Chapter 2

BASIC ANALYSES OF TWO PRIORITY QUEUEING MODELS

CHAPTER 2

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2.1 Introduction

The analysis of a particular single server feed back preemptive priority queueing model and the analysis of another priority queueing model involving alternate service pattern are carried out in this Chapter. In Section 2.2, the steady-state equations of a particular single server feed back preemptive priority queueing model are obtained by using the standard arguments. Statistical quantities such as mean, variance and covariance for the queue lengths of the model under consideration are subsequently obtained with the help of generating functions. These results are reported in Muthu and Sampathkumar (1992). In Section 2.3, another priority queueing model involving alternate service pattern is considered and its ergodicity aspects are discussed. The Laplace transforms of the waiting time distributions and the joint distribution of the number of completed busy periods and the number of units served are subsequently obtained for the model under consideration.

2.2 Basic analysis of a particular single server feed back preemptive priority queueing model

We now proceed to consider the analysis of a particular single server feed back preemptive priority queueing model.

We consider here a priority queueing model with an exponential server who serves the two priority classes at the rates β_1 and β_2 respectively. The priority-1 units arrive at the system from an outside infinite source at the rate α. As soon as the service of a priority-1 unit is over, it feeds back to the queueing system as a priority-2 unit who requires an exponentially distributed service time with mean β_2. Let $X_k(t)$ denote the number of priority-k units in the system at time t.

Let $P_{uv}(t)$ be the probability that there are u priority-1 units and v priority-2 units in the system at time t and let P_{uv} denote the corresponding steady-state probability. Let $P_{uv,rs}(\Delta t)$ be the one step transition probability that the system moves from the state (u,v) to the state (r,s) in the infinitesimal time interval of length Δt. In order to develop the governing equations of the system at time t, the possible occurences of events in the interval $(t,t+\Delta t)$ shall be postulated as follows:

$$P_{u-1v,uv}(\Delta t) = \alpha\Delta t + 0(\Delta t)$$

$$P_{u+1v,uv+1}(\Delta t) = \beta_1\Delta t + 0(\Delta t)$$

$$P_{uv+1,uv}(\Delta t) = 0 \qquad , u\geq 1, \ v\geq 0$$

$$= \beta_2\Delta t+0(\Delta t), \ u=0, \ v\geq 0$$

$$P_{u+cv,uv}(\Delta t) = 0(\Delta t),\ u\geq 0, v\geq 0, c>1$$

$$P_{0v+d,0v}(\Delta t) = 0(\Delta t),\ v\geq 0, d>1$$

$$P_{uv,u+cv}(\Delta t) = 0(\Delta t),\ u\geq 0, v\geq 0, c>1$$

$$P_{u+cv,uv+d}(\Delta t) = 0(\Delta t),\ u\geq 0, v\geq 0, c>1, d>1$$

Using the above assumptions, the governing equations of the system under consideration shall be obtained as follows:

$$P_{00}(t+\Delta t) = P_{00}(t)(1-\alpha\Delta t)+P_{01}(t)(\beta_2\Delta t)+ 0(\Delta t)$$

$$P_{0v}(t+\Delta t) = P_{0v}(t)(1-\alpha\Delta t-\beta_2\Delta t) + P_{1v-1}(t)(\beta_1\Delta t) +P_{0v+1}(\beta_2\Delta t) + 0(\Delta t)$$

$$P_{u0}(t+\Delta t) = P_{u0}(t)(1-\alpha\Delta t)-P_{u0}(t)(\beta_1\Delta t)+P_{u-10}(t)(\alpha\Delta t)+0(\Delta t)$$

$$P_{uv}(t+\Delta t) = P_{uv}(t)(1-\alpha\Delta t-\beta_1\Delta t)+P_{u+1v-1}(t)(\beta_1\Delta t) +P_{u-1v}(t)(\alpha\Delta t)+0(\Delta t)$$

Transposing, dividing through by Δt and taking limit as $\Delta t\to 0$, the difference-differential equations shall be obtained as follows:

$$\alpha P_{00}(t) = \beta_2 P_{01}(t) \quad (2.2.1)$$

$$(\alpha+\beta_2)P_{0v}(t) = \beta_1 P_{1v-1}(t) + \beta_2 P_{0v+1}(t) \quad (2.2.2)$$

$$\alpha P_{u0}(t) = -\beta_1 P_{u0}(t) + \alpha P_{u-10}(t) \quad (2.2.3)$$

$$\alpha P_{uv}(t) = -\beta_1 P_{uv}(t) + \beta_1 P_{u+1v-1}(t) + \alpha P_{u-1v}(t) \quad (2.2.4)$$

The following generating functions shall be defined in order to obtain the performance measures.

$$H_u(y,t) = \sum_{v=0}^{\infty} P_{uv}(t)\, y^u \qquad (2.2.5)$$

$$H(x,y,t) = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} P_{uv}(t)\, x^u y^v \qquad (2.2.6)$$

Multiplying both sides of (2.2.1) and (2.2.2) by y^0 and y^v respectively and summing over v, we obtain

$$\partial H_0(y,t)/\partial t = [-\alpha-\beta_2(1-(1/y))]H_0(y,t)+\beta_1 y H_1(y,t)+\beta_2(1-(1/y))\, P_{00}(t) \qquad (2.2.7)$$

Multiplying both sides of (2.2.3) and (2.2.4) by y^0 and y^v respectively and summing over v, we obtain

$$\partial H_u(y,t)/\partial t = -\alpha H_u(y,t)-\beta_1 H_u(y,t)+\alpha H_{u-1}(y,t)+\beta_1 y\, H_{u+1}(y,t) \qquad (2.2.8)$$

Then, multiplying both sides of (2.2.7) and (2.2.8) by x^0 and x^u respectively and summing over u, we obtain

$$\begin{aligned}\partial H(x,y,t)/\partial t = {} & (-\alpha+(\beta_1 y/x)+\alpha x-\beta_1)\, H(x,y,t) \\ & +(-\beta_1(y/x)+\beta_1-\beta_2+(\beta_2/y))\, H_0(y,t) \\ & +\beta_2(1-(1/y))P_{00}(t) \qquad (2.2.9)\end{aligned}$$

The equation (2.2.9) can not be solved directly. The results for the limiting state are easier to obtain than the time dependent solution which involves inversion of transformed equation. In the steady-state, the generating function shall be obtained in the following form.

$$H(x,y) = [[\beta_1((y/x)-1)+\beta_2(1-(1/y))]H_0(y)+\beta_2((1/y)-1)P_{00}]/[\alpha(x-1)+\beta_1((y/x)-1)] \quad (2.2.10)$$

The statistical quantities such as mean queue lengths, variances and covariance shall now be obtained by using (2.2.10) without solving it directly.

Differentiating (2.2.10) with respect to x and substituting x=y=1, we get

$$(\alpha-\beta_1)\ H(1,1) + \beta_1\ H_0(1) = 0$$

Thus, $H_0(1) = (\beta_1-\alpha)/\beta_1$ (2.2.11)

Differentiating (2.2.10) with respect to y and substituting x=y=1, we get

$$-(\beta_1+\beta_2)\ H_0(1) + \beta_2 P_{00}+\beta_1 = 0$$

Thus, $P_{00} = 1-(\alpha/\beta_1) - (\alpha/\beta_2)$ (2.2.12)

This result concerning the probability that the system is empty coincides with the result discussed in Saaty (1961).

The mean number of priority-1 units in the system, $E(x_1)$ shall be obtained as follows.

Differentiating (2.2.10) with respect to x twice and substituting x=y=1, we get

$$2(\alpha-\beta_1)E(x_1) - 2\beta_1\ H_0(1) + 2\beta_1 = 0$$

Thus, $E(x_1) = \alpha/(\beta_1-\alpha)$ (2.2.13)

The mean number of priority-2 units in the system, $E(x_2)$ shall be obtained as follows:

Differentiating (2.2.10) with respect to x and y and substituting x=y=1, we get

$$\beta_1 E(x_1) + (\alpha-\beta_1)E(x_2) + \beta_1 Ho(1) + \beta_1 Ho'(1) - \beta_1 = 0 \qquad (2.2.14)$$

Differentiating (2.2.10) with respect to y twice and substituting x=y=1, we get

$$2\beta_1 E(x_2) - 2(\beta_1+\beta_2)Ho'(1) + 2\beta_2 Ho(1) - 2\beta_2 Poo = 0 \qquad (2.2.15)$$

Solving (2.2.14) and (2.2.15) with the help of (2.2.11), (2.2.12) and (2.2.13), we obtain

$$E(X_2) = [\beta_1^3(\beta_1-2\alpha)+(\beta_1-\alpha)^2((\alpha-\beta_1)(\beta_2/\beta_1)+(\beta_1\beta_2-\alpha\beta_2-\alpha\beta_1)) + \beta_1(2\alpha-\beta_1)(\beta_1^2-\beta_1-\beta_2)]/[(\alpha-\beta_1)^2(\beta_1^2-\beta_1-\beta_2)] \qquad (2.2.16)$$

From the equation (2.2.14), the value of $Ho'(1)$ shall be obtained as

$$Ho'(1) = [\beta_1^3(\beta_1-2\alpha)+(\beta_1-\alpha)^2((\alpha-\beta_1)(\beta_2/\beta_1)+(\beta_1\beta_2-\alpha\beta_2-\alpha\beta_1))]/[\beta_1(\beta_1-\alpha)^2(\beta_1^2-\beta_1-\beta_2)] \qquad (2.2.17)$$

The variance of the priority-1 units shall be obtained as follows:

Differentiating (2.2.10) with respect to x thrice and substituting x=y=1, we get

$$3(\alpha-\beta_1)\ E(x_1^2) + 6\beta_1 E(x_1) + 6\beta_1\ Ho(1) - 6\beta_1 = 0$$

Thus, $E(x_1^2) = 2\alpha^2/(\beta_1-\alpha)^2$ (2.2.18)

Using (2.2.13) and (2.2.18), the variance of priority-1 units shall be obtained as

$$var(x_1) = \alpha^2/(\beta_1-\alpha)^2 \qquad (2.2.19)$$

In order to obtain the variance of priority-2 units and the covariance between the number of priority-1 and priority-2 units, the values of $Ho''(1)$ and $E(x_1x_2)$ are found.

Differentiating (2.2.10) with respect to x twice and then with respect to y and substituting x=y=1, we get

$$\beta_1 E(x_1^2) - 2\beta_1 E(x_1) + 2\beta_1 E(x_2) + 2(\alpha-\beta_1)E(x_1x_2) + 2\beta_1 Ho(1) - 2\beta_1 Ho'(1) + 2\beta_1 = 0$$

Thus,

$$E(x_1\ x_2) = [2\alpha^2\beta_1^2 - 2\alpha\beta_1^2(\beta_1-\alpha)+2\beta_1^2(\beta_1-\alpha)^2E(x_2)+2\beta_1(\beta_1-\alpha)^3 -2\beta_1^2(\beta_1-\alpha)^2Ho'(1)+2\beta_1^2(\beta_1-\alpha)^2]/[2\beta_1(\beta_1-\alpha)^3] \qquad (2.2.20)$$

Differentiating (2.2.10) with respect to x and then with respect to y twice and substituting x=y=1, we get

$$2\beta_1 E(x_1x_2)-2\beta_1 E(x_2)+(\alpha-\beta_1)E(x_2^2)+\beta_1 Ho'(1)+\beta_1 Ho''(1) = 0 \tag{2.2.21}$$

Differentiating (2.2.10) with respect to y thrice and substituting x=y=1, we get

$$3\beta_1 E(x_2^2)-3(\beta_1+\beta_2)Ho''(1)+6\beta_2 Ho'(1)-6\beta_2 Ho(1)+6\beta_2 Poo = 0 \tag{2.2.22}$$

Solving (2.2.21) and (2.2.22), the value of Ho"(1) shall be obtained as

$$Ho''(1) = [6\beta_1^2 E(x_2)-6\beta_1^2 E(x_1x_2)-(3\beta_1^2-6\beta_2(\alpha-\beta_1))Ho'(1)$$
$$-6\beta_2(\alpha-\beta_1)Ho(1)+6\beta_2(\alpha-\beta_1)Poo]/[3\beta_1^2+3(\beta_1+\beta_2)(\alpha-\beta_1)] \tag{2.2.23}$$

The value of $E(x_2^2)$ shall then be obtained as

$$E(x_2^2) = [2\beta_1 E(x_2)-2\beta_1 E(x_1x_2)-\beta_1 Ho'(1)-\beta_1 Ho''(1))]/(\alpha-\beta_1) \tag{2.2.24}$$

Using (2.2.16) and (2.2.24), the variance of the priority-2 units shall be obtained as

$$Var(x_2) = [2\beta_1 E(x_2)-2\beta_1 E(x_1x_2)-\beta_1 Ho'(1)-\beta_1 Ho''(1)$$
$$-(\alpha-\beta_1)(E(x_2))^2]/(\alpha-\beta_1) \tag{2.2.25}$$

The covariance between priority-1 units and priority-2 units shall be obtained by using (2.2.13), (2.2.16) and (2.2.20) as

$$\mathrm{cov}(x_1,x_2) = [2\alpha^2\beta_1^2-2\alpha\beta_1^2(\beta_1-\alpha)+2\beta_1^2(\beta_1-\alpha)^2E(x_2)+2\beta_1(\beta_1-\alpha)^2$$
$$-2\beta_1^2(\beta_1-\alpha)^2\ Ho'(1)+2\beta_1^2(\beta_1-\alpha)^2$$
$$-2\alpha\beta_1(\beta_1-\alpha)^2)E(x_2)]/[2\beta_1(\beta_1-\alpha)^3] \qquad (2.2.26)$$

:6)

The correlation coefficient between the number of priority-1 units and the number of priority-2 units shall be obtained as

$$r_{12} = [(\beta_1-\alpha)(2\alpha^2\beta_1^2-2\alpha\beta_1^2(\beta_1-\alpha)+2\beta_1^2(\beta_1-\alpha)^2E(x_2)+2\beta_1(\beta_1-\alpha)^2$$
$$-2\beta_1^2(\beta_1-\alpha)^2\ Ho'(1)+2\beta_1^2(\beta_1-\alpha)^2-2\alpha\beta_1(\beta_1-\alpha)^2E(x_2)]/$$
$$[2\alpha\beta_1(\beta_1-\alpha)^3\ (\mathrm{var}(x_2))^{\frac{1}{2}}] \qquad (2.2.27)$$

We now proceed to consider a particular case of the model under consideration in which both the priority-1 and priority-2 units are served at the same rate β. Substituting $\beta_i = \beta$, i=1,2 in the earlier results, the mean queue lengths and their variances in this case shall be obtained as follows:

$$Poo = (\beta-2\alpha)/\beta$$

$$E(x_1) = \alpha/(\beta-\alpha)$$

$$E(x_2) = [\beta^3(\beta-2\alpha)+(\beta-\alpha)^2(\alpha-\beta+\beta^2-2\alpha\beta)+\beta^2(\beta-2)(2\alpha-\beta)]/$$
$$[\beta(\beta-2)(\alpha-\beta)^2]$$

$$\mathrm{Var}(x_1) = \alpha^2/(\beta-\alpha)^2$$

$$\mathrm{Var}(x_2) = [2\beta E(x_2)-2\beta E(x_1x_2)-\beta Ho'(1)-\beta Ho''(1)-(\alpha-\beta)(E(x_2))^2]/$$
$$(\alpha-\beta)$$

$$\mathrm{cov}(x_1,x_2) = [2\alpha^2\beta^2-2\alpha\beta^2(\beta-\alpha)+2\beta^2(\beta-\alpha)^2\ E(x_2)+2\beta(\beta-\alpha)^3$$
$$-2\beta^2(\beta-\alpha)^2Ho'(1)+2\beta^2(\beta-\alpha)^2-2\alpha\beta(\beta-\alpha)^2E(x_2)]/$$
$$[2\beta(\beta-\alpha)^3]$$

where

$$Ho'(1) = [\beta^3(\beta-2\alpha)+(\beta-\alpha)^2(\alpha-\beta+\beta^2-2\alpha\beta)]/\beta^2(\beta-2)(\beta-\alpha)^2]$$

$$Ho''(1) = [6\beta^2E(x_2)-6\beta^2E(x_1x_2)-(3\beta^2-6\alpha\beta+6\beta^2)Ho'(1)$$
$$-6\beta(\alpha-\beta)Ho(1)-6\beta(\alpha-\beta)Poo]/(3\beta^2+6\alpha\beta-6\beta^2)$$

$$E(x_1x_2) = [2\alpha^2\beta^2-2\alpha\beta^2(\beta-\alpha)+2\beta^2(\beta-\alpha)^2\ E(x_2) + 2\beta(\beta-\alpha)^3$$
$$-2\beta^2(\beta-\alpha)^2Ho'(1) + 2\beta^2(\beta-\alpha)^2]/[2\beta(\beta-\alpha)^3]$$

We now proceed to consider the numerical results for the model under consideration for various values of the parameters α, β_1 and β_2. The mean number of priority-1 units in the system are provided in Table 2.2.1 for varying values of α and β_1 and fixed value of β_2, from which it is evident that the value of $E(x_1)$ decreases as β_1 increases. The values of $E(x_1)$ exhibit an increasing trend when the value of α increases. The mean number of priority-2 units in the system for varying values of α and β_1 are provided in Table 2.2.2, in which it shall be observed that $E(x_2)$ steadily increases as β_1 increases. The numerical results for the variance of the number of priority-1 units and

Table 2.2.1

Expected values of priority-1 queue size

α	$\beta_2 = 6$						
	$\beta_1 = 5$	$\beta_1 = 7$	$\beta_1 = 9$	$\beta_1 = 11$	$\beta_1 = 13$	$\beta_1 = 15$	$\beta_1 = 17$
0.5	0.1111	0.0769	0.0588	0.0476	0.0400	0.0345	0.0303
1.0	0.2500	0.1667	0.1250	0.1000	0.0833	0.0714	0.0625
1.5	0.4286	0.2727	0.2000	0.1579	0.1304	0.1111	0.0968
2.0	0.6667	0.4000	0.2857	0.2222	0.1818	0.1538	0.1333
2.5	1.0000	0.5556	0.3846	0.2941	0.2381	0.2000	0.1724
3.0	1.5000	0.7500	0.5000	0.3750	0.3000	0.2500	0.2143

Table 2.2.2

Expected values of priority-2 queue size

α	$\beta_2=6$						
	$\beta_1=5$	$\beta_1=7$	$\beta_1=9$	$\beta_1=11$	$\beta_1=13$	$\beta_1=15$	$\beta_1=17$
0	0.2904	0.3114	0.3528	0.4316	0.5204	0.6582	0.8217
0.5	0.3236	0.3751	0.4227	0.5108	0.6236	0.8574	1.3748
1.0	0.3714	0.4182	0.4864	0.5857	0.7486	1.0325	1.8107
1.5	0.4153	0.4682	0.5475	0.6684	0.8561	1.2108	2.2062
2.0	0.4517	0.5064	0.6072	0.7467	0.9627	1.3858	2.5627

Table 2.2.3

Variances of priority-1 queue size

α	$\beta_2=6$					
	$\beta_1=4$	$\beta_1=5$	$\beta_1=6$	$\beta_1=7$	$\beta_1=8$	$\beta_1=9$
0.8	0.0625	0.0363	0.0237	0.0167	0.0125	0.0095
1.2	0.1837	0.9970	0.0625	0.0428	0.0311	0.0237
1.6	0.4444	0.2215	0.1322	0.0878	0.0625	0.0467
2.0	1.000	0.4444	0.2500	0.1600	0.1111	0.0816
2.4	2.2500	0.8521	0.4445	0.2722	0.1837	0.1322
2.8	5.4444	1.1698	0.7656	0.4445	0.2899	0.2039

Table 2.2.4

Variances of priority-2 queue size

α	$\beta_2=6$					
	$\beta_1=4$	$\beta_1=5$	$\beta_1=6$	$\beta_1=7$	$\beta_1=8$	$\beta_1=9$
0.3	0.0382	0.0426	0.0475	0.0581	0.0674	0.1102
0.6	0.0708	0.0824	0.0986	0.1162	0.1569	0.2427
0.9	0.1104	0.1281	0.1468	0.1948	0.2626	0.4194
1.2	0.1427	0.1765	0.2181	0.2753	0.3914	0.6562
1.5	0.1916	0.2272	0.2837	0.3735	0.5457	0.0992
1.8	0.2448	0.2936	0.3642	0.4907	0.7428	1.4904

priority-2 units are provided in Table 2.2.3 and 2.2.4 The covariance here is a non-negative quantity. The corresponding values of correlation coefficient also show that the size of priority-1 queue and priority-2 queue are positively correlated. It is easy to infer the behaviour of priority-1 queue size and priority-2 queue size from the above numerical results.

2.3 Basic analysis of a priority queueing model involving alternate service pattern

We now proceed to consider the analysis of a priority queueing model involving alternate service pattern. A priority queueing system in which the single counter is manned by two servers in turn is considered here.

The units arrive in Poisson streams to the priority and nonpriority queues with means α_1^{-1} and α_2^{-1} and $\alpha = \alpha_1 + \alpha_2$.

The service times of each server are independently and identically distributed random variables with probability density function $f_i(x)$, i=1,2, having β_i^{-1}, i=1,2 as the first moments. It is assumed that all service times and all interarrival times are independent. The units are selected for service according to the nonpreemptive priority discipline, so that the service of an unit is never interrupted once it is started and the priority is considered only when a service has just been completed and the unit to be serviced next is to be determined.. As shifting of servers during a busy period results in unnecessary hardships, it is assumed that they are shifted alternately at the termination of busy periods.

Let $Z_t^{(i)}$, i=1,2 represent the number of priority-i units present in the system at time t. Let $Z_t = \sum_i Z_t^{(i)}$ be the total number of units present in the system at time t. Let r_n' denote the time instant of the n^{th} departure from the queueing system after the system starts functioning. The future state of the process $\{Z_t, t\in[0,\infty)\}$ depends only on the state of the system at time r_n', $Y_n = Z_{r_n'+}$, n=1,2,... The process $\{Y_n, n=1,2,...\}$ is an imbedded process of the Z_t process with the set of nonnegative integers as its state space. Let $Y_n^{(i)}$ be the number of priority-i units left behind in the queueing system by the n-th departing unit. Let L be an indicator variable which assumes the value i when the i-th server is busy. The one step transition probabilities p_{ij} for the Y_n process are given as

$$\begin{aligned} p_{ij} &= \Pr(A_k=j-1+i), \quad i=1,2,\ldots;\ j=i-1,i,\ldots;\ k=1,2 \\ &= 0 \quad , \ i=2,3,\ldots;\ j=0,1,\ldots;\ k=1,2 \\ &= \Pr(A_2=j) \quad , \ i=0;\ j=0,1,\ldots;\ k=1 \\ &= \Pr(A_1=j) \quad , \ i=0;\ j=0,1,\ldots;\ k=2 \end{aligned} \qquad (2.3.1)$$

where A_k (k=1,2) is the number of arrivals to the system during a service time generated by the distribution $f_k(t)$.

Since the p_{ij} values indicate that any two states of the Y_n process can be reached from each other, this process is irreducible. Since $p_{ii}>0$ for all i, all states of the Y_n process are aperiodic. Further,

$$\sum_{m=0}^{\infty} m\, p_{im} = (i-1) \sum_{m=0}^{\infty} p_{0m} + \sum_{m=1}^{\infty} m\, p_{0m}$$

$$= i + \sum_{i=1}^{2} \rho_{ki} - 1, \quad k=1,2$$

$$\sum_{m=0}^{\infty} m\, p_{0m} = \sum_{i=1}^{2} \rho_{2i}, \qquad k=1$$

$$= \sum_{i=1}^{2} \rho_{1i}, \qquad k=2$$

where $\rho_{ki} = \alpha_i/\beta_k$

The empty state of the Y_n process is a persistent non-null state if only if $\sum_{i=1}^{2} \rho_{ki} < 1$, (k=1,2). Thus, the empty state of the $\{(Y_n^{(1)}, Y_n^{(2)}),\ n=1,2,\ldots\}$ process will be a persistent non-null state if and only if $\sum_{i=1}^{2} \rho_{ki} < 1$, (k=1,2). It is thus possible to obtain an invariant distribution for the joint probability density function of $X_t^{(1)}$ and $X_t^{(2)}$.

Let U(t) and V(t) denote the number of busy periods completed during the time interval (0,t) and the number of units served during those busy periods. Let M(t) assume the value i when server i has just completed a busy period and the system is idle. Let the random variable T_i denote the length of a busy period of server i. The expected length of a busy period T_i shall be obtained by using the well-known results of the M/G/1 queueing system as

$$E(T_i) = \beta_i^{-1}/(1-\rho_i)$$

where $\rho_i = \alpha/\beta_i$

Let h(i,t,n), t>0, $n \geq 1$ be the joint probability density function of the length of a busy period of server i and the number of units served during such a busy period. The Laplace transform of h (i,t,n) is

$$\tilde{h}(i,s,n) = \int_0^\infty e^{-st} h(i,t,n)\, dt, \quad i=i,2$$

The generating function $\tilde{H}(i,s,y)$ shall be written as

$$\tilde{H}(i,s,y) = \sum_{n=1}^{\infty} y^n \tilde{h}(i,s,n), \; i=1,2; \; Re(s)>0; \; |y| \leq 1 \quad (2.3.2)$$

where y is the unique root in $|x|<1$ of the equation $x = y$ $f_i(s+\alpha-\alpha x)$.

Let the joint state probability $d(i,u,v,M(t),t)$ when a busy period is initiated by server i $(i=1,2)$ be defined as

$$d(i,u,v,M(t),t) = Pr[U(t)=u,\ V(t)=v,\ M(t)=j/Y\ (0)=i],$$
$$(i=1,2;\ j=1,2;\ u\geq 0,\ v\geq 0) \qquad (2.3.3)$$

Let the generating function $D(i,y,z,j,t)$ be written as

$$D(i,y,z,j,t) = \sum_{u=e}^{\infty} \sum_{v=e}^{\infty} y^v z^u d(i,u,v,j,t),\ (i=1,2;\ j=1,2) \qquad (2.3.4)$$

where e=1 when i=j and e=2 when $i\neq j$.

From the available results of the M/G/1 queueing system, the Laplace transform of $D(1,y,z,j,t)$ shall be written as

$$\tilde{D}(1,y,z,1,s) = z[1/(s+\alpha)]\ \tilde{H}(1,s,y)[1-z^2\{\alpha/(s+\alpha)\}^2 \tilde{H}(1,s,y)\ \tilde{H}(2,s,y)]^{-1} \qquad (2.3.5)$$

$$\tilde{D}(1,y,z,2,s) = z^2[\alpha/(s+\alpha)^2]\ \tilde{H}(1,s,y)\ \tilde{H}(2,s,y) [1-z^2\{\alpha/(s+\alpha)\}^2\ \tilde{H}(1,s,y)\ \tilde{H}(2,s,y)]^{-1} \qquad (2.3.6)$$

The expression for the Laplace transform $\tilde{D}(2,y,z,j,s)$, $(j=1,2)$ shall be written in a similar manner.

Consider the case in which the service time of server-i is exponentially distributed with the mean β_i^{-1}. In this case, from equation (2.3.2), $\tilde{H}(i,s,y)$ is obtained as

$$\tilde{H}(i,s,y) = [1/(2\alpha)]\ [(s+\alpha+\beta_i)-\sqrt{(s+\alpha+\beta_i)^2-4\alpha\beta_i y}\ ],\ i=1,2 \qquad (2.3.7)$$

Using (2.3.7) in the equations (2.3.5) and (2.3.6) and expressing the righthand sides of these two equations in series form, the coefficients of y^v and z^u, $(v \geq 0,\ u \geq 0)$, shall be extracted in order to obtain the joint distribution of the number of completed busy periods during the interval $(0,t)$ and the number of units served during those busy periods. The inverse Laplace transforms of the coefficients of y^v and z^u are given by

$$d(1,2u+1,\ 2u+1+v,\ 1,t) =$$

$$[u(u+1)\beta_1^{u+1}\ \beta_2^{u}\ \alpha^{2u+v}]/(2u!)$$
$$\sum_{e=0}^{v} (\beta_1^{e}\ \beta_2^{v-e})/[e!(v-e)!(u+e+1)!(u+v-e)!]$$
$$[r(2u+1,\alpha,t)*r(u+2e+1,\alpha+\beta_1,t)*r(u+2v-2e,\alpha+\beta_2,t)]$$
$$(v \geq 0,\ u \geq 0) \qquad (2.3.8)$$

$$d(1,2u+2,\ 2u+2+v,2,t) =$$

$$[(u+1)^2\beta_1^{u+1}\beta_2^{u+1}\alpha^{2u+v+1}]/(2u+1)!$$
$$\sum_{e=0}^{v} (\beta_1^{e}\ \beta_2^{v-e})/[e!(v-e)!(u+e+1)!(u+v-e+1)!]$$
$$[r(2u+2,\alpha,t)*r(u+2e+1,\alpha+\beta_1,t)*r(u+2v-2e+1,\alpha+\beta_2,t)],$$
$$(v \geq 0,\ u \geq 0) \qquad (2.3.9)$$

where $r(j,b,t) = t^{j-1}\exp(-bt)$

Let $B_k{*}(s)$ be the Laplace-Stieltjes transform of the service time distribution $f_k(x)$ for the i-th server. For an arriving priority-2 unit, the waiting time T is the sum of the time w required to serve n_k (k=1,2) units which it finds in the queue upon its arrival and the time w' required to serve n_1' priority-1 units who arrive during w, w' being a residual busy period starting with n_1'. Let V(t) be the virtual waiting time in a non-priority M/G/1 system with arrival rate α, whose Laplace transform is

$$V^*(s) = [s(1-\rho)]/[s-\alpha+\alpha B^*(s)] \qquad (2.3.10)$$

where $B^*(s)$ is the Laplace transform of the service time density function.

Let $G^*(s)$ be the Laplace-Stieltjes transform of the busy period of a nonpriority M/G/1 system with arrival rate α, where $G^*(s)$ is the unique solution to the functional equation

$$B^*[s+\alpha(1-y)] = y, \quad |y| < 1 \qquad (2.3.11)$$

Then, the Laplace-Stieltjes transform of the waiting time distribution of priority-2 units, $w_2^*(s)$, is obtained by using (2.3.10) and (2.3.11) with $B^*(s)$ being replaced by $B_2^*(s)$ when k-th server is busy. Thus,

$$W_2^*(s) = \int_{y=0}^{\infty} \exp(-sy) \sum_{n=0}^{\infty} [G_1^*(s)]^n (y\alpha_1)^n (1/n!) \exp(-y\alpha_1) dV(y)$$

where $G_1^*(s)$ is the unique solution to the equation

$$G_1^*(s) = B_k^*[s+\alpha_1(1-G_1^*(s))] \text{ with } |G_1^*(s)|<1 \qquad (2.3.12)$$

Thus, $W_2^*(s) = \int_0^\infty \exp(-sy)\exp[-(\alpha_1-\alpha_1 G_1^*(s))y]\, dV(y)$

Using (2.3.10) with (2.3.12),

$$W_2^*(s) = V^*[s+\alpha_1-\alpha_1 G_1^*(s)]$$

$$= [\{s+\alpha_1(1-G_1^*(s))\}(1-\rho)]/$$

$$[\{s+\alpha_1(1-G_1^*(s))\} - \alpha\{1-B^*(s+\alpha_1(1-G_1^*(s)))\}]$$

$$= [\{s+\alpha_1(1-G_1^*(s))\}(1-p)]/[\{s+\alpha_1(1-G_1^*(s))\}-\alpha(1-G_1^*(s))]$$

During the waiting time T' in which priority-1 units wait, the queue of priority-1 units acts as a non-priority M/G/1 queue with arrival rate α_1 acts during a busy period. The probability of a priority-1 unit arriving during T' or starting a period T' itself as ρ^* . Thus the Laplace transform of $W_1(t)$ is obtained as

$$w_1^*(s) = (1-\rho)+[\alpha_1\rho(1-B_k^*(s))(1-\rho_1)]/[\rho_1\{s-\alpha_1(1-B_k^*(s))\}]$$