Chapter 3

POINT AND INTERVAL ESTIMATION ASPECTS OF TWO MANY SERVER PRIORITY QUEUEING MODELS

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3.1 Introduction

The statistical inference aspects of priority queueing models are important from the practical point of view. For example, when the management of a service facility involving priority service mechanism, which is subject to congestion, wishes to design an efficient queueing system, it should be in a position to estimate the various parameters and performance measures of the model on the basis of the data collected at the facility. In this Chapter, the point and interval estimation problems of two many server priority queueing models with negative exponential interarrival and service time distributions are discussed. In this Section, the introduction to the content of this Chapter is provided. In Section 3.2, the maximum likelihood estimates for the parameters involved in a two server nonpreemptive priority queueing model are obtained. Subsequently, the confidence interval for the intensity parameters, two particular cases of interest and numerical results pertaining to this model are discussed in this Section. In Section 3.3, all these results are discussed in detail for the many server priority queueing model, which is an obvious remedy for a heavily congested system. These results are reported in Muthu and Sampathkumar (1993).

3.2 Maximum likelihood and interval estimation aspects of a two server nonpreemptive priority queueing model

We now proceed to introduce the notations and assumptions of the model to be discussed.

We consider here a two server nonpreemptive priority queueing model, in which each arriving unit is designated to be a member of one of the two priority classes. High-priority units belong to class-1 and low- priority units belong to class-2. High-priority and low--priority units arrive in Poisson streams at the rates λ_1 and λ_2. The service time distributions of the class-1 and class-2 units are independently and exponentially distributed with mean service rates β_1^{-1} and β_2^{-1} respectively. When a low-priority unit is in service and a high-priority unit arrives the low-priority unit is allowed to complete service but the high-priority unit is placed ahead of any other low-priority units.

It is supposed that the queue begins operation with n_o high-priority units and m_o low-priority units present. The times between transitions are negative exponentially distributed, with mean $(\lambda_1+\lambda_2)^{-1}$ when zero state is occupied, with mean $(\lambda_1+\lambda_2+\beta_1)^{-1}$ or $(\lambda_1+\lambda_2+\beta_2)^{-1}$ when zero state is not occupied and one server is busy serving a high-priority or a low-priority unit and with mean $(\lambda_1+\lambda_2+\beta_1+\beta_2)^{-1}$ or $(\lambda_1+\lambda_2+2\beta_1)^{-1}$ or $(\lambda_1+\lambda_2+2\beta_2)^{-1}$ when zero state is not occupied and both servers are busy serving high-priority and/or low-priority units. When any one of the two servers alone is busy, the queue is considered to be partially busy. The queue is considered to be completely busy when both servers are busy.

It is also assumed that the queue is in equilibrium and that the queueing system is continuously observed for a finite interval of duration, T, where T is sufficiently large to provide enough number of observations. Let T_e denote the amount of time during which the queue is empty. Let $m_e^{(i)}$ (i=0,1) denote number of class-i arrivals during the time T_e.

Let T_{f_j} (j=1,2) denote the amount of time during which the queue is partially busy with a class-j unit in service. Let $m_{f_j}^{(i)}$ (i=1,2) represent the number of class-i arrivals during the time T_{f_j} (j=1,2). Let $n_f^{(1)}$ and $n_f^{(2)}$ denote respectively the number of class-1 and class-2 departures during the times T_{f_1} and T_{f_2}.

Let T_{s_j} (j=1,2) denote the amount of time during which the queue is completely busy with two class-j units in service. Let T_{s_3} denote the amount of time during which one class-1 unit and one class-2 unit are in service. Let $m_{s_j}^{(i)}$ (i=1,2) represent the number of class-i arrivals during the time T_{s_j} (j=1,2,3). Further, let $n_{s_1}^{(1)}$ and $n_{s_1}^{(2)}$ denote the number of class-1 departures and number of class-2 departures during T_{s_1} and T_{s_2} respectively. Let $n_{s_2}^{(i)}$ (i=1,2) denote the number of class-i departures during the time T_{s_3}.

The likelihood function is made up of components which are formed from the following kinds of information.

i) When system is empty

a) Intervals of time preceeding the transitions of type (0,0) -> (1,0) and (0,0) -> (0,1).

b) A number $m_e^{(i)}$ (i=1,2) of class-i arrivals in a sequence of $\sum_i m_e^{(i)}$ Bernoulli trials.

ii) When system is partially busy

a) Intervals corresponding to transition times when there is only one class-1 unit in the queue.

b) Intervals corresponding to transition times when there is only one class-2 unit in the queue.

c) A number $m_{f_1}^{(i)}$ (i=1,2) of class-i arrivals and a number $n_f^{(1)}$ of class-1 departures in a sequence of $(\sum_i m_{f_1}^{(i)} + n_f^{(1)})$ multinomial trials.

d) A number $m_{f_2}^{(i)}$ (i=1,2) of class-i arrivals and a number $n_f^{(2)}$ of class-2 departures in a sequence of $(\sum_i m_{f_2}^{(i)} + n_f^{(2)})$ multinomial trials.

iii) When system is completely busy

a) Intervals corresponding to transition times when more than one class-1 unit are present in the queue with or without one or more class-2 units also present.

b) Intervals corresponding to transition times when more than one class-2 unit alone are in the queue.

c) Intervals corresponding to transition times when one class-1 unit and one or more class-2 units are in the queue.

d) A number $m_{s_1}^{(i)}$ (i=1,2) of class-i arrivals and a number $n_{s_1}^{(1)}$ class-1 departures in a sequence of $(\sum_i m_{s_1}^{(i)} + n_{s_1}^{(1)})$ multinomial trials.

e) A number $m_{s_2}^{(i)}$ (i=1,2) of class-i arrivals and a number $n_{s_1}^{(2)}$ class-2 departures in a sequence of $(\sum_i m_{s_1}^{(i)} + n_{s_1}^{(2)})$ multinomial trials.

f) A number $m_{s_3}^{(i)}$ (i=1,2) of class-i arrivals and a number $n_{s_2}^{(i)}$ (i=1,2) of class-i departures in a sequence of $(\sum_i m_{s_3}^{(i)} + \sum_i n_{s_2}^{(i)})$ multinomial trials.

iv) Very last interval:

The very last unended interval whose length is negative exponentially distributed with mean $(\lambda_1+\lambda_2)^{-1}$ if the number of units is zero at the end of the observation period, with mean $(\lambda_1+\lambda_2+\beta_i)^{-1}$ (i=1,2) if a class-i unit alone is in service at the end of the observation period, with mean $(\lambda_1+\lambda_2+2\beta_i)^{-1}$ (i=1,2) if two class-i units are in service and with mean $(\lambda_1+\lambda_2+\sum_i \beta_i)$ otherwise.

v) Initial state:

The probability distribution of the initial state (n_o,m_o) of the priority queueing system, $Pr(n_o,m_o)$.

When the priority queue which is completely busy is in state (n,m), (n>0, m>0), transitions of type (n,m)->(n,m-1) are not possible. The contribution to the likelihood in all such situations involves only the function of the form $(\lambda_1+\lambda_2+2\beta_1)\exp[-(\lambda_1+\lambda_2+2\beta_1)x_i]$ and this exerts its influence on the values observed during busy intervals.

Combining the contributions of all the above components to the likelihood function, the following log-likelihood function is obtained with the very last interval being properly included in either T_e or T_f or T_s, where $T_f = \sum_j T_{f_j}$ and $T_s = \sum_j T_{s_j}$.

$$\mathcal{L}(\theta) = C + \sum_i m^{(i)} \log \lambda_i + \sum_i n^{(i)} \log \beta_i - \sum_i \lambda_i T_e$$
$$- \sum_j (\sum_i \lambda_i + \beta_j) T_{f_j} - \sum_j (\sum_i \lambda_i + 2\beta_j) T_{s_j}$$
$$- (\sum_i \lambda_i + \sum_j \beta_j) T_{s_3} - \log Pr(n_o, m_o)$$

where,

$$m_f^{(i)} = \sum_j m_{f_j}^{(i)}; \quad m_s^{(i)} = \sum_j m_{s_j}^{(i)}; \quad n_s^{(i)} = \sum_j n_{s_j}^{(i)};$$
$$m^{(i)} = m_e^{(i)} + m_f^{(i)} + m_s^{(i)}; \quad n^{(i)} = n_f^{(i)} + n_s^{(i)} \text{ and}$$
$$\Theta = (\lambda_1, \lambda_2, \beta_1, \beta_2) \qquad (3.2.1)$$

Because of the assumption that the queue is in equilibrium, the effect of initial queue size on the likelihood function may be ignored, and then from (3.2.1), the maximum likelihood estimates of $\lambda_1, \lambda_2, \beta_1$ and β_2 are provided by the solution to the equations.

$$\partial\mathcal{L}/\partial\hat{\lambda}_i = 0 \quad (i=1,2)$$
$$\partial\mathcal{L}/\partial\hat{\beta}_i = 0 \quad (i=1,2)$$

Thus,

$$\hat{\beta}_i = n^{(i)}/(T_{f_i} + 2T_{s_i} + T_{s_3}), \quad (i=1,2) \qquad (3.2.2)$$
$$\hat{\lambda}_i = m^{(i)}/T, \quad (i=1,2) \qquad (3.2.3)$$

When the effect of initial state (n_o, m_o) is to be considered, the following form of maximum likelihood estimates are obtained so as to remove the effect of the difference of (n_o, m_o) from the steady-state means.

$$\hat{\beta}_1 = n^{(1)}/(T_{f_1}+2T_{s_1}+T_{s_3}) - [n_0-\rho_1/(1-\rho_1)]/(T_{f_1}+2T_{s_1}+T_{s_3})$$

$$\hat{\beta}_2 = n^{(2)}/(T_{f_2}+2T_{s_2}+T_{s_3}) - [m_0-\rho_0/(1-\rho_0)]/(T_{f_2}+2T_{s_2}+T_{s_3})$$

$$\hat{\lambda}_1 = m^{(1)}/T + [n_0-\rho_1/(1-\rho_1)]/T$$

$$\hat{\lambda}_2 = m^{(2)}/T + [m_0-\rho_0/(1-\rho_0)]/T$$

The estimate of percentage system busy time is obtained as

$$\hat{\rho}^* = [m^{(1)}n^{(2)}(T_{f_1}+2T_{s_1}+T_{s_3})+m^{(2)}n^{(1)}(T_{f_2}+2T_{s_2}+T_{s_3})] \; / \; (T\, n^{(1)}n^{(2)})$$

Assuming $\hat{\beta}_i = \hat{\beta}$, $i=1,2$, the estimate of the steady state system size is obtained as

$$\hat{L}^{(i)} = (\hat{\lambda}_i/A)(1-S_{i-1})(1-S_i) + \hat{\lambda}_i/\hat{\beta} \quad (i=1,2)$$

where

$$A = [2(2\hat{\beta}-\hat{\lambda})(1+\hat{\rho})/\hat{\rho}^{\,2}] + 2\hat{\beta}$$

$$S_i = \sum_{k=1}^{i} (\hat{\lambda}_i/2\hat{\beta})$$

$$S_o = 0 \text{ and } \hat{\rho} = \sum_i \hat{\lambda}_i/\hat{\beta}$$

We now proceed to consider two particular cases of interest. We set $\beta_1-\beta_2 = \psi$, where ψ is a known and fixed positive constant. Here, ψ is a measure of quickness with which service is offered to a high-priority class unit, as compared to the service offered to a low-priority class unit. In this case, after substituting $\beta_1 = \beta_2+\psi$ in the log-likelihood function (3.2.1), $\hat{\beta}_2$ is provided by the solution of the following equation.

$$T_1\hat{\beta}_2^{\,2} + (T_1\psi - n^{(1)} - n^{(2)})\hat{\beta}_2 - n^{(2)}\psi = 0$$

where $T_1 = T_f+T_s+T_{s_2}+T_{s_1}$

The positive real value of $\hat{\beta}_2$ should be taken in order to give the maximum log-likelihood. Further, $\hat{\beta}_1$ can be obtained from the relationship $\hat{\beta}_1 = \hat{\beta}_2+\psi$.

As another particular case, we set $\lambda_2-\lambda_1=\theta$, where θ is a known and fixed positive constant. Here, θ is a measure of quickness with which low-priority class units arrive, as compared to the rate of arrival of high-priority class units. In this case, after substituting $\lambda_2=\lambda_1+\theta$ in the log-likelihood function (3.2.1), $\hat{\lambda}_1$ is provided by the solution of the following equation.

$$2T\hat{\lambda}_1^2 + (2T\theta - m^{(1)} - m^{(2)})\hat{\lambda}_1 - m^{(1)}\theta = 0$$

The positive real value of $\hat{\lambda}_1$ should be taken in order to give the maximum log-likelihood. Further, $\hat{\lambda}_2$ can be obtained from the relationship $\hat{\lambda}_2 = \hat{\lambda}_1+\theta$.

We now proceed to obtain the interval estimates for the parameters involved in the priority queueing model under discussion by using their distribution properties. In order to make use of the estimates obtained earlier, it is necessary to know the precision of the estimates as measured by the appropriate confidence intervals.

Here, $\hat{\rho}_1 = m^{(1)} \, (T_{f_1} + 2T_{s_1})/(n^{(1)}T)$

Thus, $\hat{\rho}_1/\rho_1 = [2\beta_1(T_{f_1}+2T_{s_1}+T_{s_3})/2n^{(1)}]/(2\lambda_1 T/2m^{(1)})$

Since the arrival and service times are independent and $2\beta_1 T_{b_1}$ and $2\lambda_1 T$, where $T_{b_1} = T_{f_1}+2T_{s_1}+T_{s_3}$, have chi-square distributions with degrees of freedom $2n^{(1)}$ and $2m^{(1)}$ respectively, the ratio $(\hat{\rho}_1/\rho_1)$ has an F distribution with $2n^{(1)}$ and $2m^{(1)}$ degrees of freedom.

Thus, the following probability statement shall be made in order to get the interval estimate for ρ_1.

$$Pr[F_{1-\alpha/2}(2n^{(1)},2m^{(1)}) \le \hat{\rho}_1/\rho_1 \le F_{\alpha/2}(2n^{(1)},2m^{(1)})] = 1-\alpha$$

Hence, the $(1-\alpha)$ upper and $(1-\alpha)$ lower confidence limits for ρ_1 are obtained as

$$\hat{\rho}_1/F_{\alpha/2}(2n^{(1)},2m^{(1)}) \le \rho_1 \le \hat{\rho}_1/F_{1-\alpha/2}(2n^{(1)},2m^{(1)}) \qquad (3.2.4)$$

The $(1-\alpha)$ upper and $(1-\alpha)$ lower confidence limits for ρ_2 shall be obtained in a similar manner.

We now proceed to consider a numerical example illustrating the results discussed earlier, based on the data generated by simulation with $\lambda_1=0.14$, $\lambda_2=0.17$, $\beta_1=0.33$, $\beta_2=0.24$.

Table 3.2.1

Estimation of intensity parameters

T	$\hat{\rho}_1$	$\hat{\rho}_2$
1250	0.1518	0.3249
1600	0.1622	0.3386
2000	0.1749	0.3562
2700	0.1924	0.3817

Table 3.2.1 presents the maximum likelihood estimates of intensity parameters ρ_1 and ρ_2 for various values of T of the simulated model, where T denotes the amount of observation time. This table indicates that maximum likelihood estimates approach the true values of the parameters with increase in the amount of observation time.

3.3 Maximum likelihood and interval estimation aspects of a c-server non-preemptive priority queueing model

We now proceed to introduce the notations and assumptions of the model to be discussed.

Units arrive at a nonpreemptive priority queueing model with multiple channels in parallel and each arrival is designated to be a member of one of two priority classes. High-priority units belong to class-1 and low-priority units belong to class-2. The high-priority and low-priority units arrive in Poisson streams at the rates λ_1 and λ_2 and they have full access to the group of C servers. The service time distributions of high-priority and low-priority units are independently and exponentially distributed with mean service rates β_1^{-1} and β_2^{-1}. The low-priority unit which is already in service is allowed to complete service but the waiting high-priority unit is placed ahead of any other low-priority units.

It is supposed that the queue begins operation with n_0 priority units and m_0 non-priority units present. The times between transitions are negative exponentially distributed, with mean $(\lambda_1+\lambda_2)^{-1}$ when zero state is occupied, with mean $(\lambda_1+\lambda_2+p\beta_1+(r-p)\beta_2)^{-1}$, $(0\leq p\leq r)$, when zero state is not occupied and $r<c$ servers alone are busy and with mean $(\lambda_1+\lambda_2+p\beta_1+(c-p)\beta_2)^{-1}$, $(0\leq p\leq c)$ where zero state is not occupied and all c servers are busy. When any $r<c$ servers alone are busy, the queue is considered to be partially busy. The queue is considered to be completely busy when all the C servers are busy.

It is assumed that the queue is in equilibrium and that the queueing system is observed for a fixed, sufficiently large amount of time, T, completely extracting information from this observation period, without regard to cost of observation. The time interval T may be decomposed into three random sequences of intervals: the free intervals consisting of all times when the queue size $n(t)=0$, the partially busy intervals consisting of all times when the queue size $n(t)=r$, $(r<c)$, and the completely busy intervals consisting of all times when the queue size $n(t)\geq C$.

Let T_e denote the amount of time during which the queue is empty. Let $m_e^{(i)}$ $(i=1,2)$ denote number of class-i arrivals during the time T_e. Further, let T_{s_p} $(0\leq p\leq c)$ denote the amount of time during which the queue is completely busy with p class-1 units and $(c-p)$, $0\leq p\leq c$, class-2 units in service. Let $m_{s_p}^{(i)}$ and $n_{s_p}^{(i)}$, $(i=1,2)$ denote respectively the number of class-i arrivals and class-i departures during the time T_{s_p}. Let $m_s^{(i)}$ and $n_s^{(i)}$, $(i=1,2)$ denote the total number of class-i arrivals and class-i departures during the time $T_s = \sum_p T_{s_p}$.

Let $T_{f_{rp}}$ $(1\leq r\leq c-1,\ 0\leq p\leq r)$ denote the amount of time during which the queue is partially busy with p class-1 units and $(r-p)$, class-2 units in service. Let $m_{f_{rp}}^{(i)}$, $n_{f_{rp}}^{(i)}$, (i=1,2) denote respectively the number of class-i arrivals and number of class-i departures during the time $T_{f_{rp}}$. Let $m_f^{(i)}$ and $n_f^{(i)}$ (i=1,2) denote total number of class-i arrivals and class-i departures during the time $T_f = \sum_r \sum_p T_{f_{rp}}$.

We now proceed to obtain the maximum likelihood estimates for the parameters of the model under consideration.

The likelihood function may be constituted stepwise considering the components which are found from the following kinds of information:

i) Intervals of time preceeding the transitions of type (0,0) -> (1,0) and (0,0) -> (0,1);

ii) Intervals corresponding to transition times when p, $0 \leq p < r$, class-1 units and (r-p), $1 \leq r \leq c-1$, class-2 units are present in the queue;

iii) Intervals corresponding to transition times when p, $0 \leq p \leq c$, class-1 units and (c-p) class-2 units are present in the queue;

iv) A number $m_e^{(i)}$ (i=1,2) of class-i arrivals in a sequence of $\sum_i m_e^{(i)}$ Bernoulli trials;

v) A number $m_{f_{rp}}^{(i)}$ (i=1,2) of class-i arrivals and a number $n_{f_{rp}}^{(i)}$ (i=1,2) of class-i departures in a sequence of $(\sum_i m_{s_p}^{(i)} + \sum_i n_{s_p}^{(i)})$ multinomial trails where $0 \leq p \leq c$;

vi) The very last unended interval whose length is negative exponentially distributed with mean $(\lambda_1+\lambda_2)^{-1}$ if the number of units is zero at the end of observation period, with mean $(\lambda_1+\lambda_2+p\beta_1+(r-p)\beta_2)^{-1}$ if r servers alone are in service at the end of the observation period and with mean $(\lambda_1+\lambda_2+p\beta_1+(c-p)\beta_2)^{-1}$, $(0\leq p\leq c)$ otherwise; and

vii) The probability distribution of the initial state (n_o,m_o) of the priority queueing system $Pr(n_o,m_o)$.

When the priority queue which is completely busy is in state (n,m), n>0, m>0, transitions of type (n,m)->(n,m-1) are not possible. The contribution to the likelihood in all such situations involves only the function of the form $(\lambda_1+\lambda_2+c\beta_1).\exp[-(\lambda_1+\lambda_2+c\beta_1)x_i]$.

Since the mathematical structure of the model discussed is fully specified except for the values of the unknown parameters, a likelihood can be assigned to any sequence of events in the model during the interval (0,T). Combining the contributions of all the above components to the likelihood function, the following log-likelihood function is obtained with the very last interval being properly included in either T_e or T_f or T_s.

$$\mathcal{L}(\theta) = C + \sum_i m^{(i)} \log \lambda_i + n^{(1)} \log(p\beta_1) + n_f^{(2)} \log[(r-p)\beta_2]$$

$$+ n_s^{(2)} \log[(c-p)\beta_2] - (\lambda_1+\lambda_2)T_e$$

$$- \sum_r \sum_p (\lambda_1+\lambda_2+p\beta_1-(r-p)\beta_2)T_{f_{rp}}$$

$$- \sum_p (\lambda_1+\lambda_2+p\beta_1-(c-p)\beta_2)\, T_{s_p} + \log \Pr(n_o,m_o)$$

where $m^{(i)} = m_e^{(i)}+m_f^{(i)}+m_s^{(i)}$; $n^{(1)} = n_f^{(1)}+n_s^{(1)}$ and

$$\theta = (\lambda_1,\lambda_2,\beta_1,\beta_2) \qquad (3.3.1)$$

Because of the assumption that the queue is in equilibrium, the initial state may be ignored and then on differentiating $\mathcal{L}$ specified in (3.3.1) with respect to β_i, λ_i (i=1,2) and setting the derivatives equal to zero, after some simplification, the following equations for the maximum likelihood estimates $\hat{\beta}_i$ and $\hat{\lambda}_i$ (i=1,2) are obtained.

$$\hat{\beta}_1 = n^{(1)}/(\sum_r \sum_p pT_{f_{rp}} + \sum_p p\, T_{s_p}) \qquad (3.3.2)$$

$$\hat{\beta}_2 = n^{(2)}/(\sum_r \sum_p (r-p)\, T_{f_{rp}} + \sum_p (c-p)T_{s_p}) \qquad (3.3.3)$$

$$\hat{\lambda}_i = m^{(i)}/T, \quad i=1,2 \qquad (3.3.4)$$

When c=2, the above estimates attain the form

$$\hat{\beta}_1 = n^{(1)}/(\sum_{p=1}^{2} p\, T_{f_p} + \sum_{p=1}^{2} p\, T_{s_p})$$

$$\hat{\beta}_2 = n^{(2)}/(\sum_{p=1}^{2} (1-p) T_{f_p} + \sum_{p=1}^{2} (2-p) T_{s_p})$$

$$\hat{\lambda}_i = m^{(i)}/T, \quad i=1,2$$

which correspond to the estimates obtained earlier for the two-server case.

When the initial state (n_o, m_o) is to be considered, the maximum likelihood estimates are obtained in the following form so as to remove the effect of the difference of (n_o, m_o) from the steady state means.

$$\hat{\beta}_1 = [n^{(1)} - n_O + \rho_1/(1-\rho_1)]/(\sum_{r}\sum_{p} p\, T_{f_{rp}} + \sum_{p} p\, T_{s_p})$$

$$\hat{\beta}_2 = [n^{(2)} - m_O + \rho_2/(1-\rho_2)]/(\sum_{r}\sum_{p} (r-p)\, T_{f_{rp}} + \sum_{p} (c-p) T_{s_p})$$

$$\hat{\lambda}_1 = [m^{(1)} + n_O - \rho_1/(1-\rho_1)]/T$$

$$\hat{\lambda}_2 = [m^{(2)} + m_O - \rho_2/(1-\rho_2)]/T$$

Assuming $\beta_i = \beta$ $(i=1,\ldots,c)$, the estimate of the steady state system size for the queueing model under consideration is obtained as

$$\hat{L}^{(k)} = [(\hat{\lambda}_k/A)(1-S_{k-1})(1-S_k)] + (1/\hat{\beta}) \quad (k=1,2)$$

where

$$A = C!\ [(c\hat{\beta}-\hat{\lambda})/\hat{\rho}^c] \sum_{j=0}^{c-1} \hat{\rho}^j/j! + c\hat{\beta}$$

$$S_k = \sum_{i=1}^{k} \hat{\lambda}_i/c\hat{\beta}$$

$$S_o = 0 \text{ and } \hat{\rho} = \sum_i \hat{\lambda}_i/\hat{\beta}$$

We now proceed to discuss two particular cases of interest. We set $\beta_1 - \beta_2 = \psi$, where ψ is a known and fixed positive constant. Here, ψ is a measure of quickness with which service is offered to a low-priority unit. In this case, after substituting $\beta_1 = \beta_2 + \psi$ in the log-likelihood function (3.3.1), $\hat{\beta}_2$ is provided by the solution of the following equation.

$$T_1\hat{\beta}_2^2 + (T_1\psi - n^{(1)} - n^{(2)})\hat{\beta}_2 - n^{(2)}\psi = 0$$

$$\text{where } T_1 = \sum_r \sum_p (2p-r)\, T_{f_{rp}} + \sum_p (2p-c)\, T_{s_p} \qquad (3.3.5)$$

The positive real value of $\hat{\beta}_2$ should be taken in order to give the maximum log-likelihood. Further, $\hat{\beta}_1$ can be obtained from the relationship $\hat{\beta}_1 = \hat{\beta}_2 + \psi$.

As another particular case, we set $\lambda_2 - \lambda_1 = \theta$, where θ is a known and fixed positive constant. Here, θ is a measure of quickness with which low-priority units arrive, as compared to the rate of arrival of high-priority units. In this case, after substituting $\lambda_2 = \lambda_1 + \theta$ in the log-likelihood function (3.3.1), $\hat{\lambda}_1$ is provided by the solution of the following equation.

$$2T\hat{\lambda}_1^2 + (2T\theta - m^{(1)} - m^{(2)})\hat{\lambda}_1 - m^{(1)}\theta = 0 \qquad (3.3.6)$$

Further, $\hat{\lambda}_2$ can be obtained from the relationship $\hat{\lambda}_2 = \hat{\lambda}_1 + \theta$.

We now proceed to obtain the interval estimates for the parameters involved in the queueing model under consideration by considering their distribution properties. Here

$$\hat{\rho}_1 = m^{(1)}\left(\sum_r \sum_p \beta\, T_{f_{rp}} + \sum_p p\, T_{s_p}\right) / (n^{(1)}\, T)$$

The ratio

$$\hat{\rho}_1 / \rho_1 = [2\beta_1 (\sum_r \sum_p p\, T_{f_{rp}} + \sum_p p\, T_{s_p}) / 2n^{(1)}](2\lambda_1 T / 2m^{(1)})$$

Since the interarrival times and service times are independent and $2\beta_1 T_{b_1}$, where $T_{b_1} = \sum_r \sum_p p\, T_{f_{rp}} + \sum_p p\, T_{s_p}$, and $2\lambda_1\, T$ have chi-square distributions with degrees of freedom $2n^{(1)}$ and $2m^{(1)}$ respectively, the ratio $\hat{\rho}_1 / \rho_1$ has an F distribution. Therefore, the interval estimate for $\hat{\rho}_1$ shall be obtained from the following probability statement.

$$\Pr[F_{1-\alpha/2}(2n^{(1)}, 2m^{(1)}) \le \hat{\rho}_1 / \rho_1 \le F_{\alpha/2}(2n^{(1)}, 2m^{(1)})] = 1-\alpha$$

Thus, the 100(1-α)% interval estimate for ρ_1 is given by

$$\hat{\rho}_1 / F_{\alpha/2}(2n^{(1)}, 2m^{(1)}) \le \rho_1 \le \hat{\rho}_1 / F_{1-\alpha/2}(2n^{(1)}, 2m^{(1)})$$

The 100(1-α)% interval estimate for ρ_2 shall be obtained in a similar manner.

We now proceed to discuss some numerical aspects of the priority queueing model under discussion by simulation studies with λ_1=0.14, λ_2=0.17, β_1=0.33 and β_2=0.24 and c=4.

Table 3.3.1

Estimation of intensity parameters

T	$\hat{\rho}_1$	$\hat{\rho}_2$
1250	0.1518	0.3249
1600	0.1622	0.3386
2000	0.1749	0.3562
2700	0.1924	0.3817

The maximum likelihood estimates of intensity parameters ρ_1 and ρ_2 for various values of T of the simulated model, where T denotes the amount of observation time, are provided in Table 3.2.1. This table indicates that maximum likelihood estimates approach the true values of the parameters with increase in the amount of observation time.