambda, \mu, \nu) = p_0 (p\lambda, \mu, \nu) + \sum_{n=1}^{\infty} p_n(p\lambda, \mu, \nu) \]
\[ \Rightarrow 1 = p_0 (p\lambda, \mu, \nu) + \sum_{n=1}^{\infty} (p\lambda)^n / n! \mu^n + \sum_{n=k+1}^{\infty} \left[ (p\lambda)^n p^{(n-k)(n-k+1)/2} / k! \mu^k \prod_{r=k+1}^{n} \{\mu + (r-k)\nu} \right] p_0 (p\lambda, \mu, \nu) \]
\[ \Rightarrow 1 = p_0 (p\lambda, \mu, \nu) + \left\{ (P(p) - p_0) / p_0 \right\} p_0 (p\lambda, \mu, \nu) \]
\[ \Rightarrow P(p) = p_0 / p_0 (p\lambda, \mu, \nu). \]

(3.4.2.7)

Again let \( K_{p, \lambda, \mu, \nu} = \sum_{n=0}^{\infty} p_n (p\lambda, \mu, \nu) \)
\[ = \sum_{n=k}^{\infty} \left[ (p\lambda)^n p^{(n-k)(n-k+1)/2} / k! \mu^k \prod_{r=k+1}^{n} \{\mu + (r-k)\nu} \right] p_0 (p\lambda, \mu, \nu) \]
\[ = \sum_{n=k}^{\infty} p_n p^{n} \left\{ p_0 (p\lambda, \mu, \nu) / p_0 \right\}. \]
Therefore,
\[ \sum_{n=k}^\infty p_n p^n = p_0 K_{R_{-BOS}}(p\lambda, \mu, \nu) / p_0(p\lambda, \mu, \nu). \] (3.4.2.8)

Using (3.4.1.7), (3A2.7) and (3A2.8) in (3.4.2.3) we obtain
\[
P'((l) = (1/\nu) \left[ \lambda (1 - K_{R_{-BOS}}) + \left( \lambda p_0 K_{R_{-BOS}}(p\lambda, \mu, \nu) / p^{k+1} p_0(p\lambda, \mu, \nu) \right) - (\mu - \nu) \sum_{n=1}^k np_n \right] - k(\mu - \nu)(K_{R_{-BOS}} - p_k),
\]

where \( p_0(p\lambda, \mu, \nu) \) is given in (3.4.2.5).

Therefore, the mean system size under \( R_{BOS} \) is given by
\[
L_{R_{-BOS}} = (1/\nu) \left\{ \lambda (1 - K_{R_{-BOS}}) + \left( \lambda p_0 K_{R_{-BOS}}(p\lambda, \mu, \nu) / p^{k+1} p_0(p\lambda, \mu, \nu) \right) - (\mu - \nu) \sum_{n=1}^k np_n \right\} - k(\mu - \nu)(K_{R_{-BOS}} - p_k).
\]

Mean number of customers in the queue is denoted by \( L_q \) and under \( R_{BOS} \) it is given by
\[
L_{q(R_{-BOS})} = \sum_{n=k}^\infty (n - k) p_n = L_{R_{-BOS}} - \sum_{n=1}^k np_n - k(K_{R_{-BOS}} - p_k).
\]

To calculate the mean system size and mean queue size under \( R_{EOS} \) we proceed as follows

Let \( Q(s) \) denote the probability generating function, defined by \( Q(s) = \sum_{n=0}^\infty q_n s^n \)

and let \( K_{R_{-EOS}}(p\lambda, \mu, \nu) = \sum_{n=k}^\infty q_n (p\lambda, \mu, \nu) \).
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From equation (3.4.1.8) we have,

$$\lambda q_{n-1} + (n + 1)(\mu + \nu)q_{n+1} = \lambda q_n + n(\mu + \nu)q_n; \quad n=1,2,\ldots,k-1$$

Multiplying both sides of this equation by $s^n$ and summing over $n$ we get

$$\lambda s \sum_{n=0}^{k-1} q_{n-1} s^{n-1} - \lambda \sum_{n=0}^{k-1} q_n s^n = (\mu + \nu) \sum_{n=0}^{k-1} n q_n s^n - \frac{1}{s} (\mu + \nu) \sum_{n=0}^{k-1} (n + 1) q_{n+1} s^{n+1} \quad (3.4.2.9)$$

From equation (3.4.1.9)

$$\lambda p^{n-1} q_{n-1} + [k\mu + (n + 1)\nu] q_{n+1} = \lambda p^{n-k+1} q_n + [k\mu + n\nu] q_n; \quad n=k+1, k+2,\ldots.$$

Multiplying both sides of this equation by $s^n$ and summing over $n$ we get

$$\lambda s \sum_{n=k+1}^{\infty} p^{n-k} q_{n-1} s^{n-1} - \lambda \sum_{n=k+1}^{\infty} p^{n-k+1} q_n s^n = \sum_{n=k+1}^{\infty} [k\mu + n\nu] q_n s^n - \frac{1}{s} \sum_{n=k+1}^{\infty} [k\mu + (n + 1)\nu] q_{n+1} s^{n+1}$$

$$n=k+1 \quad (3.4.2.10)$$

Adding (3.4.2.9) and (3.4.2.10) and proceeding in a manner similar to mean system size under R_BOS, we obtain,

$$Q(l) = \frac{1}{\nu} \left[ \lambda (1-K_{R_{\text{EOS}}}) + \{\lambda q_0 K_{R_{\text{EOS}}} (p\lambda, \mu, \nu)\} / p^{k-1} q_0 (p\lambda, \mu, \nu) - \mu \sum_{n=1}^{k} n q_n - k\mu (K_{R_{\text{EOS}}} - q_k) \right]$$

where

$$q_0 (p\lambda, \mu, \nu) = \left[ \sum_{n=0}^{k} (p\lambda)^n / n! \mu^n + \sum_{n=k+1}^{\infty} (p\lambda)^n p^{(n-k)(n-k+1)/2} / k! \mu^k \prod_{r=k+1}^{n} \{\mu + r\nu \} \right]^{-1} \quad (3.4.2.11)$$

Thus, the mean system size under R_EOS is given by

$$L_{R_{\text{EOS}}} = (1/\nu) [\lambda (1-K_{R_{\text{EOS}}}) + \{\lambda q_0 K_{R_{\text{EOS}}} (p\lambda, \mu, \nu)\} / p^{k-1} q_0 (p\lambda, \mu, \nu) - \mu \sum_{n=1}^{k} n q_n - k\mu (K_{R_{\text{EOS}}} - q_k)]$$

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and the mean queue size under R_EOS is given by

\[ L_{R_{EOS}} = L_{R_{EOS}} - \sum_{n=1}^{k} n q_n - k(K_{R_{EOS}} - q_k) \]

\[ = (1/\nu) [\lambda (1-K_{R_{EOS}}) + (\lambda q_0 K_{R_{EOS}}(p \lambda, \mu, \nu)) / p^{k-1} q_0(p \lambda, \mu, \nu) - (\mu + \nu) \sum_{n=1}^{k} n q_n - k(\mu + \nu)(K_{R_{EOS}} - q_k)] \]

Using Little’s formula, one can calculate the average waiting time in

the system and average waiting time in queue from the above mean lengths both

under R_BOS and R_EOS. Under R_BOS the mean waiting time in system and

queue are given by

\[ W_{R_{BOS}} = \frac{L_{R_{BOS}}}{\lambda} \]

\[ = (1/\lambda \nu) [\lambda (1 - K_{R_{BOS}}) + (\lambda p_0 K_{R_{BOS}}(p \lambda, \mu, \nu)) / p^{k-1} q_0(p \lambda, \mu, \nu) - (\mu + \nu) \sum_{n=1}^{k} n p_n - k(K_{R_{BOS}} - p_k) (\mu - \nu)] \]

\[ W_{q(R_{BOS})} = \frac{L_{q(R_{BOS})}}{\lambda} \]

\[ = (1/\lambda \nu) [\lambda (1 - K_{R_{BOS}}) + (\lambda p_0 K_{R_{BOS}}(p \lambda, \mu, \nu)) / p^{k-1} q_0(p \lambda, \mu, \nu) - (\mu + \nu) \sum_{n=1}^{k} n p_n - k\mu (K_{R_{BOS}} - p_k)] \]

and under R_EOS these are given by

\[ W_{R_{EOS}} = \frac{L_{R_{EOS}}}{\lambda} \]

\[ = (1/\lambda \nu) [\lambda (1 - K_{R_{EOS}}) + (\lambda q_0 K_{R_{EOS}}(p \lambda, \mu, \nu)) / p^{k-1} q_0(p \lambda, \mu, \nu) - (\mu + \nu) \sum_{n=1}^{k} n q_n - k\mu (K_{R_{EOS}} - q_k)] \]
Customers arrive into the system at rate $\lambda$. However all the customers who arrive do not join the system because of balking. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$W_{q(R_{-EOS})} = \frac{L_{q(R_{-EOS})}}{\lambda}$$

$$=(1/\lambda)[\lambda(1-K_{R_{-BOS}})+\{q_0 K_{R_{-BOS}}(p\lambda,\mu,\nu)/\{p^{k+1} q_0(p\lambda,\mu,\nu)\}-(\mu+\nu)\sum_{n=1}^{k} n p_n$$

$$-k(\mu+\nu)(K_{R_{-EOS}}-q_k)].$$

Similarly in case of $R_{-EOS}$

$$W_{q(R_{-EOS})} = \lambda(1-K_{R_{-EOS}}) + \{q_0 K_{R_{-EOS}}(p\lambda,\mu,\nu)/\{p^{k+1} q_0(p\lambda,\mu,\nu)\}.$$

We have assumed that each customer has a random patience time following $\text{exp}(\nu)$. Clearly then, the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rates ($\text{avgrr}$) under $R_{-BOS}$ and $R_{-EOS}$ are given by

$$\text{avgrr}_{(R_{-BOS})} = \sum_{n=0}^{k} (n-k)w_p_n$$

$$= \lambda(1-K_{R_{-BOS}}) + \{q_0 K_{R_{-BOS}}(p\lambda,\mu,\nu)/\{p^{k+1} q_0(p\lambda,\mu,\nu)\} -$$

$$= \mu \sum_{n=1}^{k} n p_n - k\mu(K_{R_{-BOS}} - p_k).$$
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\[ \text{Avg. } rr_{R_{EOS}} = \sum_{n=1}^{\infty} n q_n \]
\[ = V L_{R_{EOS}} \]
\[ = \lambda (1 - K_{R_{EOS}}) + [\lambda q_0 K_{R_{EOS}}(\lambda, \mu, \nu)] \left( p^{k-1} q_0(p, \lambda, \mu, \nu) - \mu \sum_{n=1}^{k} n q_n - k \mu (K_{R_{EOS}} - q_k) \right). \]

Average balking rates under R_BOS and R_EOS are given by

\[ \text{Avg. } \text{br}_{R_{BOS}} = \lambda \sum_{n=1}^{\infty} (1 - p^{n-k+1}) p_n \]
\[ = K_{R_{BOS}} - \{ p_0 K(p, \lambda, \mu, \nu) \} \left( p^{k-1} p_0(p, \lambda, \mu, \nu) \right). \]
\[ \text{Avg. } \text{br}_{R_{EOS}} = \lambda \sum_{n=1}^{\infty} (1 - p^{n-k+1}) q_n \]
\[ = K_{R_{EOS}} - \{ q_0 K(p, \lambda, \mu, \nu) \} \left( p^{k-1} q_0(p, \lambda, \mu, \nu) \right). \]

In a real life situation, customers who balk or renege represent the business lost. It is therefore of interest to determine the proportion of customers lost, both out of those joining the system as well as out of those arriving into the system. These performance measures are given below

Proportion of customer lost due to reneging out of those joining the system (under R_BOS) is

\[ = \frac{\text{Avg. } rr_{R_{BOS}}}{{X}_{(R_{BOS})}} \]
\[ = 1 - [\lambda (1 - K_{R_{BOS}}) + (\lambda p_0 K_{R_{BOS}} p, \lambda, \mu, \nu)] \left( p^{k-1} p_0(p, \lambda, \mu, \nu) \right)] \left( k \mu (K_{R_{EOS}} - p_k) \right). \]

Proportion of customer lost due to reneging out of those joining the system (under R_EOS) is
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\[ \frac{\text{Avgrr}_{(R\_EOS)}}{X_{(R\_EOS)}} \]

\[ = 1 - \frac{1}{\lambda} (1 - K_{R\_EOS}) + \left( \frac{\lambda q_o K_{R\_EOS} p \lambda \mu \nu}{p^{k+1} q_0 p \lambda \mu \nu} \right) \left( \mu \sum_{n=1}^{k} nq_n + k\mu(K_{R\_EOS} - q_k) \right). \]

Proportion of customer lost due to reneging out of total customers arriving in the system (under R\_BOS) is

\[ \frac{\text{Avgrr}_{(R\_BOS)}}{X} \]

\[ = (1 - K_{R\_BOS}) + \left( \frac{p_0 K_{R\_BOS} (p \lambda \mu \nu)}{p^{k+1} (p \lambda \mu \nu)} \right) - \left( \frac{1}{\lambda} \right) \left( \mu \sum_{n=1}^{k} nq_n + k\mu(K_{R\_BOS} - p_k) \right). \]

Proportion of customer lost due to reneging out of total customers arriving in the system (under R\_EOS) is

\[ \frac{\text{Avgrr}_{(R\_EOS)}}{X} \]

\[ = (1 - K_{R\_EOS}) + \left( \frac{q_0 K_{R\_EOS} (p \lambda \mu \nu)}{p^{k+1} q_0 (p \lambda \mu \nu)} \right) - \left( \frac{1}{\lambda} \right) \left( \mu \sum_{n=1}^{k} nq_n + k\mu(K_{R\_EOS} - q_k) \right). \]

Customers are lost to the system in two ways, due to balking and due to reneging. Management would like to know the proportion of total customers lost in order to have an idea of total business lost.

Hence the mean rate at which customers are lost (under R\_BOS) is

\[ \lambda - X_{(R\_BOS)} + \text{Avgrr}_{(R\_BOS)} \]

\[ = \lambda - \mu \sum_{n=1}^{k} nq_n - k\mu(K_{R\_BOS} - p_k). \]

and the mean rate at which customers are lost (under R\_EOS) is

\[ \lambda - X_{(R\_EOS)} + \text{Avgrr}_{(R\_EOS)} \]

\[ = \lambda - \mu \sum_{n=1}^{k} nq_n - k\mu(K_{R\_EOS} - q_k). \]
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These rates help in the determination of proportion of customers lost which is of interest to the system manager as also an important measure of business lost. This proportion (under $R_{BOS}$) is given by

\[
\left\{ \lambda - \lambda'(R_{BOS}) + Avgrr(R_{BOS}) \right\}/\lambda = 1 - \left(1/\lambda \right) \left[ \mu \sum_{n=1}^{k} n p_n + k \mu (K_{R_{BOS}} - p_k) \right].
\]

and the proportion (under $R_{EOS}$) is given by

\[
\left\{ \lambda - \lambda'(R_{EOS}) + Avgrr(R_{EOS}) \right\}/\lambda = 1 - \left(1/\lambda \right) \left[ \mu \sum_{n=1}^{k} n q_n + k \mu (K_{R_{EOS}} - q_k) \right].
\]

The proportion of customer completing receipt of service can now be easily determined from the above proportion.

The customers who leave the system from the queue do not receive service. Consequently, only those customers who reach the service station constitute the actual load of the server. From the server's point of view, this provides a measure of the amount of work the server has to do. Let us call the rate at which customers reach the service station as $\lambda^s$. Then under $R_{BOS}$

\[
\lambda^s(R_{BOS}) = \lambda^s(R_{BOS})(1 - \text{proportion of customers lost due to reneging out of those joining the system})
\]

\[
= \lambda'(R_{BOS}) \left\{ 1 - \sum_{n=k+1}^{\infty} (n-k) p_n / \lambda'(R_{BOS}) \right\}
\]

\[
= \lambda'(R_{BOS}) - Avgrr(R_{BOS})
\]

\[
= \mu \sum_{n=1}^{k} n p_n + k \mu (K_{R_{BOS}} - p_k).
\]
In case of R_EOS, we know that customers may renege even while being served and only those customers who renege from the queue will not constitute any work for the server. Thus

\[ \lambda^e (R_{E_{OS}}) = \lambda^e (R_{E_{OS}}) (1 - \text{proportion of customers lost due to reneging from the queue out of those joining the system}) \]

\[ = X_{(R_{E_{OS}})} \left\{ 1 - \sum_{n=1}^{\infty} (n-k)q_n / X_{(R_{E_{OS}})} \right\} \]

\[ = X_{(R_{E_{OS}})} - \nu \left\{ Q'(1) - \sum_{n=1}^{\infty} nq_n \right\} + k \nu \left\{ 1 - \sum_{n=0}^{\infty} q_n \right\} \]

\[ = X_{(R_{E_{OS}})} - \nu Q'(1) + \nu \sum_{n=1}^{\infty} nq_n + k \nu (K_{R_{E_{OS}} - q_k}) \]

\[ = (\mu + \nu) \sum_{n=1}^{\infty} nq_n + k(\mu + \nu)(K_{R_{E_{OS}} - q_k}) \]

In order to ensure that the system is in steady state, it is necessary for the rate of customers reaching the service station to be less than the system capacity. This translates to

\[ (\lambda^e / k \mu) < 1. \]

3.4.3. Sensitivity Analysis:

Here also we have assumed three parameters viz: \( \lambda, \mu \) and \( \nu \) relating to the stochastic nature of arrival, service and reneging patterns as we assumed in section 3.3.3.

Let \( p_n (\lambda, \mu, \nu) \) and \( q_n (\lambda, \mu, \nu) \) will denote the probability that there are 'n' customers in a system with parameters \( \lambda, \mu, \nu \) in steady state under R_BOS and R_EOS respectively.
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i) If \( \lambda_i > \lambda_0 \), then

\[
\frac{p_0(\lambda_i, \mu, v)}{p_0(\lambda_0, \mu, v)} < 1
\]

\[
\Rightarrow \frac{(\lambda_0 - \lambda_i)}{\mu} + \frac{(\lambda_0^\mu - \lambda_i^\mu)}{k!\mu^k} + \frac{p(\lambda_0^\mu - \lambda_i^\mu)}{k!\mu^k(k\mu + v)} + \ldots < 0
\]

which is true. Hence \( p_0 \downarrow \) as \( \lambda \uparrow \).

ii) If \( \mu_i > \mu_0 \), then

\[
\frac{p_0(\lambda, \mu_i, v)}{p_0(\lambda, \mu_0, v)} > 1
\]

\[
\Rightarrow \lambda \left( \frac{1}{\mu_0} - \frac{1}{\mu_i} \right) + \frac{\lambda^k}{k!} \left( \frac{1}{\mu_0^k} - \frac{1}{\mu_i^k} \right) + \frac{\lambda^{k+1}}{k!} \left( \frac{1}{\mu_0^k(k\mu_0 + v)} - \frac{1}{\mu_i^k(k\mu_i + v)} \right) + \ldots > 0
\]

which is true. Hence \( p_0 \uparrow \) as \( \mu \uparrow \).

iii) If \( v_i > v_0 \), then

\[
\frac{p_0(\lambda, \mu, v_i)}{p_0(\lambda, \mu, v_0)} > 1
\]

\[
\Rightarrow \lambda^{k+1} \left( \frac{1}{(k\mu + v_0)} - \frac{1}{(k\mu + v_i)} \right) + \lambda^{k+2} \left( \frac{1}{(k\mu + v_0)(k\mu + 2v_0)} - \frac{1}{(k\mu + v_i)(k\mu + 2v_i)} \right) + \ldots > 0
\]

which is true. Hence \( p_0 \uparrow \) as \( v \uparrow \).

The following can similarly be shown.

iv) \( q_0 \downarrow \) as \( \lambda \uparrow \)

v) \( q_0 \uparrow \) as \( \mu \uparrow \)

vi) \( q_0 \uparrow \) as \( v \uparrow \)
These results state that an increase in arrival rate would result in lowering of the fraction of time the server is idle. An increase in service rate would mean the server is able to work efficiently so that it can process same amount of work quickly. This translates to higher server idle time. An increase in reneging rate would mean the server has fewer work to do and hence higher fraction of idle time.

3.4.4. Numerical Example:

To illustrate the use of our results, we apply them to a queuing problem. We quote below an example from Allen (2005, page 352).

'Customers arrive randomly (during the evening hours) at the Kittenhouse, the local house of questionable services, at an average rate of five per hour. Service time is exponential with a mean of 20 minutes per customer. There are two servers on duty.

So many queuing theory students visits the Kittenhouse to collect data for this book that proprietress, Kitty Callay (also known as the Cheshire Cat) make some changes. She trains her kittens to provide more exotic but still exponentially distributed service and add three more servers, for a total of five. Her captivated, customers still complain that the queue is too long. Kitty commissions her most favoured customer Gralre K. Renga to make a study of her establishment. He is to determine the..., the number of servers she should provide so that..., the probability that an arriving customers must wait for service will not exceed 0.25.'

This is a design problem. Here \( \lambda = 5/hr \) and \( \mu = 3/hr \). As required by the owner of the Kittenhouse, we examine the minimum number of servers with different choices of \( k \). Though not explicitly mentioned, it is necessary to assume reneging and balking.
Let us consider the possible Markovian reneging rates of \( v = 0.5 \)\text{/hr} and \( v = 0.25 \)\text{/hr}. We further assume that balking rate is dependent of state and are 0.1 and 0.95.

Various performance measures of interest computed are given in the following tables. These measures were arrived at using a FORTRAN 77 program coded by the authors. Different choices of \( k \) were considered. Results relevant with regard to the requirement that the Kittenhouse should provide servers so that the probability that an arriving customers will find all servers busy should be <0.25 are presented in Table 3.2 and Table 3.3. (All rates in the table as per hour rates).

**Table 3.2: Table of Performance Measures (with \( \lambda = 5, \mu = 3, v = 0.5 \) and \( p = 0.9 \))**

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Number of servers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( k = 2 )</td>
</tr>
<tr>
<td>( \sum_{n=1}^{\infty} p_n )</td>
<td>0.55616</td>
</tr>
<tr>
<td>( \lambda^2 ) (i.e. arrival rate of customers reaching service station)</td>
<td>4.16915</td>
</tr>
<tr>
<td>Effective mean arrival rate(( \lambda^2 ))</td>
<td>4.47006</td>
</tr>
<tr>
<td>Fraction of time server is idle (( p_0 ))</td>
<td>0.16644</td>
</tr>
<tr>
<td>Average length of queue</td>
<td>0.60181</td>
</tr>
<tr>
<td>Average length of system</td>
<td>1.99152</td>
</tr>
<tr>
<td>Mean reneging rate</td>
<td>0.30090</td>
</tr>
<tr>
<td>Average balking rate</td>
<td>0.10599</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Number of servers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean rate of customers lost</td>
<td>0.83085 0.28029 0.08599</td>
</tr>
<tr>
<td>Proportion of customers lost due to reneging, and balking.</td>
<td>0.16615 0.05606 0.01719</td>
</tr>
</tbody>
</table>

Table 3.3: Table of Performance Measures (with $\lambda=5$, $\mu=3$, $v=0.25$ and $p=0.95$)

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Number of servers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{n=k+i} P_n$</td>
<td>0.61060 0.26891 0.09605</td>
</tr>
<tr>
<td>$\lambda^x$ (i.e. arrival rate of customers reaching service station)</td>
<td>4.39373 4.82377 4.94990</td>
</tr>
<tr>
<td>Effective mean arrival rate($\lambda^x$)</td>
<td>4.63282 4.88061 4.96344</td>
</tr>
<tr>
<td>Fraction of time server is idle ($p_0$)</td>
<td>0.14602 0.18027 0.18726</td>
</tr>
<tr>
<td>Average length of queue</td>
<td>0.95638 0.22735 0.05414</td>
</tr>
<tr>
<td>Average length of system</td>
<td>2.42095 1.83527 1.70411</td>
</tr>
<tr>
<td>Mean reneging rate</td>
<td>0.23909 0.05684 0.01354</td>
</tr>
<tr>
<td>Average balking rate</td>
<td>0.07344 0.02388 0.00731</td>
</tr>
<tr>
<td>Mean rate of customers lost</td>
<td>0.60627 0.17623 0.05009</td>
</tr>
<tr>
<td>Proportion of customers lost due to reneging, and balking.</td>
<td>0.12125 0.03525 0.01002</td>
</tr>
</tbody>
</table>
Chapter 3

From the above tables it is clear that an ideal choice of k could be k=4 with \( \sum_{n=k+1}^\infty P_n = 0.09145 \) (from Table 3.2) and \( \sum_{n=k+1}^\infty P_n = 0.09605 \) (from Table 3.3). Under the assumption of balking and reneging, it appears that the proprietress need not increase the number of servers to five. Her design requirement would be met with four servers. She may therefore increase the number of servers by two.

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