Chapter 6

OPTIMALITY ASPECTS OF CERTAIN NONPREEMPTIVE PRIORITY QUEUEING MODELS

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6.1 Introduction

The optimality aspects of certain priority queueing models that arise in the study of inventory and reliability problems are discussed in this Chapter. In Section 6.2, the optimality of design parameters involved in a finite capacity nonpreemptive priority queueing model that arises in an inventory situation is discussed. Subsequently, in Section 6.3, the optimality aspects of a finite source nonpreemptive priority queueing model which is involved in system reliability studies are discussed. A numerical example is subsequently provided. These types of priority queueing models will be useful in analysing inventory and reliability problems involving priority queueing mechanism.

6.2 Optimality aspects of a finite capacity nonpreemptive priority queueing model

We now proceed to discuss the optimality aspects of a finite capacity nonpreemptive priority queueing model that arises in an inventory situation.

We consider here a situation in which s items constituting an inventory are rented out to the units waiting in a finite capacity queueing system on priority basis. Let the priority and nonpriority units arrive in accordance with independent Poisson processes with rates α_1 and α_2 and let $\alpha = \sum_i \alpha_i$. If an item is available for being rented out, it is rented out to the arriving unit. Otherwise, the arriving unit joins a finite capacity queue and waits to rent according to a nonpreemptive priority discipline. If a nonpriority unit finds the queue to be full, it leaves without getting the renting facility. If a priority unit finds on its arrival that the queue is full with b or more units being present in the non-priority queue it is allowed to replace a non-priority unit in order to get a berth for itself. Let the rentel times be independently and exponentially distributed with the mean β^{-1}. Let the maximum capacity of the queueing system be M, where M=s+n, n being the maximum queue length.

Let P_{uvr} be the steady state probability that there are u priority units and v non-priority units in the system with r priority units availing the rental service, where $r \leq s$ and $u \geq r$. A system of differential-difference equations shall be established for $p_{uvr}(t)$ = Pr[at time t, u priority units and v non-priority units are in the system, and r priority units get the rental service]. The steady-state difference equations shall then be obtained in the event that $\rho = \alpha/s\beta<1$.

$$p_{000}(t+\Delta t) = p_{000}(t)[1-\alpha\Delta t]+p_{101}(t)[\beta\Delta t]+p_{010}(t)[\beta\Delta t]+O(\Delta t)$$

$$= p_{000}(t)-\alpha\Delta t\ p_{000}(t)+\beta\Delta t[p_{101}(t)+p_{010}(t)]+O(\Delta t)$$

Then, $$0 = -\alpha\ p_{000}+\beta(p_{101}+p_{010}) \qquad (6.2.1)$$

Similarly,

$$0 = -(\alpha+\beta)p_{101}+2\beta\ p_{202} + \beta\ p_{111} + \alpha_1\ p_{000}$$

$$0 = -(\alpha+\beta)p_{010} + \beta\ p_{111} + 2\beta\ p_{020} + \alpha_2\ p_{000}$$

$$0 = -(\alpha+2\beta)p_{111}+2\beta p_{212}+2\beta p_{121}+\alpha_1 p_{010}+\alpha_2 p_{101}$$

The above equations shall be rewritten as

$$0 = -[\alpha+(u+v)\beta]p_{uvr}+(r+1)\beta p_{u+1vr+1}+(v+1)\beta p_{uv+1r}$$
$$+\alpha_1\ d(u-1)p_{u-1vr-1}+\alpha_2\ d(v-1)p_{uv-1r}$$
$$(0<u+v<s,\ u\geq 0,\ v<s,\ r=u)$$

where $d(y)=1,\quad y\geq 0$

$$=0,\quad y<0 \qquad (6.2.2)$$

Similarly, the remaining equations shall be obtained as follows:

$$0 = -(\alpha+su)p_{uvr} + \alpha_1 d(u-1)p_{u-1vr-1} + \alpha_2 d(v-1)p_{uv-1r}$$
$$+r\beta\ p_{u+1vr} + (s-r)\beta\ p_{uv+1r} + (s-r+1)\beta\ d(r-1)p_{uv+1r-1}$$
$$(u+v=s,\ u\geq 0,\ v<s,\ r=u)$$

$$(6.2.3)$$

$$0 = -(\alpha+s\beta)p_{uvr}+\alpha_1 d(u-1-r)p_{u-1vr}+\alpha_2 d(v-1-s+r)p_{uv-1r}$$
$$+r\beta p_{u+1vr}+(s-r)\beta\ d(r-u)p_{uv+1r}+(s-r+1)\beta\ d(u-r)d(r-1)$$
$$p_{uv+1r-1}$$
$$(s<u+v<M,\ u\geq 0,\ v<M,\ \min[g(u-v),g(s-v)]\leq r\leq \min[u,s]) \tag{6.2.4}$$

$$0 = -s\beta p_{uvr} + \alpha_1 d(u-1-r)p_{u-1vr} + \alpha_2 d(v-1-s+r)p_{uv-1r}$$
$$(u+v=M,\ u\geq 0,\ v<b,\ \min[g(u-v),g(s-v)]\leq r\leq \min[u,s]) \tag{6.2.5}$$

$$0 = -(\alpha_1+s\beta)p_{uvr} +\alpha_1 d(u-1-r)p_{u-1vr} + \alpha_2 d(v-1-s+r)p_{uv-1r}$$
$$+\alpha_1\ d(u-1-r)p_{u-1v+1r}$$
$$(u+v=M,\ u\geq 0,\ b\leq v<M,\ \min[g(u-v),g(s-v)]\leq r\leq \min[u,s])$$

where $g(y) = y,\ y>0$

$$= 0,\ y\leq 0 \tag{6.2.6}$$

Since the traffic intensity $\rho = \alpha/s\beta$ of the system is less than unity, the set of difference equations (6.2.1) to (6.2.5) will provide a genuine probability distribution as its unique solution. Since triple subscripts are involved in these equations, the steady-state probabilities shall be obtained in an indirect manner.

From equation (6.2.1), the probability that there is one unit in the system shall be obtained as

$$P_1 = P_{101} + P_{010} = (\alpha/\beta)\ P_{000} \tag{6.2.7}$$

From equation (6.2.2), the probability that the system has two units shall be obtained.

$$(\alpha+\beta)p_{101}+(\alpha+\beta)p_{010} = 2\beta p_{202}+\alpha_1 p_{000}+\beta p_{111}+\beta p_{111}+2\beta p_{020} +\alpha_2 p_{000}$$

Using equation (6.2.7) in the above equation,

$$p_2 = p_{202}+p_{111}+p_{020} = [\alpha^2/(\beta(2\beta))]p_{000} \qquad (6.2.8)$$

The probability that there are three units in the system shall be obtained from equation (6.2.3) as

$$p_3 = p_{303}+p_{030}+p_{212}+p_{121} = [\alpha^3/[\beta(2\beta)(3\beta)]]p_{000} \qquad (6.2.9)$$

The probability of s units in the system shall be obtained as

$$p_s = p_{s0s}+p_{s-11s-1}+p_{s-22s-2}+\dots+p_{1s-11}+p_{0s0}$$
$$= [\alpha^s/(\beta(2\beta)(3\beta)\dots(s\mu))]p_{000} \qquad (6.2.10)$$

The probability that there are s+1, s+2, ..., s+n-1 units in the system shall be obtained as

$$p_j = \alpha^j/[\beta(2\beta)\dots(s\beta)(s\beta)^{j-s}]p_{000},\ s<j<s+n-1 \qquad (6.2.11)$$

The probability that the system has s+n units in the system, P_{s+n}, shall be obtained as

$$p_{s+n} = [\alpha^{s+n}/(s!\beta^s(s\beta)^n)]p_{000} \qquad (6.2.12)$$

The value of P_0 is provided by the normality condition as

$$p_{000}+p_{010}+p_{101}+p_{202}+p_{111}+p_{020}+\cdots+p_{0s+n0} = 1 \qquad (6.2.13)$$

Let $\{\eta(T), T \geq 0\}$ denote an integer valued stochastic process representing the number of items out on rent at time t. Let R_u be the income per unit on rental service per unit time. Also, let a(s) denote the purchase cost for the s items initially purchased, which is usually a non-linear function of s. Further, let q(t,s) denote the overhead costs and the holding costs covering the depreciation of the s items, repairs and the idle time cost during the length of time t. The total profit, R, accrued in the interval [0,t] shall be written as

$$R = R_u \int_0^{\alpha} \eta(T)\, dT - a(s) - q(t,s) \qquad (6.2.14)$$

When expectation is taken in (6.2.14) and the integral and expectation are interchanged subsequently applying Fibini's theorem, (6.2.14) yields

$$E(R) = R_u \int_0^{t} E[\eta(T)]\, dT - a(s) - q(t,s) \qquad (6.2.15)$$

If $\eta(T)$ yields the steady state distribution p_k^* as $T \to \infty$, and further $\lim_{T \to \infty} [q(T+t,s)-q(T,s)]$ yields $q^*(t,s)$, (6.2.15) attains the following form after ignoring a(s).

$$E(R) = R_u t \sum_{k=0}^{s} k\, p_k^* - q^*(t,s) \qquad (6.2.16)$$

Using the values of $p_k(k=0,\ldots,s+n)$ obtained in equations (6.2.7) to (6.2.13) in the place of p_k^*, equation (6.2.16) yields

$$E(R) = R_u t \sum_{k=1}^{s-1} (k/k!)(\alpha/\beta)^k p_{000}$$

$$+ sR_u t[\sum_{k=s}^{s+n-1} (1/s!)(1/s^{k-s})(\alpha/\beta)^k p_{000}$$

$$+ \alpha^{s+n}(1/s!)(1/\beta^s)(1/(s\beta)^n)p_{000}]-q^*(t,s)$$

Setting $\partial E(R)/\partial s$ and $\partial E(R)/\partial\beta$ to zero and finding the roots of the resultant polynomial equations by an appropriate numerical method, the desired optimal values of s and β shall be obtained for the inventory problem under consideration.

6.3 Optimality aspects of a finite source nonpreemptive priority queueing model

We now proceed to discuss the optimality aspects of a finite source nonpreemptive priority queueing model which is involved in system reliability studies. We consider here a situation in which a repairman is assigned to maintain two batteries of machines with M_i machines in battery-i (i=1,2). Each machine in the battery-i (i=1,2) is liable to two types of stoppages, where the first type of stoppage is assigned priority in repair over the other. The repair of machines is done on nonpreemptive priority basis so that the repair of a machine is never interrupted once it is started. All working times and all repair times are mutually independent. The type-j (j=1,2) failure occurs at the rate α_j (j=1,2) in the battery-1 and at the rate θ_j (j=1,2) in the battery-2. The clearing times of type-j (j=1,2) failure are exponentially distributed with mean β_j^{-1}. Among the type-j (j=1,2) stoppages, arrivals from battery-1 are given preference in repair over the arrivals from battery-2. Let $P_{ij}(u_1,v_1,u_2,v_2)$ denote the steady-state probability that there are u_k (k=1,2) type-k stoppages from battery-1 and v_k (k=1,2) type-k stoppages from battery-2 in the system and a type-j (j=1,2) stoppage from battery-i (i=1,2) is getting service. The governing equations of the model under consideration shall be formed as follows:

$$\begin{aligned}
&[(1-d(u_1))\beta_1 + (M_1-u_1)(\alpha_1+\alpha_2) + (M_2-v_1-v_2)(\theta_1+\theta_2)]\\
&[(1-d(u_1))P_{11}(u_1,v_1,0,v_2) + d(u_1)P_{21}(0,v_1,0,u_2)]\\
&\quad = q(u_1-1)(M_1-u_1+1)\alpha_1 P_{11}(u_1-1,v_1,0,v_2)\\
&\qquad +d(u_1+1-M_1)\beta_1 P_{11}(u_1+1,v_1,0,v)\\
&\qquad +d(u_1+1-M_1)\beta_2 P_{21}(u_1,v_1,1,v_2)
\end{aligned}$$

$$\begin{aligned}
&[(1-d(v_1)]\beta_1+(M_1-u_1)(\alpha_1+\alpha_2) + (M_2-v_1-v_2)(\theta_1+\theta_2)]\\
&[(1-d(v_1)]P_{21}(u_1,v_1,0,v_2) + d(v_1)P_{11}(u_1,v_1,0,v_2)]\\
&\quad = q(v_1-1)(M_2-v_1+1)\theta_1 P_{21}(u_1,v_1-1,u_2,0)
\end{aligned}$$

$$[\beta_1+(M_1-u_1-u_2)(\alpha_1+\alpha_2) + (M_2-v_1-v_2)(\theta_1+\theta_2)]P_{11}(u_1,v_1,u_2,v_2)$$
$$= (M_1-u_1-u_2+1)\alpha_1 P_{11}(u_1-1,v_1,u_2,v_2)$$
$$+(M_1-u_1-u_2+1)\alpha_2 P_{11}(u_1,v_1,u_2-1,v_2)$$
$$+\beta_1 P_{11}(u_1+1,v_1,u_2,v_2)+\beta_2 P_{12}(u_1,v_1,u_2+1,v_2),$$
$$u_1,u_2\neq 0,M_1$$

$$[\beta_1+(M_1-u_1-u_2)(\alpha_1+\alpha_2) + (M_2-v_1-v_2)(\theta_1+\theta_2)]P_{21}(u_1,v_1,u_2,v_2)$$
$$= (M_2-v_1-v_2+1)\theta_1 P_{21}(u_1,v_1-1,u_2,v_2)$$
$$+(M_2-v_1-v_2+1)\theta_2 P_{21}(u_1,v_1,u_2,v_2-1), \quad v_1,v_2\neq 0$$

$$[(1-d(v_2))\beta_2+(M_1-u_1)(\alpha_1+\alpha_2)+(M_2-v_1-v_2)(\theta_1+\theta_2)]$$
$$[(1-d(v_2))P_{22}(u_1,v_1,0,v_2)+d(v_2)P_{11}(u_1,v_1,0,v_2)]$$
$$= q(v_1-1)(M_2-v_1+1)\theta_1 P_{22}(u_1,v_1-1,0,v_2)$$

$$[(1-d(v_1))\beta_1+(M_1-u_1-u_2)(\alpha_1+\alpha_2)+(M_2-v_1)(\theta_1+\theta_2)]$$
$$[(1-d(v_1))P_{21}(u_1,v_1,u_2,0)+d(v_1)P_{11}(u_1,0,u_2,0)]$$
$$= q(v_1-1)(M_2-v_1+1)\theta_1 P_{21}(u_1,v_1-1,u_2,0)$$

$$[\beta_2+(M_1-u_1-u_2)(\alpha_1+\alpha_2)+(M_2-v_1-v_2)(\theta_1+\theta_2)]P_{12}(u_1,v_1,u_2,v_2)$$
$$= (M_1-u_1-u_2+1)\alpha_1 P_{12}(u_1-1,v_1,u_2,v_2)$$
$$+(M_1-u_1-u_2+1)\alpha_2 P_{12}(u_1,v_1,u_2-1,v_2), \quad u_1,u_2\neq 0, M_1$$

$$[(1-d(u_1))\beta_1+(M_1-u_1-u_2)(\alpha_1+\alpha_2) + (M_2-v_1)(\theta_1+\theta_2)]$$

$$[(1-d(u_1))P_{11}(u_1,v_1,u_2,0) + d(u_1)P_{21}(0,v_1,u_2,0)]$$

$$= d(u_1+1-M_1)\beta_1\ P_{11}(u_1+1,v_1,0,v_2)$$

$$+d(u_2+1-M_1)\beta_2\ P_{12}(u_1,v_1,1,v_2)$$

$$+q(u_1-1)(M_1-u_1+1)\alpha_1\ P_{11}(u_1-1,v_1,0,v_2)$$

$$[(1-d(u_2))\beta_2 + (M_1-u_1-u_2)(\alpha_1+\alpha_2) + (M_2-v_1)(\theta_1+\theta_2)]$$

$$[(1-d(u_2))P_{12}(u_1,v_1,u_2,0) + d(u_2)P_{11}(u_1,v_1,0,0)]$$

$$= q(u_1-1)(M_1-u_1+1)\alpha_1\ P_{12}(u_1-1,\ v_1,0,v_2)$$

$$[(1-d(v_1))\beta_1+(M_1-u_2)(\alpha_1+\alpha_2) + (M_2-v_1-v_2)(\theta_1+\theta_2)]$$

$$[(1-d(v_1))P_{21}(0,v_1,u_2,v_2) + d(v_1)P_{12}(0,0,u_2,v_2)]$$

$$= q(v_1-1)(M_2-v_1+1)\theta_1 P_{21}(0,v_1-1,u_2,v_2)$$

$$+d(u_2+1-M_1)\beta_1 P_{21}(1,v_1,u_2,v_2)$$

$$+d(u_2+1-M_1)\beta_2 P_{12}(0,v_1,u_2+1,v_2)$$

$$[(1-d(v_2))\beta_2+(M_1-u_2)(\alpha_1+\alpha_2) + (M_2-v_1-v_2)(\theta_1+\theta_2)]$$

$$[(1-d(v_2))P_{22}(0,v_1,u_2,v_2) + d(v_2)\ P_{21}(0,v_1,u_2,0)]$$

$$= q(v_2-1)(M_2-v_2+1)\theta_2\ P_{22}(0,v_1,u_2,v_2-1)$$

$$[\beta_2+(M_1-u_1-u_2)(\alpha_1+\alpha_2)+(M_2-v_1-v_2)(\theta_1+\theta_2)]P_{22}(u_1,v_1,u_2,v_2)$$

$$= (M_2-v_1-v_2+1)\theta_1\ P_{22}(u_1,v_1-1,u_2,v_2)$$

$$+(M_2-v_1-v_2+1)\theta_2\ P_{22}(u_1,v_1,u_2,v_2-1),\quad v_1,v_2 \neq 0,M_2$$

$$[(1-d(u_2))\beta_2+(M_1-u_2)(\alpha_1+\alpha_2)+(M_2-v_1-v_2)(\Theta_1+\Theta_2)]$$
$$[(1-d(u_2))P_{12}(0,v_1,u_2,v_2)+d(u_2)P_{21}(0,v_1,0,v_2)]$$
$$= q(u_2-1)(M_1-u_2+1)\alpha_2\ P_{12}(0,v_1,u_2-1,v_2)$$

$$[(1-d(v_2))\beta_2+(M_1-u_1-u_2)(\alpha_1+\alpha_2)+(M_2-v_2)(\Theta_1+\Theta_2)]$$
$$[(1-d(v_2))P_{22}(u_1,0,u_2,v_2)+d(v_2)\ P_{11}(u_1,0,u_2,0)]$$
$$= q(v_2-1)(M_2-v_2+1)\Theta_2\ P_{22}(u_1,0,u_2,v_2-1)$$

$$[(1-d(u_1))\beta_1 + (M_1-u_1-u_2)(\alpha_1+\alpha_2) + (M_2-v_2)(\Theta_1+\Theta_2)]$$
$$[(1-d(u_1))P_{11}(u_1,0,u_2,v_2) + d(u_1)\ P_{12}(0,0,u_2,v_2)]$$
$$= q(u_1-1)(M_1-u_1+1)\alpha_1\ P_{11}(u_1-1,0,u_2,v_2)$$
$$+d(u_1+u_2+1-M_1)\beta_1\ P_{11}(u_1+1,0,u_2,v_2)$$
$$+d(u_1+u_2+1-M_1)\beta_2\ P_{12}(u_1,0,u_2+1,v_2)$$
$$+d(v_2+1-M_2)\beta_2\ P_{22}(u_1,0,u_2,v_2+1)$$

$$[(1-d(u_2))\beta_2+(M_1-u_1-u_2)(\alpha_1+\alpha_2) + (M_2-v_2)(\Theta_1+\Theta_2)]$$
$$[(1-d(u_2))P_{12}(u_1,0,u_2,v_2) + d(u_2)\ P_{11}(u_1,0,0,v_2)]$$
$$= q(u_2-1)(M_1-u_2+1)\alpha_2\ P_{12}(u_1,0,u_2-1,v_2)$$

where $q(x)=1,\ x\geq 0$ and $d(x) = 1,\ x\leq 0$
$=0,\ x<0 \qquad = 0,\ x>0$

For the case $\alpha_i = \Theta_i$ (i=1,2) and $\sum_i M_i = M$, the following reduced set of steady-state equations shall be obtained.

$$[(1-d(u_1))\beta_1+(M-u_1)(\alpha_1+\alpha_2)]$$
$$[(1-d(u_1))\ P_1(u_1,0) + d(u_1)P(0,0)]$$
$$= q(u_1-1)(M-u_1+1)\alpha_1\ P_1(u_1-1,0)$$
$$+d(u_1+1-M)\beta_1\ P_1(u_1+1,0) + d(u_1+1-M)\beta_2\ P_2(u_1,1) \quad (6.3.1)$$

$$[(1-d(u_2))\beta_2 + (M-u_2)(\alpha_1+\alpha_2)]$$
$$[(1-d(u_2))P_2(0,u_2) + d(u_2)\ P(0,0)]$$
$$= q(u_2-1)(M-u_2+1)\alpha_2\ P_2(0,u_2-1)$$
$$+d(u_2+1-M)\beta_1\ P_1(1,u_2) + d(u_2+1-M)\beta_2\ P_2(0,u_2+1) \quad (6.3.2)$$

$$[\beta_1+(M-u_1-u_2)(\alpha_1+\alpha_2)]\ P_1(u_1,u_2)$$
$$= (M-u_1-u_2+1)\alpha_1\ P_1(u_1-1,u_2) + (M-u_1-u_2+1)\alpha_2\ P_1(u_1,u_2-1)$$
$$+ \beta_1\ P_1(u_1+1,u_2) + \beta_2\ P_2(u_1,u_2+1),$$
$$u_1,u_2 \neq 0,M \quad (6.3.3)$$

$$[\beta_2+(M-u_1-u_2)(\alpha_1+\alpha_2)]\ P_2(u_1,u_2)$$
$$= (M-u_1-u_2+1)\alpha_1\ P_2(u_1-1,u_2) + (M-u_1-u_2+1)\alpha_2\ P_2(u_1,u_2-1)$$
$$u_1,u_2 \neq 0,M \quad (6.3.4)$$

where $d(x) = 1,\ x \leq 0$
$\qquad\quad = 0,\ x > 0$

and $\quad q(x) = 1,\ x \geq 0$
$\qquad\quad = 0,\ x < 0$

and $P(0,0) + \sum_{u_1=1}^{M} \sum_{u_2=0}^{M-u_1} P_1(u_1,u_2) + \sum_{u_2=1}^{M} \sum_{u_1=0}^{M-u_2} P_2(u_1,u_2)=1$

Let us consider the following generating functions:

$$H_{u_1 1}(y) = \sum_{u_2} P_1(u_1,u_2)\, y^{u_2}$$

$$H_{u_1 2}(y) = \sum_{u_2} P_2(u_1,u_2)\, y^{u_2}$$

$$H_1(x,y) = \sum_{u_1} H_{u_1 1}(y)\, x^{u_1}$$

$$H_2(x,y) = \sum_{u_1} H_{u_1 2}(y)\, x^{u_1}$$

$$H(x,y) = H_1(x,y) + H_2(x,y) + P_0$$

Multiplying equations (6.3.1), (6.3.2), (6.3.3) and (6.3.4) by the appropriate powers of x and y and summing accordingly, the generating functions $H_1(x,y)$ and $H_2(x,y)$ shall be obtained as follows:

$$\begin{aligned}
&[\beta_1-\alpha_1 x+(M-u_1)(\alpha_1+\alpha_2)-u_2(\alpha_1+\alpha_2)-(M-u_1-u_2+1)\alpha_2 y \\
&\quad + (M-u_1)\alpha_1 x + (\beta_1/x)]\, H_1(x,y) \\
&\qquad = (\beta_2/y)H_2(x,y) + \alpha_1 x\, H_{01}(y) - \alpha_1 x\, H_{M1}(y)x^M \\
&\qquad\quad +[(M-u_1)(\alpha_1+\alpha_2)-u_2(\alpha_1+\alpha_2)-(M-u_1-u_2+1)\alpha_2 y]\, H_{M1}(y)x^M \\
&\qquad\quad -(\beta_2/y)H_{02}(y)+(M-u_1-u_2)\alpha_1 H_{01}(y)-(M-u_1-u_2)\alpha_1 H_{M1}(y)x^M \\
&\qquad\quad -(M-u_1-u_2)\alpha_1 H_{(M-1)1}(y)x^M-\beta_1 H_{11}(y) \\
&\qquad\quad +\beta_2 y^{M-u_1} \sum_{u_1} P_2(u_1,M-u_1+1)\, x^{u_1}-u_2(\alpha_1+\alpha_2) \sum_{u_1} P_1(u_1,0)x^{u_1} \\
&\qquad\quad +u_2\alpha_1 \sum_{u_1} P_1(u_1-1,0)x^{u_1}-(M-u_1-u_2+1)\alpha_2\, y^{M-u_1+1} \\
&\qquad\qquad \sum_{u_1} P_1(u_1,M-u_1)x^{u_1}
\end{aligned}$$

$$[\beta_2-u_1+(u_1-1)(\alpha_1+\alpha_2)]\ H_2(x,y)$$

$$= [(\beta_2/y)-u_1+(u_1+u_2-M-1)\alpha_1$$

$$+(u_1+u_2-M-1+y+(M-u_2)y)\alpha_2]\ H_{02}(y)$$

$$+\beta_1 H_{11}(y)-\alpha_2 P_2(0,M)y^{M+1}+(M-u_2)(\alpha_1+\alpha_2)P_2(0,M)y^M$$

$$-(M-u_2)\alpha_2 P_2(0,M-1)y^M-(M-u_2)\alpha_2\ P_2(0,M)y^{M+1}$$

$$-\beta_1 P_1(1,0)-\beta_2 P_2(0,1)$$

The solution of the joint probability generating function $H(x,y)$ of this model can not be obtained easily as it involves again the steady state probabilities, which are unknown quantities. The process of finding the expected queue sizes can not be carried out due to the presence of the unknown probability functions. However, for relatively smaller values of M, the underlying equations shall be solved for $P(u)$, where $P(u)$ is the steady-state probability that the overall queue size is u, and as a result, the expected over all queue size $E(u)$, and machine availability, ψ and operative efficiency, π, shall be obtained for the common service rate β, where

$$E(u) = \Sigma u P(u)$$

$$\psi = 1-E(u)/M$$

$$\pi = 1-P(0)$$

Table 6.3.1

Values of E(u), ψ and π

Number of Machines	3			4			5			6		
	E(u)	M.A.	O.E.	E(u)	M.A.	O.E.	E(u)	M.A.	O.E.	E(u)	M.A.	O.E
α_1=0.07, α_2=0.09, β =0.33	0.5193	0.8269	0.7129	0.8668	0.7833	0.9304	1.2995	0.7401	0.9567	1.8234	0.6961	0.9933
α_1=0.09, α_2=0.10, β =0.48	0.4743	0.8419	0.5715	0.7884	0.8029	0.8027	1.1905	0.7619	0.9133	1.6266	0.7289	0.9650
α_1=0.10, α_2=0.12, β =0.48	0.4404	0.8532	0.6660	0.7364	0.8159	0.9401	1.1261	0.7748	0.9430	1.5522	0.7413	0.9795

Using c_s to denote the marginal cost of the server per unit time, c_w to denote the waiting time cost per machine and p_m to denote the income obtained per machine, the profit function shall be formed as

$$P = p_m \psi - c_s(1-\pi)\beta - c_w E(u) \tag{6.3}$$

The values of E(u), ψ and π calculated earlier shall be used in combination with the known values of c_s, c_w and p_m to compute the profit P for varying values of β and M and hence the optimal value β^* shall be identified for which the net profit p will be maximum.