Chapter 6
Chapter 6

ANALYSIS OF IMPATIENCE IN AN ERLANGIAN SERVICE MODEL

6.1. Introduction:

In this chapter, we discuss the analysis of a single server finite buffer queuing system having Erlangian service discipline with the additional restriction that customers may renege. "The Erlangian distribution can be used to model service times with a low coefficient of variation (less than one), but it can also arise naturally. For instance, if a job has to pass, stage by stage, through a series of 'k' independent production stages, where each stage takes a exponentially distributed time" (www.cs.duke.edu.com.). To the best of our knowledge, analysis of a queuing system with Erlangian service discipline with the concept of reneging has not been considered in queuing literature. A similar work was carried out by Shawky (2005) and El-Paoumy and Ismail (2009). In both of these papers, closed form expressions of performance measures were not available. This formed the motivation of our work. We provide closed form expressions of the performance measures of M/E_k/1/1 model in this chapter.

6.2. Assumptions:

In this chapter, we analyze M/E_k/1/1 model with the assumptions that customers may renege. The assumptions of this model are:

1) Customers arriving into the system follow Markovian law with rate $\lambda$. 

183
2) The service distribution is assumed to be Erlangian with mean \( \mu \) and stage \( k \). The Erlang type \( k \) distribution is made up of \( k \) independent and identical stages each with mean \( (1/k\mu) \).

3) There is only one server who offers service in \( k \)-stages.

4) Each customer individually has patience or reneging distribution following \( \exp (\nu) \).

5) System capacity is restricted to one customer only.

6) Service discipline is considered as FIFO.

7) The size of calling population is infinite.

Since the number of servers is equal to the system capacity, a customer arriving at it either goes straight into the service or is turned away without service (as there is no waiting space) if the server is busy. As waiting is not allowed, balking is not possible in this model. As regards reneging, it is natural that the reneging rule in the queuing model can only be of \( \text{R}_\text{EOS} \) type. Here we assume that each customer individually has patience or reneging distribution following \( \exp (\nu) \) which commences at the instant the customer joins the system.

6.3. The System States Analysis.

Let \( p_n(t) \) denote the probability that there are ‘\( n \)’ phases in the system at time ‘\( t \)’ under \( \text{R}_\text{EOS} \). Then we can have 0 phase at time \( t+\Delta t \) in the following mutually exclusive ways:

1) 0 phase at time ‘\( t \)’, no arrival, no service and no customer leaving the system during next \( \Delta t \). The probability is
Chapter 6

\[ p_0(t)\{1 - \lambda \Delta t + 0(\Delta t)\} \] (6.3.1)

2) 1 phase at time 't', no arrival, one service and no customer leaving the system during next \( \Delta t \) and another possibility is 1 phase at time 't', no arrival, no service and one customer leaving the system during next \( \Delta t \). Thus the probability is

\[ p_1(t)\{k \mu \Delta t + 0(\Delta t)\} \{1 - \nu \Delta t + 0(\Delta t)\} + p_1(t)\{1 - k \mu \Delta t + 0(\Delta t)\} \{\nu \Delta t + 0(\Delta t)\} \]

= \[ p_1(t)\{k \mu \Delta t + \nu \Delta t + 0(\Delta t)\} \] (6.3.2)

3) 2 phases at time 't', no arrival, no service and one customer leaving the system during next \( \Delta t \). The probability is

\[ = p_2(t)\{1 - k \mu \Delta t + 0(\Delta t)\} \{\nu \Delta t + 0(\Delta t)\} \]

= \[ p_2(t)\{\nu \Delta t + 0(\Delta t)\} \] (6.3.3)

And so on. Similarly,

4) \( k \) phases at time 't', no arrival, no service and one customer leaving the system during next \( \Delta t \). The probability is

\[ = p_k(t)\{1 - k \mu \Delta t + 0(\Delta t)\} \{\nu \Delta t + 0(\Delta t)\} \]

= \[ p_k(t)\{\nu \Delta t + 0(\Delta t)\} \] (6.3.4)

From (6.3.1), (6.3.2), (6.3.3) and (6.3.4) we have,

\[ p_0(t + \Delta t) = p_0(t)\{1 - \lambda \Delta t + 0(\Delta t)\} + p_1(t)\{k \mu \Delta t + \nu \Delta t + 0(\Delta t)\} + p_2(t)\{k \mu \Delta t + \nu \Delta t + 0(\Delta t)\} + \ldots +
\]

\[ p_k(t)\{k \mu \Delta t + \nu \Delta t + 0(\Delta t)\} \]

\[ \Rightarrow p_0(t + \Delta t) - p_0(t) = -\lambda \Delta t p_0(t) + k \mu \Delta t p_1(t) + \nu \sum_{i=1}^{k} p_i(t)0(\Delta t) \]
Now dividing both sides of this equation by $\Delta t$ and taking limit $\Delta t \to 0$, we get

$$p_0' (t) = -\lambda p_0 (t) + k\mu p_n(t) + \sum_{i=1}^{k} p_i(t) \quad (6.3.5)$$

There can be $n$ phases where $0 < n < k$, $n = 1, 2, ..., k - 1$ at time $t + \Delta t$ in the following mutually exclusive ways.

1) There are $n$ phases at time $t$; there is no arrival, no service and no customer leaving the system during next $\Delta t$. The probability is

$$p_n(t) \left[ 1 - k\mu\Delta t + O(\Delta t) \right] \left[ 1 - \nu\Delta t + O(\Delta t) \right]$$

$$= p_n(t) \left[ 1 - k\mu\Delta t - \lambda(1 - p)\Delta t + O(\Delta t) \right] \left[ 1 - \nu\Delta t + O(\Delta t) \right] \quad (6.3.6)$$

$$= p_n(t) \left[ 1 - \nu\Delta t - k\mu\Delta t + O(\Delta t) \right]$$

2) There are $(n+1)$ phases at time $t$; no arrival, one service and no customer leaving the system during next $\Delta t$. The probability is

$$p_{n+1}(t) \left[ k\mu\Delta t + O(\Delta t) \right] \left[ 1 - \nu\Delta t + O(\Delta t) \right]$$

$$= p_{n+1}(t) \left[ k\mu\Delta t + O(\Delta t) \right] \left[ 1 - \nu\Delta t + O(\Delta t) \right] \quad (6.3.7)$$

$$= p_{n+1}(t) \left[ k\mu\Delta t + O(\Delta t) \right]$$

From (6.3.6) and (6.3.7) we have,

$$p_n(t + \Delta t) = p_n(t) \left[ 1 - \nu\Delta t - k\mu\Delta t + O(\Delta t) \right] + \left[ k\mu\Delta t + O(\Delta t) \right] p_{n+1}(t)$$

$$\Rightarrow p_n(t + \Delta t) - p_n(t) = -[\nu + k\mu] \Delta t p_n(t) + k\mu \Delta t p_{n+1}(t) + O(\Delta t)$$

Dividing both sides of this equation by $\Delta t$ and taking limit $\Delta t \to 0$, we get

$$p_n'(t) = -[\nu + k\mu] p_n(t) + k\mu p_{n+1}(t) \quad ; \quad 0 < n < k, \quad n = 1, 2, ..., k \quad (6.3.8)$$
Chapter 6

There can be \( k \) phases at time \( t+\Delta t \) in the following mutually exclusive ways.

1) There are 0 phase at time 't'; one arrival, no service and no customer leaving the system during next \( \Delta t \). The probability is

\[
p_0(t)\{\lambda \Delta t + 0(\Delta t)\}
\]

(6.3.9)

2) There are \( k \) phases at time 't', no arrival, no service and no customer leaving the system during next \( \Delta t \). The probability is

\[
p_k(t)\{1 - k\mu \Delta t + 0(\Delta t)\} \{1 - \nu \Delta t + 0(\Delta t)\}
\]

\[
= p_k(t)\{1 - k\mu \Delta t - \nu \Delta t + 0(\Delta t)\}
\]

(6.3.10)

From (6.3.9) and (6.3.10) we have,

\[
p_k(t + \Delta t) = p_0(t)\{\lambda \Delta t + 0(\Delta t)\} + p_k(t)\{1 - k\mu \Delta t - 0(\Delta t)\}
\]

\[
\Rightarrow p_k(t + \Delta t) - p_k(t) = \lambda p_0(t) - (\nu + k\mu) p_k(t) + 0(\Delta t)
\]

Dividing both sides of this equation by \( \Delta t \) and taking limit \( \Delta t \to 0 \), we get

\[
p_k'(t) = \lambda p_0(t) - (\nu + k\mu) p_k(t)
\]

(6.3.11)

Under steady state, the differential equations (6.3.5), (6.3.8) and (6.3.11) become

\[
\lambda p_0 = k\mu p_1 + \nu \sum_{i=1}^{k} p_i
\]

(6.3.12)

\[
(\nu + k\mu) p_n = k\mu p_{n+1}; n=1,2,...,k-1
\]

(6.3.13)

\[
(\nu + k\mu) p_k = \lambda p_0
\]

(6.3.14)

From (6.3.12) we have
Chapter 6

\[ p_1 = (1/k \mu)(\lambda p_0 - \nu) \sum_{i=1}^{k} p_i \]  \hspace{1cm} (6.3.15)

Now multiplying both sides of equation (6.3.13) by \( z^n \) and summing over \( n \) we get,

\[ (\nu + k \mu) \sum_{n=1}^{k} p_n z^n = (k \mu / z) \sum_{n=1}^{k} p_n z^{n+s} \]  \hspace{1cm} (6.3.16)

Again multiplying both sides of equation (6.3.14) by \( z^n \) and summing over \( n \) we get,

\[ (\nu + k \mu) p_k z^k = \lambda p_0 z^k \]  \hspace{1cm} (6.3.17)

Adding (6.3.16) and (6.3.17), we get

\[ (\nu + k \mu) \sum_{n=1}^{k} p_n z^n = (k \mu / z) \sum_{n=1}^{k} p_n z^{n+s} + \lambda p_0 z^k \]
\[ \Rightarrow (\nu + k \mu) \{P(z) - p_0\} = (k \mu / z) \{P(z) - p_0 - p_1 z\} + \lambda p_0 z^k \]
\[ \Rightarrow \{ (\nu + k \mu) - (k \mu / z) \} P(z) = \{ (\nu + k \mu - (k \mu / z)) p_0 + \lambda p_0 z^k - k \mu p_1 \} \]
Using (6.3.15)
\[ \Rightarrow P(z) = [\{ \lambda z (z^k - 1) + k \mu (z - 1) \} p_0 + \nu z]/(z (\nu + k \mu) - k \mu) \]
\hspace{1cm} (6.3.18)

Differentiating (6.3.18) with respect to \( z \) we get,

\[ P'(z) = \{ z (\nu + k \mu) - k \mu \} [\lambda p_0 (k + z^{k-1}) + k \mu p_0 + \nu] - \{ \lambda (z^k - 1) + k \mu (z - 1) \} p_0 + \nu \}
\[ = [\lambda (z^k - 1) + k \mu (z - 1)] p_0 + \nu \}
\[ \Rightarrow P'(z) = (\lambda + \mu) k p_0 - k \mu) / \nu \]  \hspace{1cm} (6.3.19)

Now putting \( z = 1 \) we get,

\[ P'(1) = ((\lambda + \mu) k p_0 - k \mu) / \nu \]  \hspace{1cm} (6.3.20)

From (6.3.12), (6.3.13) and (6.3.14) we can determine the probability that there are 'n' phases in the system and it is given by
Chapter 6

\[ P_n = \frac{\lambda (k \mu)^{k-n}}{(v + k \mu)^{k-n+1}} p_0; \; n=1,2,\ldots,k \]

where \( p_0 = \left[ 1 + \sum_{n=1}^{k} \frac{\lambda (k \mu)^{k-n}}{(v + k \mu)^{k-n+1}} \right]^{-1} \) \hspace{1cm} (6.3.21)

6.4. Performance Measures:

An important measure is the mean number of customers in the system, which is usually denoted by \('L'. From (6.3.20), the mean system size is given by

\[ L = \frac{(\lambda + \mu) kp_0 - k \mu}{v} \]

Customers arrive into the system at the rate \( \lambda \). However all the customers who arrive do not join the system because of finite buffer restriction. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

\[ \lambda' = \lambda p_0 + \lambda p_1 + \cdots + \lambda p_{t-1} + 0 \cdot p_k \]

\[ = \lambda \sum_{n=0}^{k-1} p_n \]

\[ = \lambda (1 - p_k) \]

where \( p_k = \frac{\lambda}{(k \mu + v)} p_0 \)

'\( p_0 \) is given in (6.3.21)

We have assumed that each customer has a random patience time following \( \text{exp} \ (v) \). Since there is no reneging from an empty system, the reneging
rate of the system would depend on the state of the system. The average reneging rate \((\text{avg} \; \text{rr})\) is given by

\[
\text{Avg} \; \text{rr} = \sum_{n=1}^{k} V p_n = \nu (1 - p_0)
\]

In system management, customers who renege represent business lost. It is therefore of interest to determine the proportion of customers lost, both out of these joining the system as well as out of those arriving into the system. These are given below:

Proportion of customer lost due to reneging out of those arriving and joining the system is

\[
= \frac{\text{Avg} \; \text{rr}}{\lambda} = \frac{\nu (1 - p_0)}{\lambda (1 - p_k)}
\]

= \frac{\nu (1 - p_0)}{\lambda (1 - p_k)} \frac{(\nu + k \mu)(1 - p_0)}{(\nu + k \mu - \lambda p_0)}

Proportion of customer lost due to reneging out of total customers arriving in the system is

\[
= \frac{\text{Avg} \; \text{rr}}{\lambda} = \frac{\nu (1 - p_0)}{\lambda}
\]

In totality, customers are lost to the system in two ways, due to finite buffer and due to reneging. The management would like to know the proportion of total customers lost in order to have an idea of total business lost. Hence the mean rate at which customers are lost is
Chapter 6

Rate of loss due to finite buffer +Avgrr

\[ = \lambda - \lambda^e + Avgrr \]

\[ = (\lambda / \nu)(1 - p_0) + P_k \]

\[ = (\lambda / \nu)(1 - (k\mu p_0) / (\nu + k\mu)) \]

This rate helps in the determination of proportion of customers lost which is

\[ = (\lambda - \lambda^e + Avgrr) / \lambda \]

\[ = (1/\nu)(1 - (k\mu p_0) / (\nu + k\mu)) \]

The proportion of customers completing service is its complement.

6.5. Sensitivity Analysis:

We place below the effect of change in system parameters on server utilization. For this purpose, let us consider that \( p_n (\lambda, k\mu, \nu) \) will denote the probability that there are \( n \) phases in a system with parameters \( \lambda, k\mu, \nu \) in steady state under R_EOS. It can be shown that

1) Let \( \lambda_1 > \lambda_0 \) then

\[ \frac{p_0 (\lambda_1, k\mu, \nu)}{p_0 (\lambda_0, k\mu, \nu)} < 1 \]

\[ \Rightarrow (\lambda_0 - \lambda_1) \sum_{n=1}^k \frac{(k\mu)^{k-n}}{(\nu + k\mu)^{k-n+1}} < 0 \]

which is true and hence \( p_0 \downarrow as \lambda \uparrow. \)
2) Let $v_i > v_0$ then

\[
\frac{p_0(\lambda, k\mu, v_i)}{p_0(\lambda, k\mu, v_0)} > 1
\]

\[
= (k\mu)^{k-1}\left(\frac{1}{(k\mu + v_0)^k} - \frac{1}{(k\mu + v_i)^k}\right) + (k\mu)^{k-2}\left(\frac{1}{(k\mu + v_0)^{k-1}} - \frac{1}{(k\mu + v_i)^{k-1}}\right) + \ldots
\]

\[
(k\mu)^{k-1}\left(\frac{1}{(k\mu + v_o)} - \frac{1}{(k\mu + v_i)}\right) > 0
\]

which is true and hence $p_0 \uparrow$ as $\nu \uparrow$.

These results state that an increase in arrival rate would result in lowering of the fraction of time the server is idle. Similarly, an increase in reneging rate would mean the server has fewer work to do and hence higher fraction of idle time.

**********