Chapter 4

ESTIMATION AND TESTING OF HYPOTHESES ASPECTS OF CERTAIN NONPREEMPTIVE PRIORITY QUEUEING MODELS

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4.1 Introduction

Statistical inference, both estimation and hypothesis testing, plays an important role in the analysis of queues, particularly when a practical congestion problem is to be tackled by empirical methods. The estimation and testing of hypotheses aspects of a finite capacity priority queueing model involving general service time distributions and the estimation aspects of a multiple finite source priority queueing model are discussed in this Chapter. In this Section, a brief introduction to the content of this Chapter is furnished. In Section 4.2, the maximum likelihood estimates of the parameters involved in a finite capacity priority queueing model involving general service time distributions and state dependent hysterectic service rates are obtained. Subsequently, the precision of the maximum likelihood estimates is studied with the help of the likelihood theory for Markov processes. These results are reported in Muthu and Sampathkumar (1994a). In Section 4.3, the maximum likelihood estimates are obtained in the case of a multiple finite source priority queueing model involving general service time distributions by making use of the imbedded Markov chain approach. A particular case is discussed subsequently in this Section.

4.2 Estimation and testing of hypothesis aspects of a finite capacity priority queueing model

We now proceed to obtain the maximum likelihood estimates of the parameters involved in a finite capacity priority queueing model involving general service time distributions.

We consider here a single server queue with two priority classes. High-priority units belong to class-1 and low-priority units belong to class-2 and they are assumed to arrive in Poisson processes at rates λ_1 and λ_2. The two arrival processes are taken to be independent. The service times of successive class-1 and class-2 units form two independently distributed sequences determined by the distribution functions Fu(t) and Fv(t) with the first moments β_u^{-1} and β_v^{-1} respectively. The service of a unit from class-2 may not be interrupted once it is begun but the class-1 units are placed ahead of any other class-2 units. It is also assumed that the finite waiting room capacity is M.

Let X(t) and Y(t) be the number of high-priority and low-priority units in the system at time t and let $\{r_s\}$ be the sequence of service-completion instants. Then, $\{R_s, S_s\} = \{X(r_s+), Y(r_s+)\}$ is an imbedded Markov chain which will be ergodic and hence will possess a steady-state distribution when $\sup\{\rho_u\} + \sup\{\rho_v\} < 1$, where $\rho_u = \lambda_1/\beta_u$ and $\rho_v = \lambda_2/\beta_v$. The behaviour of the chain $\{R_s, S_s\}$ shall be determined by the transition probability matrix which is expressible in terms of p_{iju}, where p_{iju} is the probability of i high-priority units and j low-priority units arriving during the service time generated by Fu(t) and q_{ijv}, where q_{ijv} is the probability of i high-priority units and j low-priority units arriving during the service time generated by $F_v(t)$. Thus,

$$p_{iju} = \int_0^\infty (\lambda_1 t)^i (\lambda_2 t)^j \exp[-(\lambda_1+\lambda_2)t]/(i!j!)dFu(t)$$

$$q_{ijv} = \int_0^\infty (\lambda_1 t)^i (\lambda_2 t)^j \exp[-(\lambda_1+\lambda_2)t]/(i!j!)dFv(t)$$

The probabilities of the possible transitions of $\{R_s, S_s\}$ are given as

$$P[(u,v) \rightarrow (u',v')] = 0 \text{ for } u'<u-1,\ u>1, \text{ all } v,v' \quad (4.2.1)$$

$$P[(u,v) \rightarrow (u',v')] = 0 \text{ for } v'<v,\ u\geq 1, \text{ all } u' \quad (4.2.2)$$

$$P[(u,v) \rightarrow (u-1+i,v+j)] = p_{iju} \text{ for } i,j\geq 0,\ u\geq 1, \text{all } v \quad (4.2.3)$$

$$P[(0,v) \rightarrow (u',v')] = 0 \text{ for } v'<v-1, \text{ all } u' \quad (4.2.4)$$

$$P[(0,v) \rightarrow (i,v-1+j)] = q_{ijv} \text{ for } i,\ j\geq 0,\ v>0 \quad (4.2.5)$$

$$P[(0,0) \rightarrow (i,j)] = r_1 p_{ij0} + r_2 q_{ij0} \text{ for } i,j \geq 0$$

$$\text{where } r_i = \lambda_i / \sum_i \lambda_i, i=1,2 \qquad (4.2.6)$$

$$P[(u,v) \rightarrow (c,y)] = 1 - \sum_{i=0}^{y} \sum_{j=0}^{z} p_{iju}, \quad u>0 \qquad (4.2.7)$$

$$P[(0,v) \rightarrow (0,m)] = 1 - \sum_{i=0}^{m} \sum_{j=0}^{h} q_{ijv} \qquad (4.2.8)$$

$$P[(0,0) \rightarrow (0,m)] = 1 - \sum_{i=0}^{m} \sum_{j=0}^{w} (r_1 p_{ij0} + r_2 q_{ij0}) \qquad (4.2.9)$$

where c=u-1, h=M-v-i, m=M-1, w=m-i, y=M-u and z=y-v-i.

If the current state is (0,0), the queue is next observed at the end of the service period for the first arrival and so the probability of moving to the new state (u',v') depends on which type of unit was first to arrive. The single-step transition matrix is truncated at the point u'+v'=M-1, since the observation is made just after a departure.

When the imbedded Markov chain for the priority queueing system under consideration is observed until the total number of departures reaches a preassigned value, the likelihood function of the whole realization shall be expressed as given below, after ignoring the initial state's probability distribution following the approach of Basawa and Prakasa Rao (1980).

$$\log L = \sum_{u,v,u'v'} \log P[(u,v) \rightarrow (u',v')] \qquad (4.2.10)$$

Let $f_{u,v,u',v'}$ denote the number of transitions of the form $(u,v) \to (u',v')$. Then, combining the contributions to the likelihood function (4.2.10) of the transitions (4.2.1) through (4.2.9), the following form of likelihood function is obtained.

$$\begin{aligned}
\mathcal{L} = & \sum_{u=1}^{m} \sum_{v=0}^{b} \sum_{i=0}^{a} \sum_{j=0}^{n} f_{u,v,u-1+i,v+j} \log p_{iju} \\
& + \sum_{v=1}^{m} \sum_{i=0}^{m} \sum_{j=0}^{s} f_{0,v,i,v-1+j} \log q_{ijv} \\
& + \sum_{i=0}^{m} \sum_{j=0}^{w} f_{0,0,i,j} \log (r_1 p_{ij0} + r_2 q_{ij0}) \\
& + \sum_{u=1}^{m} \sum_{v=0}^{b} f_{u,v,u-1,a} \log \Big(1 - \sum_{i=0}^{a} \sum_{j=0}^{n} p_{iju}\Big) \\
& + \sum_{v=1}^{m} f_{0,v,0,m} \log \Big(1 - \sum_{i=0}^{m} \sum_{j=0}^{s} q_{ijv}\Big) \\
& + f_{0,0,0,m} \log \Big(1 - \sum_{i=0}^{m} \sum_{j=0}^{w} (r_1 p_{ij0} + r_2 q_{ij0})\Big)
\end{aligned}$$

where $b = M-u-1$, $n=s-u$, $a=M-u$, $s=M-v-i$, $m=M-1$ (4.2.11)

Suppose the service time distributions of high-priority and low-priority units are taken as

$$\begin{aligned}
F_u(t) &= 1-\exp(-\beta_1 t), \quad u \le M/2 \\
&= 1-\exp(-\beta_2 t), \quad u > M/2 \qquad (4.2.12)
\end{aligned}$$

$$\begin{aligned}
F_v(t) &= 1-\exp(-\beta_3 t), \quad v \le M/2 \\
&= 1-\exp(-\beta_4 t), \quad v > M/2 \qquad (4.2.13)
\end{aligned}$$

where β_2 and β_4 are hysterectic service rates for the high-priority and low-priority queues respectively. The values of p_{iju} and q_{ijv} are obtained as

$$p_{iju} = \lambda_1^i \lambda_2^j \beta_1 (i+j)!/(i!j!(\lambda_1+\lambda_2+\beta_1)^{i+j+1}),\ u \leq M/2$$
$$= \lambda_1^i \lambda_2^j \beta_2 (i+j)!/(i!j!(\lambda_1+\lambda_2+\beta_2)^{i+j+1}),\ u > M/2$$

$$q_{ijv} = \lambda_1^i \lambda_2^j \beta_3 (i+j)!/(i!j!(\lambda_1+\lambda_2+\beta_3)^{i+j+1}),\ v \leq M/2$$
$$= \lambda_1^i \lambda_2^j \beta_4 (i+j)!/(i!j!(\lambda_1+\lambda_2+\beta_4)^{i+j+1}),\ v > M/2$$

which yield

$$\partial p_{iju}/\partial \lambda_1 = d(i,j,\lambda_1,\lambda_2,\beta_1),\ u \leq M/2$$
$$= d(i,j,\lambda_1,\lambda_2,\beta_2),\ u > M/2$$
$$\partial q_{ijv}/\partial \lambda_1 = d(i,j,\lambda_1,\lambda_2,\beta_3),\ v \leq M/2$$
$$= d(i,j,\lambda_1,\lambda_2,\beta_4),\ v > M/2$$
$$\partial p_{iju}/\partial \beta_1 = g(i,j,\lambda_1,\lambda_2,\beta_1),\ u \leq M/2$$
$$= 0,\ u > M/2$$
$$\partial p_{iju}/\partial \beta_2 = 0,\ u \leq M/2$$
$$= g(i,j,\lambda_1,\lambda_2,\beta_2),\ u > M/2$$
$$\partial p_{iju}/\partial \lambda_2 = e(i,j,\lambda_1,\lambda_2,\beta_1),\ u \leq M/2$$
$$= e(i,j,\lambda_1,\lambda_2,\beta_2),\ u > M/2$$
$$\partial q_{ijv}/\partial \lambda_2 = e(i,j,\lambda_1,\lambda_2,\beta_3),\ v \leq M/2$$
$$= e(i,j,\lambda_1,\lambda_2,\beta_4),\ v > M/2$$

$$\partial q_{ijv}/\partial\beta_3 = g(i,j,\lambda_1,\lambda_2,\beta_3),\ v \leq M/2$$
$$= 0 \qquad\qquad ,\ v > M/2$$
$$\partial q_{ijv}/\partial\beta_4 = 0 \qquad\qquad ,\ v \leq M/2$$
$$= g(i,j,\lambda_1,\lambda_2,\beta_4),\ v > M/2$$

where

$$d(i,j,\lambda_1,\lambda_2,\beta_c) = [\lambda_2^j\ \beta_c(i+j)!\lambda_1^{i-1}/(i!j!)]$$
$$[(\lambda_1+\lambda_2+\beta_c)i-\lambda_1(i+j+1)]/(\lambda_1+\lambda_2+\beta_c)^{i+j+2}$$
$$g(i,j,\lambda_1,\lambda_2,\beta_c) = [\lambda_1^i\ \lambda_2^j(i+j)!/(i!j!)]$$
$$[(\lambda_1+\lambda_2+\beta_c)-\beta_c(i+j+1)]/(\lambda_1+\lambda_2+\beta_c)^{i+j+2}$$
$$e(i,j,\lambda_1,\lambda_2,\beta_c) = [\lambda_1^i\beta_c(i+j)!\lambda_2^{j-1}/(i!j!)]$$
$$[(\lambda_1+\lambda_2+\beta_c)j-\lambda_2(i+j+1)]/(\lambda_1+\lambda_2+\beta_c)^{i+j+2}$$

Substituting these partial derivatives in the equations

$$\partial\mathcal{L}/\partial\hat{\lambda}_e = 0\ (e=1,2)$$
$$\partial\mathcal{L}/\partial\hat{\beta}_e = 0\ (e=1,2)$$

which are based on the likelihood function provided by (4.2.11), equations in the unknowns $\hat{\lambda}_e(e=1,2)$ and $\hat{\beta}_e(e=1,2)$ shall be obtained and they shall be solved in order to obtain the maximum likelihood estimates of $\lambda_e(e=1,2)$ and $\beta_e(e=1,2)$ which will under suitable regularity conditions be asymptotically efficient for the distributions under consideration being equal to the true underlying distributions.

When the service time distributions of high-priority and low-priority units are taken as two-component exponential mixture probability distributions specified as

$$f_1(t) = \sum_{c=1}^{2} p_c\beta_c \exp(-\beta_c t), \; t \geq 0$$

$$f_2(t) = \sum_{c=3}^{4} p_c\beta_c \exp(-\beta_c t), \; t \geq 0$$

where $\sum_{c=1}^{2} p_c = 1$, $\sum_{c=3}^{4} p_c = 1$, $p_c > 0$, $c=1,2,3,4$ and $\beta_c > 0$, $c=1,2,3,4$, the values of p_{ij} and q_{ij} shall be obtained as

$$p_{ij} = [\lambda_1^i \lambda_2^j (i+j)!/(i!j!)] \sum_{c=1}^{2} p_c\beta_c/(\lambda_1+\lambda_2+\beta_c)^{i+j+1}$$

$$q_{ij} = [\lambda_1^i \lambda_2^j (i+j)!/(i!j!)] \sum_{c=1}^{2} p_{c+2}\beta_{c+2}/(\lambda_1+\lambda_2+\beta_{c+2})^{i+j+1}$$

which yield

$$\partial p_{ij}/\partial\lambda_1 = \sum_{c=1}^{2} r_c(p_c,i,j,\lambda_1,\lambda_2,\beta_c)$$

$$\partial q_{ij}/\partial\lambda_1 = \sum_{c=3}^{4} r_c(p_c,i,j,\lambda_1,\lambda_2,\beta_c)$$

$$\partial p_{ij}/\partial\beta_c = s_c(p_c,i,j,\lambda_1,\lambda_2,\beta_c), \; c=1,2$$

$$\partial q_{ij}/\partial\beta_c = s_c(p_c,i,j,\lambda_1,\lambda_2,\beta_c), \; c=3,4$$

$$\partial p_{ij}/\partial\lambda_2 = \sum_{c=1}^{2} y_c(p_c,i,j,\lambda_1,\lambda_2,\beta_c)$$

$$\partial q_{ij}/\partial\lambda_2 = \sum_{c=3}^{4} y_c(p_c,i,j,\lambda_1,\lambda_2,\beta_c)$$

where

$$r_c(p_c,i,j,\lambda_1,\lambda_2,\beta_c) = [p_c\lambda_2^j\ \beta_c(i+j)!\lambda_1{}^{i-1}/(i!j!)]$$
$$[(\lambda_1+\lambda_2+\beta_c)i-\lambda_1(i+j+1)]/$$
$$(\lambda_1+\lambda_2+\beta_c)^{i+j+2}$$

$$s_c(p_c,i,j,\lambda_1,\lambda_2,\beta_c) = [p_c\lambda_1^i\lambda_2^j(i+j)!/(i!j!)]$$
$$[(\lambda_1+\lambda_2+\beta_c)-\beta_c(i+j+1)]/$$
$$(\lambda_1+\lambda_2+\beta_c)^{i+j+2}$$

$$y_c(p_c,i,j,\lambda_1,\lambda_2,\beta_c) = [p_c\lambda_1^i\beta_c(i+j)!\lambda_2^{j-1}/(i!j!)]$$
$$[(\lambda_1+\lambda_2+\beta_c)j-\lambda_2(i+j+1)]/$$
$$(\lambda_1+\lambda_2+\beta_c)^{i+j+2}$$

When these partial derivatives are substituted in the following equations based on the log-likelihood function and the resultant equations in $\hat{\lambda}_e(e=1,2)$ and $\hat{\beta}_e(e=1,2,3,4)$ are solved, the maximum likelihood estimates of $\lambda_e(e=1,2)$ and $\beta_e(e=1,2,3,4)$ shall be obtained.

$$\partial \mathcal{L}/\partial\hat{\lambda}_e = 0 \quad , \ e=1,2$$

$$\partial \mathcal{L}/\partial\hat{\beta}_e = 0 \quad , \ e=1,2,3,4$$

When the service times of high-priority and low-priority units are assumed to have Erlang K type distributions specified as

$f_c(t) = (\beta_c k)^k \, t^{k-1} \exp(-k\beta_c t)/(k-1)!, \quad t \geq 0, \; \beta_c > 0, \; c=1,2$

the values of p_{ij} and q_{ij} shall be obtained as

$$p_{ij} = \lambda_1^i \lambda_2^j (k\beta_1)^k (i+j+k-1)!/ [i!j!(k-1)!(\lambda_1+\lambda_2+k\beta_1)^{i+j+k}]$$

$$q_{ij} = \lambda_1^i \lambda_2^j (k\beta_2)^k (i+j+k-1)!/ [i!j!(k-1)!(\lambda_1+\lambda_2+k\beta_2)^{i+j+k}]$$

Since both p_{ij} and q_{ij} are functions of $\lambda_e(e=1,2)$ and $\beta_e(e=1,2)$, the required partial derivatives of p_{ij} and q_{ij} with respect to $\lambda_e(e=1,2)$ and $\beta_e(e=1,2)$ shall be obtained and used in the following equations based on the log-likelihood function so as to obtain the maximum likelihood estimates $\hat{\lambda}_e(e=1,2)$ and $\hat{\beta}_e(e=1,2)$.

$$\partial \mathcal{L}/\partial\hat{\lambda}_e = 0 \quad (e=1,2)$$

$$\partial \mathcal{L}/\partial\hat{\beta}_e = 0 \quad (e=1,2)$$

The statistical hypotheses testing theory shall be used to test the precision of the maximum likelihood estimates obtained earlier. Suppose the hypothesis is framed that the vector $\theta = (\lambda_1, \lambda_2, \beta_1, \beta_2, \beta_3, \beta_4)$ is equal to some particular vector of interest $\theta^0 = (\lambda_1^0, \lambda_2^0, \beta_1^0, \beta_2^0, \beta_3^0, \beta_4^0)$.

When Ho: $\theta=\theta^0$ is true, using Billingsley's (1961) result, the quantity $2[\max_{\theta} \log L(\theta) - \log L(\theta^0)]$ follows chi-square distribution with two degrees of freedom. The critical region can be determined such that $Pr(\chi^2(2) \geq C) = \alpha$. For the exponential service time distributions, the quantity $2[\max_{\Theta} \log L(\theta) - \log L(\theta^0)]$ can be computed from the sample. When the computed numeric value exceeds c for $\alpha = 0.05$, the hypothesis $Ho: \theta = \theta^0$ is rejected. If Ho is false such that $\theta = \theta^1$, say, then the power of the test can be determined as a function of θ^0 and θ^1.

We now proceed to discuss some numerical aspects of the priority queueing model under discussion by simulation studies with M=7, $\lambda_1=0.12$, $\lambda_2=0.17$ and simple exponential service time distributions with parameters $\beta_1=0.33$ and $\beta_2=0.25$.

Table 4.2.1

Estimates of intensity parameters $\hat{\rho}_k$, k=1,2

N	$\hat{\rho}_1$	$\hat{\rho}_2$
382	0.2947	0.6261
468	0.3109	0.6459
576	0.3246	0.6687
769	0.3437	0.7098

Table 4.2.1 provides the maximum likelihood estimates of the intensity parameters ρ_1 and ρ_2 for various values of N of the simulated model, where N denotes the number of

observations made. This table indicates that maximum likelihood estimates approach the true values of the parameters with the increase in the number of observations over the time axis.

4.3 Estimation aspects in a multiple finite source nonpreemptive priority queueing model

We now proceed to discuss the estimation aspects in a multiple finite source priority queueing model. We consider here a priority queueing model in which there are two classes of units emanating from two independent finite sources of sizes M_1 and M_2 respectively. Each unit of class-k (k=1,2) calls for service at the service facility having a single server after spending at the source a random amount of time which is exponentially distributed with a mean of α_k^{-1}. The service times S_k of class-k (k=1,2) are independently and identically distributed with an arbitrary distribution function $F(S_k)$ with a mean of β_k^{-1}. Nonpreemptive priority service discipline is adopted by the server with class-1 units getting priority in service over the class-2 units. Within the same class, units are selected for service on the basis of their order of arrival.

An imbedded Markov chain shall be observed at the departure instants of the units. Let Y_m and Y_{m+1} denote the states of the Markov chain at the m-th and (m+1)-th departure instants. Let u_k (k=1,2) and v_k(k=1,2) denote the number of units of the k-th class in the system at the m-th and (m+1)-th departure instants. The elements of the transition probability matrix, P, corresponding to this imbedded Markov Chain are given by the conditional probabilities $P[Y_{m+1}=(v_1,v_2)/Y_m=(u_1,u_2)]$. The number of possible states of this Markov Chain is given by $[(M_1+1)(M_2+1)-1]$, which shall be numbered from 1,2,...,n. Since this finite Markov Chain is aperiodic and irreducible, it possesses a unique set of steady state probabilities.

Since the units arrive at the priority queueing system under consideration from independent finite sources after spending exponentially distributed times in the sources, the number of arrivals of units of one class to the system within a given time interval is binomially distributed, independent of the arrivals of units of the other class. The steady state probabilities shall now be described as given below.

$$P[Y_{m+1}=(v_1,v_2)/Y_m=(u_1,u_2),u_1>0,u_2,\geq 0]$$

$$=\int_0^\infty \binom{M_1-u_1}{v_1-u_1+1}\{1-\exp(-\alpha_1 t)\}^{v_1-u_1+1}\{\exp(-\alpha_1 t)\}^{M_1-v_1-1}$$

$$\binom{M_2-u_2}{v_2-u_2}\{1-\exp(-\alpha_2 t)\}^{v_2-u_2}\{\exp(-\alpha_2 t)\}^{M_2-v_2}\, dF(s_1) \tag{4.3.1}$$

$$P[Y_{m+1}=(v_1,v_2)/Y_m=(0,u_2),u_2>0]$$

$$=\int_0^\infty \binom{M_2-u_2}{v_2-u_2+1}\{1-\exp(-\alpha_2 t)\}^{v_2-u_2+1}\{\exp(-\alpha_2 t)\}^{M_2-v_2-1}$$

$$\binom{M_1}{v_1}\{1-\exp(-\alpha_1 t)\}^{v_1}\{\exp(-\alpha_1 t)\}^{M_1-v_1}\, dF(s_2)$$

(4.3.2)

$$P[Y_{m+1}=(v_1,v_2)/Y_m=(0,0)]$$

$$=\theta_1 P[Y_{m+1}=(v_1,v_2)/Y_m=(u_1,u_2),u_1=1,u_2=0]$$

$$+\theta_2 P[Y_{m+1}=(v_1,v_2)/Y_m=(0,u_2),u_2=1]$$

where $\theta_k = M_k\alpha_k / \sum_{j=1}^{2} M_j\alpha_j$, $k=1,2$ (4.3.3.)

When the imbedded Markov chain for the priority queueing system under consideration is observed until the total number of departures reaches a preassigned value, the likelihood function of the whole realization shall be obtained as given below, after ignoring the initial state's probability distribution

$$\log L = \sum_{r,s,r',s'} \log P[Y_{m+1}=(r',s')/Y_m=(r,s)] \qquad (4.3.4)$$

Let $f_{rs,r's'}$ denote the number of transitions of the form (r,s)->(r',u'). Then, combining the contributions to o the likelihood function (4.3.4) of the transitions (4.3.1) through (4.3.3), the following form of likelihood function shall be obtained.

$$\mathcal{L} = \sum_{r=1}^{M_1} \sum_{s=0}^{M_2} \sum_{r'=0}^{M_1} \sum_{s'=0}^{M_2} f_{rs,r's'} \log P[Y_{m+1}=(r',s')/ Y_m=(r,s)\ ,r>0,s\geq 0]$$

$$+ \sum_{s=1}^{M_2} \sum_{r'=0}^{M_1} \sum_{s'=0}^{M_2} f_{0s,r's'} \log P[Y_{m+1}=(r',s')/ Y_m=(0,s),s>0]$$

$$+ \sum_{r'=0}^{M_1} \sum_{s'=0}^{M_2} f_{00,r's'} P[Y_{m+1}=(r's')/Y_m=(0,0)] \qquad (4.3.5)$$

When the case of class-1 and class-2 units having the constant service rates β_1 and β_2 with the same arrival rate α is considered, the probabilities specified in (4.3.1), (4.3.2) and (4.3.3) shall be obtained as

$$P[Y_{m+1}=(v_1,v_2)/Y_m=(u_1,u_2),u_1>0,u_2\geq 0] =$$

$$[(M_1-u_1)!(M_2-u_2)!\beta_1/ \{(v_1-u_1+1)!(M_1-v_1-1)!(v_2-u_2)!(M_2-v_2)!\alpha\}]$$

$$[(v_1+v_2-u_1-u_2+1)!(M_1+M_2-v_1-v_2-2)!/(M_1+M_2-u_1-u_2)!]$$

$$P[Y_{m+1}=(v_1,v_2)/Y_m=(0\ ,u_2),u_2>0]$$

$$[(M_2-u_2)!M_1!\beta_2/\{(M_1-v_1)!v_1(v_2-u_2+1)!(M_2-v_2-1)!\alpha\}]$$

$$[(v_1+v_2-u_2+1)!(M_1-v_1+M_2-v_2-2)!/(M_2+M_2-u_2)!]$$

$$P[Y_{m+1}=(v_1,v_2)/Y_m=(0,0)]=$$

$$[\theta_1(M_1-1)!M_2\beta_1/\{(M_1-v_1-1)!v_1!v_2!(M_2-v_2)!\alpha\}]$$

$$[(v_1+v_2)!(M_1+M_2-v_1-v_2-2)!/(M_1+M_2-1)!]$$

$$+[\theta_2(M_2-1)!M_1!\beta_2/\{(M_1-v_1)!v_1!v_2(M_2-v_2-1)!\alpha\}]$$

$$[(v_1+v_2)!(M_1+M_2-v_1-v_2-2)!/(M_1+M_2-1)!]$$

Since the above probabilities are functions of the unknowns α and β_e (e=1,2), their partial derivatives with respect to α and β_e (e=1,2) shall be obtained and used in the equations

$$\partial \mathcal{L}/\partial\hat{\alpha} = 0$$

$$\partial \mathcal{L}/\partial\hat{\beta}_e = 0 \quad (e=1,2)$$

which are based on the likelihood function $\mathcal{L}$ specified in (4.3.5) so as to obtain the required likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$ (e=1.2).