Chapter 4

Bi-spectra associated with local and non-local features in the primordial scalar power spectrum

In the last two chapters, we have discussed as to how, though nearly scale invariant primordial spectra as is generated in slow roll inflationary scenarios are rather consistent with the CMB observations, certain features in the inflationary scalar power spectra lead to an improved fit to the data. Though the statistical significance of such features remain to be understood completely satisfactorily \[69, 70\], they gain importance from the phenomenological perspective of comparing the models with the data, because only a smaller class of single field inflationary models, which allow for departures from slow roll, can generate them.

Over the last half-a-dozen years, it has been increasingly realized that the detection of non-Gaussianities in the primordial perturbations can considerably help in constraining the inflationary models (see Refs. \[49, 51, 52\]; for early efforts in this direction, see Refs. \[56\]). In particular, the detection of a high value for the \(f_{\text{NL}}\) parameter that is used to describe the extent of non-Gaussianity [cf. Eq. (1.29)] can rule out a wide class of models. If, indeed, the extent of non-Gaussianity proves to be as large as the mean values of \(f_{\text{NL}}\) arrived at from the recent WMAP data \[19, 57, 58\], then canonical scalar field models that lead to slow roll inflation and nearly scale invariant primordial spectra will cease to be consistent with the data. But, interestingly, demanding the presence of features in the scalar power spectrum seems to generically lead to larger non-Gaussianities (see, for example, Refs. \[53\]). Therefore, features may offer the only route (unless one works with non-vacuum initial states \[102\]) for the canonical scalar fields to remain viable if \(f_{\text{NL}}\) turns out to be significant.
The above discussion raises two important issues. Firstly, if indeed the presence of features turns out to be the correct reason behind possibly large non-Gaussianities, can we observationally identify the correct underlying inflationary scenario, in particular, given the fact that different models can lead to similar features in the scalar power spectrum? In other words, to what extent can the non-Gaussianity parameter $f_{NL}$ help us discriminate between the inflationary models that permit features? To address this question, we shall consider a few typical inflationary models leading to features (including those considered in the previous two chapters), assuming that they can be viewed as representatives of such a class of scenarios. Concretely, we shall consider the Starobinsky model [71] and the punctuated inflationary scenario [77], both of which result in a sharp drop in power at large scales that is followed by oscillations. We shall also study large and small field models with an additional step introduced in the inflaton potential that we had considered in Chapter 2 [53, 74, 75, 85]. As we had seen, the step leads to a burst of oscillations in the scalar power spectrum which improve the fit to the outliers near the multipole moments of $\ell = 22$ and 40 in the CMB angular power spectrum. We shall also consider oscillating inflaton potentials such as the one that arises in the axion monodromy model discussed in the last chapter. As we had illustrated, such oscillatory potentials lead to modulations in the power spectrum over a wide range of scales and result in a considerable betterment in the fit to the data [53, 97, 98, 99, 101].

The second issue pertains to the calculation of non-Gaussianities in models where the slow roll approximation is not satisfied. Usually, the slow roll approximation is utilized to arrive at analytical expressions for the non-Gaussianity parameter $f_{NL}$. Clearly, this is no longer possible when departures from slow roll occur. We shall use a new Fortran numerical code to evaluate the non-Gaussianities in such situations. Although, some partial numerical results have already been published in the literature, we believe that it is for the first time that a general (we shall restrict ourselves to the equilateral case here, but the code can compute for any configuration), and efficient (that can arrive at results within a few minutes) code has been put together. Moreover, as we shall demonstrate, the code can also compute all the different contributions to the bi-spectrum.

The plan of this chapter is as follows. In the following section, we shall briefly describe the inflationary models of interest and discuss the scalar power spectra that arise in these models. In the succeeding section, we shall quickly describe the essential details pertaining to the evaluation of the bi-spectrum and the non-Gaussianity parameter $f_{NL}$ in inflationary models involving a single, canonical, scalar field. In Section 4.3, after demonstrating that the super-Hubble contributions to the complete bi-spectrum during inflation
prove to be negligible, we shall describe the method that we adopt to numerically compute the bi-spectrum and the non-Gaussianity parameter $f_{NL}$ in the equilateral limit. We shall also illustrate the extent of accuracy of the computations by comparing our numerical results in the equilateral limit with the bi-spectra expected in power law inflation and the analytical results that have recently been obtained in the case of the Starobinsky model (see Ref. [54]; however, in this context, also see Refs. [55]). In Section 4.4, we shall present the main results, and compare the $f_{NL}$ that arise in the various models of our interest. We shall finally conclude this chapter with a brief discussion on the implications of our results.

4.1 The inflationary models of interest and the resulting power spectra

Broadly, the models that we shall consider can be categorized into three classes. The first class shall involve potentials which admit a relatively mild and brief departure from slow roll. The second class shall contain small but repeated deviations from slow roll, while the third and the last class shall involve a short but rather sharp departure from slow roll. In this section, we shall briefly outline the different inflationary models that we shall consider under these classes and discuss the scalar power spectra that are generated in these models.

4.1.1 Inflationary potentials with a step

Under the first class, we shall consider models with a step that we had discussed in Chapter 2. We shall consider the effects of the introduction of the step (2.5) in the archetypical quadratic large field model (2.1) and the small field model governed by the potential (2.2). In the case of the small field model, we shall specifically focus on the situation wherein $p_0 = 4$ and $\mu = 15 M_{Pl}$, as we had done earlier. Moreover, in both these cases, we shall work with values of the parameters that correspond to the best fit values arrived at upon comparing the models with the WMAP seven-year data, as quoted in Table 2.3. Further, we shall assume that the field starts on the inflationary attractor at an initial value, say, $\phi_i$, such that at least 60 e-folds of inflation takes place. We choose $\phi_i$ to be 16.5 $M_{Pl}$ and 7.3 $M_{Pl}$ in the cases of the quadratic and the small field models with the step, respectively.
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4.1.2 Oscillations in the inflaton potential

The second class of models that we shall consider are those which lead to small but repeated deviations from slow roll as in the case of potentials containing oscillatory terms that we had discussed in the previous chapter. In this context, we shall consider both the potentials, viz. the quadratic potential modulated by sinusoidal oscillations (3.1) as well as the axion monodromy model (3.2), that we had discussed. Again, we shall work with the best fit values corresponding to the WMAP seven-year data, as listed in Table 3.2. Moreover, we shall assume that the field starts on the inflationary attractor at the initial value $\phi_i$ of $16M_{\text{pl}}$ and $12M_{\text{pl}}$ in the cases of the quadratic potential with sinusoidal oscillations and the axion monodromy model, respectively.

4.1.3 Punctuated inflaton and the Starobinsky model

We shall consider two models under the last class, both of which are known to lead to brief but sharp departures from slow roll. The first of the inflationary models that we shall consider in this class is described by the following potential containing two parameters $m$ and $\lambda$:

$$V(\phi) = \frac{m^2}{2} \phi^2 - \frac{\sqrt{2} \lambda (n-1)}{n} \frac{m}{\phi^n} + \frac{\lambda}{4} \phi^{2(n-1)}. \quad (4.1)$$

The third quantity $n$ that appears in the potential is an integer which takes values greater than two. Such potentials are known to arise in certain minimal supersymmetric extensions of the standard model [103]. It is worthwhile noting here that the case of $n = 3$ has been considered much earlier for reasons similar to what we shall consider here, viz. towards producing certain features in the scalar power spectrum [104]. In the above potential, the coefficient of the $\phi^n$ term is chosen such that the potential contains a point of inflection at, say, $\phi = \phi_0$ (i.e. the location where both $dV/d\phi$ and $d^2V/d\phi^2$ vanish), so that $\phi_0$ given by

$$\phi_0 = \left[ \frac{2 m^2}{(n-1) \lambda} \right]^{\frac{1}{n-2}}. \quad (4.2)$$

If one starts at a suitable value of the field beyond the point the inflection in the above potential, it is found that one can achieve two epochs of slow roll inflation sandwiching a brief period of departure from inflation (lasting for a little less than a e-fold), a scenario which has been dubbed as punctuated inflation [77]. In fact, it is the point of inflection, around which the potential exhibits a plateau with an extremely small curvature, which permits such an evolution to be possible. It is found that the following values for the potential parameters results in a power spectrum that leads to an improved fit to the
CMB data in the \( n = 3 \) case: \( m = 1.5012 \times 10^{-7} M_{\odot} \) and \( \phi_0 = 1.95964 M_{\odot} \). It should be added that the field is assumed to start from rest at an initial value of \( \phi_i = 11.5 M_{\odot} \) to arrive at the required behavior.

The second model that we shall consider is the Starobinsky model [71], which, as we shall see, leads to a scalar power spectrum that has a certain resemblance to the spectrum generated by punctuated inflation. The model consists of a linear potential with a sharp change in its slope at a given point, and can be described as follows:

\[
V(\phi) = \begin{cases} 
V_0 + A_+ (\phi - \phi_0) & \text{for } \phi > \phi_0, \\
V_0 + A_- (\phi - \phi_0) & \text{for } \phi < \phi_0.
\end{cases} \tag{4.3}
\]

Evidently, while the value of the scalar field where the slope changes abruptly is \( \phi_0 \), the slope of the potential above and below \( \phi_0 \) are given by \( A_+ \) and \( A_- \), respectively. Moreover, the quantity \( V_0 \) denotes the value of the potential at \( \phi = \phi_0 \). A crucial assumption of the Starobinsky model is that the value of \( V_0 \) is sufficiently large so that the behavior of the scale factor always remains close to that of de Sitter. The change in the slope causes a short period of deviation from slow roll as the field crosses \( \phi_0 \). However, in contrast to the case of the punctuated inflationary scenario, where one encounters a brief departure from inflation, inflation continues uninterrupted in the Starobinsky model. We have not compared the Starobinsky model with the data, and we shall work with two different sets of values for the parameters of the model. We shall choose one set to allow for comparison of the analytical results that have been obtained in this case (see Refs. [54, 55]) with the corresponding numerical ones. The other set shall be chosen to lead to a spectrum that closely mimics the power spectrum encountered in punctuated inflation. In the case of the former, we shall choose the following values of the parameters: \( V_0 = 2.36 \times 10^{-12} M_{\odot}^4 \), \( A_+ = 3.35 \times 10^{-14} M_{\odot}^3 \), \( A_- = 7.26 \times 10^{-15} M_{\odot}^3 \) and \( \phi_0 = 0.707 M_{\odot} \), while in the case of the latter, we shall work with the same values of \( A_+ \) and \( \phi_0 \), but shall set \( V_0 = 2.94 \times 10^{-13} M_{\odot}^4 \), and \( A_- = 3.35 \times 10^{-16} M_{\odot}^3 \). Also, we shall work with an initial value of \( \phi_i = 0.849 M_{\odot} \) in the first instance and with \( \phi_i = 1.8 M_{\odot} \) in the second. Further, we shall start with field velocities that are determined by the slow roll conditions in both the cases.

### 4.1.4 The power spectra

We shall now discuss the scalar power spectra that arise in the inflationary models that we have listed above. As we have mentioned, we shall provide the details concerning the numerical evolution of the governing equations and the evaluation of the scalar power spectrum a little later in Section 4.3. In Figure 4.1, we have illustrated the scalar power
Figure 4.1: The scalar power spectrum in the different types of inflationary models that we consider. The parameters of the Starobinsky model [71] has been chosen such that the resulting power spectrum closely resembles the spectrum that arises in the punctuated inflationary scenario which is known to lead to an improved fit to the CMB data [77]. As we have seen in the previous two chapters, while the models with a step [74, 75, 85] lead to a burst of oscillations over a specific range of scales, inflaton potentials with oscillating terms produce modulations over a wide range of scales in the power spectrum [101]. The inset highlights the differences in the various power spectra over a smaller range of scales. We have emphasized certain aspects of these different power spectra in some detail in the text.
parameter $\epsilon_1$ remains small in the Starobinsky model as the field crosses the transition, the second slow parameter $\epsilon_2$ turns very large briefly [54, 55]. In the case of punctuated inflation, $\epsilon_1$ itself grows to a large value thereby actually interrupting inflation for about a e-fold. It is this property that results in a sharper spike in the case of punctuated inflation than the Starobinsky model. The overall step in these models is easier to understand, and it simply arises due to the difference in the Hubble scales associated with the slow roll epoch before and after the period of fast roll. Both these models also lead to oscillations before the spectra turn nearly scale invariant on small scales. The spectra that arises in punctuated inflation, in addition to leading to a better fit to the outliers at very small multipoles (because of the drop in power on these scales), also provides an improvement in the fit to the outlier at $\ell \simeq 22$ [77]. It is interesting to notice that the spectra of punctuated inflation and the model with a step in the potential match briefly as they oscillate near scales corresponding to $\ell \simeq 22$.

4.2 The scalar bi-spectrum in the Maldacena formalism

As we had mentioned in the introductory chapter, there now exists a standard formalism, initially proposed by Maldacena, that allows one to evaluate the bi-spectrum in a given inflationary model. In the Maldacena formalism [49], the bi-spectrum is evaluated using the standard rules of perturbative quantum field theory, based on the interaction Hamiltonian that depends cubically on the curvature perturbation. For the case of the canonical scalar field of our interest, the action at the cubic order in the curvature perturbation is found to be [49, 51, 52]

$$S_3[\mathcal{R}] = M_{Pl}^2 \int d\eta \int d^3x \left[ a^2 \epsilon_1^2 \mathcal{R} \mathcal{R}'^2 + a^2 \epsilon_1^2 \mathcal{R} (\partial \mathcal{R})^2 - 2a \epsilon_1 \mathcal{R}' (\partial^i \mathcal{R}) (\partial_i \chi) 
+ \frac{a^2}{2} \epsilon_1 \epsilon_2 \mathcal{R}^2 \mathcal{R}' + \frac{\epsilon_1}{2} (\partial^i \mathcal{R}) (\partial_i \chi) (\partial^2 \chi) + \frac{\epsilon_1}{4} (\partial^2 \mathcal{R}) (\partial \chi)^2 + a \mathcal{F} \left( \frac{\delta L_2}{\delta \mathcal{R}} \right) \right], \quad (4.4)$$

where the Latin indices denote the spatial coordinates, while the function $\chi$ is defined through the relation

$$\partial^2 \chi \equiv a \epsilon_1 \mathcal{R}' \quad (4.5)$$

The quantity $\delta L_2/\delta \mathcal{R}$ denotes the variation of the Lagrangian density corresponding to the following quadratic action:

$$S_2[\mathcal{R}] = \frac{1}{2} \int d\eta \int d^3x \ z^2 \left[ \mathcal{R}'^2 - (\partial \mathcal{R})^2 \right], \quad (4.6)$$
which, for instance, leads to the equation of motion (1.15) for the curvature perturbation $\mathcal{R}$, and can be written as

$$\frac{\delta \mathcal{L}_2}{\delta \mathcal{R}} = \dot{\Lambda} + H \Lambda - \epsilon_1 (\partial^2 \mathcal{R}).$$

(4.7)

The term $\mathcal{F}(\delta \mathcal{L}_2/\delta \mathcal{R})$ that has been introduced in the above cubic order action refers to the following expression [51, 54]:

$$\mathcal{F}\left(\frac{\delta \mathcal{L}_2}{\delta \mathcal{R}}\right) = \frac{1}{2aH} \left\{ \left[ a^2 H \epsilon_2 \mathcal{R}^2 + 4 a \mathcal{R} \mathcal{R}' + (\partial^2 \mathcal{R})(\partial_i \chi) - \frac{1}{H} (\partial \mathcal{R})^2 \right] \frac{\delta \mathcal{L}_2}{\delta \mathcal{R}} + \left[ \Lambda (\partial \mathcal{R}) + (\partial^2 \mathcal{R})(\partial_i \chi) \right] \delta^{ij} \partial_j \left[ \partial^{-2} \left( \frac{\delta \mathcal{L}_2}{\delta \mathcal{R}} \right) \right] \right\}. \quad (4.8)$$

where, again, the Latin indices represent the spatial coordinates.

For convenience, we shall introduce a new quantity $G(k_1, k_2, k_3)$ that is related to the bi-spectrum $B_5(k_1, k_2, k_3)$ by a constant factor as follows:

$$G(k_1, k_2, k_3) = (2 \pi)^{9/2} B_5(k_1, k_2, k_3). \quad (4.9)$$

It can be shown that the quantity $G(k_1, k_2, k_3)$, which results from the interaction Hamiltonian corresponding to the cubic action (4.4), evaluated towards the end of inflation, say, at the conformal time $\eta_e$, can be expressed as [49, 51, 52, 53, 54]

$$G(k_1, k_2, k_3) \equiv \sum_{C=1}^{7} G_C(k_1, k_2, k_3)$$

$$\equiv \sum_{C=1}^{7} \left\{ \sum_{C=1}^{6} \left[ f_{k_1}(\eta_e) f_{k_2}(\eta_e) f_{k_3}(\eta_e) \right] G_C(k_1, k_2, k_3) \right\} + G_7(k_1, k_2, k_3), \quad (4.10)$$

where $f_k$ are the Fourier modes associated with the curvature perturbation [cf. Eq. (1.19)] that satisfy the differential equation (1.15). The quantities $G_C(k_1, k_2, k_3)$ with $C = (1, 6)$ correspond to the six terms in the interaction Hamiltonian, and are described by the inte-
where the expression (4.11a) gives the analytical results that are available in the case of the Starobinsky model. The expected form of the bi-spectrum in the equilateral limit in power law inflation and the slow roll parameters, and eventually evaluate the inflationary scalar power and bi-spectra. Also, we shall adopt to numerically evolve the equations governing the background and the perturbations, and eventually evaluate the inflationary scalar power and bi-spectra. Also, we shall outline the methods that we adopt to numerically evolve the equations governing the background and the perturbations, and eventually evaluate the inflationary scalar power and bi-spectra. Also, we shall outline the methods that we adopt to numerically evolve the equations governing the background and the perturbations, and eventually evaluate the inflationary scalar power and bi-spectra. Also, we shall illustrate the extent of accuracy of the numerical methods by comparing them with the expected form of the bi-spectrum in the equilateral limit in power law inflation and the analytical results that are available in the case of the Starobinsky model [54, 55].

4.3 The numerical computation of the scalar bi-spectrum

In this section, after illustrating that the super-Hubble contributions to the complete bi-spectrum during inflation proves to be negligible, we shall outline the methods that we adopt to numerically evolve the equations governing the background and the perturbations, and eventually evaluate the inflationary scalar power and bi-spectra. Also, we shall illustrate the extent of accuracy of the numerical methods by comparing them with the expected form of the bi-spectrum in the equilateral limit in power law inflation and the analytical results that are available in the case of the Starobinsky model [54, 55].

4.3.1 The contributions to the bi-spectrum on super-Hubble scales

It is clear from the above expressions that the evaluation of the bi-spectrum involves integrals over the mode \( f_k \) and its derivative \( f_k' \) as well as the slow roll parameters \( \epsilon_1 \),
$\epsilon_2$ and the derivative $\epsilon'_2$. While evaluating the power spectra, it is well known that it suffices to evolve the curvature perturbation from an initial time when the modes are sufficiently inside the Hubble radius to a suitably late time when the amplitude of the curvature perturbation settles down to a constant value $[81, 82]$. We shall illustrate that many of the contributions to the bi-spectrum prove to be negligible when the modes evolve on super-Hubble scales. Interestingly, we shall also show that, those contributions to the bi-spectrum which turn out to be significant at late times when the modes are well outside the Hubble radius are canceled by certain other contributions that arise. As a consequence, we shall argue that, numerically, it suffices to evaluate the integrals over the period of time during which the curvature perturbations have been conventionally evolved to arrive at the power spectra, *viz.* from the sub-Hubble to the super-Hubble scales.

**Evolution of $f_k$ on super-Hubble scales**

During inflation, when the modes are on super-Hubble scales, it is well known that the solution to $f_k$ can be written as $[6, 7]$

$$f_k(\eta) \simeq A_k + B_k \int^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})},$$  \hspace{1cm} (4.13)

where $A_k$ and $B_k$ are $k$-dependent constants which are determined by the initial conditions imposed on the modes in the sub Hubble-limit. The first term involving $A_k$ is the growing mode, which is actually a constant, while the term containing $B_k$ represents the decaying mode. Therefore, on super-Hubble scales, the mode $f_k$ simplifies to

$$f_k(\eta) \simeq A_k.$$  \hspace{1cm} (4.14)

Moreover, the leading non-zero contribution to its derivative is determined by the decaying mode, and is given by

$$f'_k(\eta) \simeq \frac{B_k}{z^2} = \frac{\bar{B}_k}{a^2 \epsilon_1},$$  \hspace{1cm} (4.15)

where we have set $\bar{B}_k = B_k/(2 M_{\text{Pl}}^2)$.

It is now a matter of making use of the above solutions for $f_k$ and $f'_k$ to determine the super-Hubble contributions to the bi-spectrum during inflation.
The various contributions

To begin with, note that, each of the integrals $G_C(k_1, k_2, k_3)$, where $C = (1, 6)$, can be divided into two parts as follows:

$$G_C(k_1, k_2, k_3) = G_{\text{is}}^C(k_1, k_2, k_3) + G_{\text{se}}^C(k_1, k_2, k_3).$$  

(4.16)

The integrals in the first term $G_{\text{is}}^C(k_1, k_2, k_3)$ run from the earliest time (i.e. $\eta_i$) when the smallest of the three wavenumbers $k_1$, $k_2$ and $k_3$ is sufficiently inside the Hubble radius [typically corresponding to $k/(aH) \simeq 100$] to the time (say, $\eta_s$) when the largest of the three wavenumbers is well outside the Hubble radius [say, when $k/(aH) \simeq 10^{-5}$]. Then, evidently, the second term $G_{\text{se}}^C(k_1, k_2, k_3)$ will involve integrals which run from the latter time $\eta_s$ to the end of inflation at $\eta_e$. In what follows, we shall discuss the various contributions to the bi-spectrum due to the terms $G_{\text{se}}^C(k_1, k_2, k_3)$. We shall show that the corresponding contribution either remains small or, when it proves to be large, it is exactly canceled by another contribution to the bi-spectrum.

The contributions due to the fourth and the seventh terms

Let us first focus on the fourth term $G_4(k_1, k_2, k_3)$ since it has often been found to lead to the largest contribution to the bi-spectrum when deviations from slow roll occur [53, 54, 55]. As the slow roll parameters turn large towards the end of inflation, we can expect this term to contribute significantly at late times. However, as we shall quickly illustrate, such a late time contribution is exactly canceled by the contribution from $G_7(k_1, k_2, k_3)$ which arises due to the field redefinition.

Upon using the form (4.14) of the mode $f_k$ and its derivative (4.15) on super-Hubble scales in the expression (4.11d), one obtains that

$$G_{\text{se}}^C(k_1, k_2, k_3) \simeq i \left( (A_{k_1}^* A_{k_2} B_{k_3}^*) + \text{two permutations} \right) \int_{\eta_s}^{\eta_e} d\eta \epsilon_2.'$$  

(4.17)

This expression can be trivially integrated to yield

$$G_{\text{se}}^C(k_1, k_2, k_3) \simeq i \left( (A_{k_1}^* A_{k_2} B_{k_3}^*) + \text{two permutations} \right) \left[ \epsilon_2(\eta_e) - \epsilon_2(\eta_s) \right],$$  

(4.18)

so that the corresponding contribution to the bi-spectrum can be expressed as

$$G_{\text{se}}^C(k_1, k_2, k_3) \simeq i M_{\pi_1}^2 \left[ \epsilon_2(\eta_e) - \epsilon_2(\eta_s) \right] \left[ |A_{k_1}|^2 |A_{k_2}|^2 (A_{k_3} B_{k_3}^* - A_{k_3}^* B_{k_3}) \right]$$

+ two permutations,

(4.19)
Since this expression depends on the value of $\epsilon_2$ at the end of inflation, it suggests that the contributions to the bi-spectrum at late times during inflation could be considerable. But, as we shall soon show, this large contribution is canceled by a similar contribution from the seventh term that arises due to the field redefinition [cf. Eq. (4.12)].

Now, consider the Wronskian

$$W = f_k f_k^* - f_k^* f_k.$$  \hspace{1cm} (4.20)

Upon using the equation of motion (1.15) for $f_k$, one can show that, $W = W/z^2$, where $W$ is a constant. It is important to note that this result is valid on all scales, even in the sub-Hubble limit during inflation. In this limit, as we had mentioned, the modes $v_k$ satisfy the Bunch-Davies initial condition (1.23). On making use of this sub-Hubble behavior in the above definition of the Wronskian $W$, one obtains that $W = i$. In the super-Hubble limit, we have, on using the corresponding solution (4.14) and its derivative (4.15),

$$W = \frac{2 M^2_{pl}}{z^2} \left( A_k \dot{B}_k^* - A_k^* \dot{B}_k \right) = \frac{i}{z^2}. \hspace{1cm} (4.21)$$

Therefore, we obtain that

$$A_k \dot{B}_k^* - A_k^* \dot{B}_k = \frac{i}{2 M^2_{pl}},$$  \hspace{1cm} (4.22)

and, hence, the expression (4.19) for $G^se_4(k_1, k_2, k_3)$ simplifies to

$$G^se_4(k_1, k_2, k_3) \simeq -\frac{1}{2} \left[ \epsilon_2(\eta_e) - \epsilon_2(\eta_s) \right] \left( |A_{k_1}|^2 |A_{k_2}|^2 + \text{two permutations} \right). \hspace{1cm} (4.23)$$

The first of these terms involving the value of $\epsilon_2$ at the end of inflation exactly cancels the contribution $G_7(k_1, k_2, k_3)$ [with $f_k$ set to $A_k$ in Eq. (4.12)] that arises due to the field redefinition. But, the remaining contribution cannot be ignored and needs to be taken into account. It is useful to note that this term is essentially the same as the one due to the field redefinition, but which is now evaluated on super-Hubble scales (i.e. at $\eta_s$) rather than at the end of inflation. In other words, if we consider the fourth and the seventh terms together, it is equivalent to evaluating the contribution to the bi-spectrum corresponding to $G^se_4(k_1, k_2, k_3)$, and adding to it the contribution due to the seventh term $G_7(k_1, k_2, k_3)$ evaluated at $\eta_s$, instead of at the end of inflation.

**The contribution due to the second term**  Let us now turn to the contribution due to the second term, which can occasionally prove to be comparable to the contribution due to
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the fourth term \([54]\). Upon making use of the behavior of the mode \(f_k\) on super-Hubble scales in the integral \((4.11b)\), we have

\[
G_2^{se}(k_1, k_2, k_3) = -2i (k_1 \cdot k_2 + \text{two permutations}) A_{k_1}^* A_{k_2}^* A_{k_3}^* I_2(\eta_e, \eta_s),
\]

(4.24)

where \(I_2(\eta_e, \eta_s)\) denotes the integral

\[
I_2(\eta_e, \eta_s) = \int_{\eta_s}^{\eta_e} d\eta \, a^2 \epsilon_1^2,
\]

(4.25)

so that the corresponding contribution to the bi-spectrum is given by

\[
G_2^{se}(k_1, k_2, k_3) = -2i M^2_{Pl} (k_1 \cdot k_2 + \text{two permutations})
\times |A_{k_1}|^2 |A_{k_2}|^2 |A_{k_3}|^2 [I_2(\eta_e, \eta_s) - I_2^*(\eta_e, \eta_s)].
\]

(4.26)

Note that, due to quadratic dependence on the scale factor, actually, \(I_2(\eta_e, \eta_s)\) is a rapidly growing quantity at late times. However, the complete super-Hubble contribution to the bi-spectrum vanishes identically since the integral \(I_2(\eta_e, \eta_s)\) is a purely real quantity. Hence, in the case of the second term, it is sufficient to evaluate the contribution to the bi-spectrum due to \(G_2^{se}(k_1, k_2, k_3)\).

The remaining terms Let us now compute the contributions due to the remaining terms, viz. the first, the third, the fifth and the sixth. Notice that, the first term \(G_1(k_1, k_2, k_3)\) and the third term \(G_3(k_1, k_2, k_3)\) involve the same integrals. Therefore, these two contributions to the bi-spectrum can be clubbed together. Similarly, the fifth and the sixth terms, viz. \(G_5(k_1, k_2, k_3)\) and \(G_6(k_1, k_2, k_3)\), also contain integrals of the same type, and hence their contributions too can be combined. On making use of the super-Hubble behavior \((4.14)\) and \((4.15)\) of the mode \(f_k\) and its derivative, we obtain that

\[
G_1^{se}(k_1, k_2, k_3) \simeq 2i \left( A_{k_1}^* \bar{B}_{k_2}^* \bar{B}_{k_3}^* + \text{two permutations} \right) I_{13}(\eta_e, \eta_s),
\]

(4.27)

and

\[
G_3^{se}(k_1, k_2, k_3) \simeq -2i \left[ \frac{k_1 \cdot k_2}{k_3^2} \right] A_{k_1}^* \bar{B}_{k_2}^* \bar{B}_{k_3}^* + \text{five permutations} \right] I_{13}(\eta_e, \eta_s),
\]

(4.28)

where the quantity \(I_{13}(\eta_e, \eta_s)\) represents the integral

\[
I_{13}(\eta_e, \eta_s) = \int_{\eta_s}^{\eta_e} d\eta \frac{a^2}{\epsilon_1^2}.
\]

(4.29)
From these results, one can easily show that the super-Hubble contributions due to the first and the third terms to the bi-spectrum can be written as

\[
G^{se}_{1}(k_1, k_2, k_3) + G^{se}_{3}(k_1, k_2, k_3) = 2iM_{pl}^2 \left[ \frac{1 - \frac{k_1 \cdot k_2}{k_2^2} - \frac{k_1 \cdot k_3}{k_3^2}}{k_2^2 \cdot k_3^2} \right] |A_{k_1}|^2 
\times \left( A_{k_2} B_{k_2}^* A_{k_3} B_{k_3}^* - A_{k_2} B_{k_2} A_{k_3}^* B_{k_3}^* \right) 
+ \text{two permutations} \right] I_{13}(\eta_e, \eta_s). \tag{4.30}
\]

The corresponding contributions due to the fifth and the sixth terms can be arrived at in a similar fashion. We find that

\[
G^{se}_{5}(k_1, k_2, k_3) + G^{se}_{6}(k_1, k_2, k_3) = \frac{iM_{pl}^2}{2} \left\{ \frac{k_1 \cdot k_2}{k_2^2} + \frac{k_1 \cdot k_3}{k_3^2} + \frac{k_1^2 \cdot (k_2 \cdot k_3)}{k_2^2 \cdot k_3^2} \right\} 
\times |A_{k_1}|^2 \left( A_{k_2} B_{k_2}^* A_{k_3} B_{k_3}^* - A_{k_2} B_{k_2} A_{k_3}^* B_{k_3}^* \right) 
+ \text{two permutations} \right\} I_{56}(\eta_e, \eta_s), \tag{4.31}
\]

where the quantity \( I_{56}(\eta_e, \eta_s) \) is described by the integral

\[
I_{56}(\eta_e, \eta_s) = \int_{\eta_e}^{\eta_0} \frac{d\eta}{a^2} \epsilon_1. \tag{4.32}
\]

Hence, the non-zero, super-Hubble contribution to the bi-spectrum is determined by the sum of the contribution due to the first, the third, the fifth and the sixth terms arrived at above. In order to illustrate that this contribution is insignificant, we shall now turn to estimating the amplitude of the corresponding contribution to the non-Gaussianity parameter \( f_{NL} \).

### 4.3.2 An estimate of the super-Hubble contribution to the non-Gaussianity parameter

Let us restrict ourselves to the equilateral limit, i.e. when \( k_1 = k_2 = k_3 = k \), for simplicity. In such a case, the super-Hubble contributions to the bi-spectrum, say, \( G^{se}_{eq}(k) \), due to the first, the third, the fifth and the sixth terms, as given by the expressions (4.30) and (4.31), add up to be

\[
G^{se}_{eq}(k) = iM_{pl}^2 |A_k|^2 \left( A_k^2 B_k^* - A_k^* B_k^2 \right) \left[ 12 I_{13}(\eta_e, \eta_s) - \frac{9}{4} I_{56}(\eta_e, \eta_s) \right]. \tag{4.33}
\]
In the equilateral limit, the expression (1.29) for the non-Gaussianity parameter $f_{\text{NL}}(k_1, k_2, k_3)$ simplifies to

$$f_{\text{NL}}^{\text{eq}}(k) = -\frac{10}{9} \frac{1}{(2\pi)^4} \frac{k^6 G_{\text{eq}}(k)}{\mathcal{P}_s^2(k)},$$

(4.34)

where $\mathcal{P}_s(k)$ is the scalar power spectrum defined in Eq. (1.21). It is straightforward to show that the $f_{\text{NL}}$ corresponding to the super-Hubble contribution to the bi-spectrum $G_{\text{eq}}(k)$ above is given by

$$f_{\text{NL}}^{\text{eq}(se)}(k) \simeq -\frac{5}{18} \frac{i M_{\text{Pl}}^2}{18} \left( \frac{A_k^2 B_k^2 - A_k^* B_k^*}{|A_k|^2} \right) \left[ 12 I_{13}(\eta_e, \eta_s) - \frac{9}{4} I_{56}(\eta_e, \eta_s) \right],$$

(4.35)

where we have made use of the fact that $f_k \simeq A_k$ at late times in order to arrive at the power spectrum.

To estimate the above super-Hubble contribution to the non-Gaussianity parameter $f_{\text{NL}}^{\text{eq(se)}}$, let us choose to work with power law inflation because it permits exact calculations, and it can also mimic slow roll inflation. During power law expansion, the scale factor can be written as

$$a(\eta) = a_1 \left( \frac{\eta}{\eta_1} \right)^{\gamma + 1},$$

(4.36)

where $a_1$ and $\eta_1$ are constants, while $\gamma$ is a free index. It is useful to note that, in such a case, the first slow roll parameter is a constant and is given by $\epsilon_1 = (\gamma + 2)/(\gamma + 1)$. The current observational constraints on the scalar spectral index suggest that $\gamma \lesssim -2$, which implies that the corresponding scale factor is close to that of de Sitter.

In power law inflation, the exact solution to Eq. (1.16) can be expressed in terms of the Bessel function $J_\nu(x)$ as follows (see, for instance, Refs. [105]):

$$v_k(\eta) = \sqrt{-k \eta} \left[ C_k J_\nu(-k \eta) + D_k J_{-\nu}(-k \eta) \right],$$

(4.37)

where $\nu = (\gamma + 1/2)$, and the quantities $C_k$ and $D_k$ are constants that are determined by the initial conditions. Upon demanding that the above solution satisfies the Bunch-Davies initial condition (1.23), one obtains that

$$C_k = -D_k e^{-i \pi (\gamma + 1/2)};$$

(4.38a)

$$D_k = \sqrt{\frac{i}{\pi}} \frac{e^{i \pi \gamma/2}}{k \cos (\pi \gamma)}.$$  

(4.38b)

Since $f_k = v_k/z$, with $z = \sqrt{2 \epsilon_1 M_{\text{Pl}} a}$, and as $\epsilon_1$ is a constant in power law inflation, we can arrive at the constants $A_k$ and $B_k$ [cf. Eqs. (4.14) and Eqs. (4.15)] from the super-Hubble
limit of the solution (4.37), which are found to be

\[ A_k = \frac{2^{-(\gamma+1)/2}}{\Gamma(\gamma+3/2)} \frac{(-k \eta_1)^{\gamma+1}}{\sqrt{2}} C_k, \]  
\[ B_k = \frac{(2 \gamma + 1) 2^{\gamma+1/2}}{\Gamma(-\gamma+1/2)} \frac{\sqrt{2} a_1 M_{pl}}{\eta_1} (-k \eta_1)^{-\gamma} D_k. \]

Then, upon inserting the above expressions for the quantities \( A_k \) and \( B_k \) in Eq. (4.35), we obtain that

\[ f_{\text{NL}}^{\text{eq}(\text{se})}(k) = \frac{5}{72 \pi} \left[ 12 - \frac{9 (\gamma + 2)}{4 (\gamma + 1)} \right] \Gamma^2 \left( \gamma + 1 \right) 2^{2 \gamma + 1} (2 \gamma + 1) (\gamma + 2) \times (\gamma + 1)^{-2(\gamma+1)} \sin(2 \pi \gamma) \left[ 1 - \frac{H_s}{H_e} e^{-3(N_e-N_s)} \right] \left( \frac{k}{a_s H_s} \right)^{-(2 \gamma+1)}. \]  

It should be mentioned that, in arriving at this expression, for convenience, we have set \( \eta_1 \) to be \( \eta_s \), which corresponds to \( a_1 \) being \( a_s \), viz. the scale factor at \( \eta_s \). Moreover, while \( N_s \) and \( N_e \) denote the e-folds corresponding to \( \eta_s \) and \( \eta_e \), \( H_s \) and \( H_e \) represent the Hubble scales at these times, respectively. Recall that, \( \eta_s \) denotes the conformal time when the largest wavenumber of interest, say, \( k_s \), is well outside the Hubble radius, i.e. when \( k_s/(a H) \approx 10^{-5} \). Since \( (N_e-N_s) \) is expected to be at least 40 for the smallest cosmological scale, it is clear that the factor involving \( \exp[-3(N_e-N_s)] \) can be completely neglected. As we mentioned above, observations point to the fact that \( \gamma \lesssim -2 \). Therefore, if we further assume that \( \gamma = -(2+\varepsilon) \), where \( \varepsilon \approx 10^{-2} \), we find that the above estimate for the non-Gaussianity parameter reduces to

\[ f_{\text{NL}}^{\text{eq}(\text{se})}(k) \lesssim -\frac{5 \varepsilon^2}{9} \left( \frac{k_s}{a_s H_s} \right)^3 \approx -10^{-19}, \]  

where, in obtaining the final value, we have set \( k_s/(a_s H_s) = 10^{-5} \). The inequality above arises due to the fact that, for larger scales, i.e. when \( k < k_s \), \( k/(a H) < 10^{-5} \) at \( \eta_s \). In models involving the canonical scalar field, the smallest values of \( f_{\text{NL}} \) are typically generated in slow roll inflationary scenarios, wherein the non-Gaussianity parameter has been calculated to be of the order of the first slow roll parameter [49, 52]. The above estimate clearly points to fact that the super-Hubble contributions to the complete bi-spectrum and the non-Gaussianity parameter \( f_{\text{NL}} \) can be entirely ignored.

In summary, to determine the scalar bi-spectrum, it suffices to evaluate the contributions to the bi-spectrum due to the quantities \( G^{\text{se}}_C(k_1, k_2, k_3) \), with \( C = (1, 6) \), which involve integrals running from the initial time \( \eta_i \) to the time \( \eta_s \) when the smallest of the
three modes reaches super-Hubble scales. Further, the addition of the contribution due to the field redefinition evaluated at $\eta_s$ ensures that no non-trivial super-Hubble contributions are ignored. In the following sub-section, with the help of a specific example, we shall also corroborate these conclusions numerically.

### 4.3.3 Details of the numerical methods

The scalar bi-spectrum and the parameter $f_{NL}$ can be easily evaluated analytically in the slow roll inflationary scenario [49]. However, barring some exceptional cases [54, 55, 72], it often proves to be difficult to evaluate the bi-spectrum analytically when departures from slow roll occur. Hence, one has to resort to numerical computations in such cases.

We solve the background as well as the perturbation equations using a Bulirsch-Stoer algorithm with an adaptive step size control routine [106]. As we had mentioned earlier in Section 2.2, we shall treat the number of e-folds as the independent variable, which allows for efficient and accurate computation. To obtain the power spectrum, we impose the standard Bunch-Davies initial conditions [cf. Eq. (1.23)] on the perturbations when the modes are well inside the Hubble radius, and evolve them until suitably late times. Typically, in the case of smooth inflaton potentials, it suffices to evolve the modes $f_k$ from an initial time when $k/(aH) = 100$. However, as we had pointed out in Subsection 3.1.3, in the case of the axion monodromy model, for the best fit values of the parameters of our interest, the modes have to be evolved from an earlier initial time, when $k/(aH) \simeq 250$, so that the resonance that occurs in these models due to the oscillations in the potential is captured [53, 101]. The scalar power spectra displayed in Figure 4.1 have been evaluated at super-Hubble scales, say, when $k/(aH) \simeq 10^{-5}$, which is typically when the amplitude of the curvature perturbations freeze in\(^1\).

Having obtained the behavior of the background and the modes, we carry out the integrals involved in arriving at the bi-spectrum using the method of adaptive quadrature [107]. Since we shall be focusing on the equilateral limit of the bi-spectrum, we can evolve each of the modes of interest independently and calculate the integrals for each of

\(^1\)Recall that, in the last two chapters, wherein we had compared certain inflationary models with the CMB data, apart from the scalar power spectrum, we had also evaluated the tensor power spectrum and had incorporated it in our analysis. We should add here that the tensor modes are evolved and the corresponding power spectrum is evaluated in the same fashion as the scalar spectrum. Moreover, since it is only the scalar modes that exhibit resonance in oscillatory inflationary potentials, no special considerations need to be paid to the tensors even in cases such as the axion monodromy model. Actually, barring cases wherein extreme departures from slow roll occur (such as, for example, in the punctuated inflationary scenario, see Refs. [77]), the tensor spectra largely prove to be nearly scale independent.
the modes separately. The integrals $G_n$ actually contain a cut-off in the sub-Hubble limit, which is essential for singling out the perturbative vacuum [49, 51, 52]. Numerically, the presence of the cut-off is fortunate since it controls the contributions due to the continuing oscillations that would otherwise occur. Generalizing the cut-off that is often introduced analytically in the slow roll case, we impose a cut-off of the form $\exp\left[-\kappa \frac{k}{(a H)}\right]$, where $\kappa$ is a small parameter. In the previous subsection, we had discussed as to how the integrals need to be carried out from the early time $\eta_i$ when the largest scale is well inside the Hubble radius to the late time $\eta_s$ when the smallest scale is sufficiently outside. In the equilateral configuration of our interest, rather than integrate from $\eta_i$ to $\eta_s$, it suffices to compute the integrals for the modes from the time when each of them satisfy the sub-Hubble condition, say, $k/(a H) = 100$, to the time when they are well outside the Hubble radius, say, when $k/(a H) = 10^{-5}$. In other words, one carries out the integrals exactly over the period the modes are evolved to obtain the power spectrum. The presence of the cut-off ensures that the contributions at early times, i.e. near $\eta_i$, are negligible. Furthermore, it should be noted that, in such a case, the corresponding super-Hubble contribution to $f_{\text{NL}}^{\text{eq}}$ will saturate the bound (4.41) in power law inflation for all modes.

With the help of specific examples, let us now illustrate that, for a judicious choice of $\kappa$, the results that we obtain are largely independent of the upper and the lower limits of the integrals. In fact, we shall demonstrate these points in two steps for the case of the standard quadratic potential (2.1). Firstly, focusing on a specific mode (recall that we are working in the equilateral limit), we shall fix the upper limit of the integral to be the time when $k/(a H) = 10^{-5}$. Evolving the mode from different initial times, we shall evaluate the integrals involved from these initial times to the fixed final time for different values of $\kappa$. This exercise helps us to identify at an optimal value for $\kappa$ when we shall eventually carry out the integrals from $k/(a H) = 100$. Secondly, upon choosing the optimal value for $\kappa$ and integrating from $k/(a H) = 100$, we shall calculate the integrals for different upper limits. For reasons outlined in the previous subsection, it proves to be necessary to consider the contributions to the bi-spectrum due to the fourth and the seventh terms together. Moreover, since the first and the third, and the fifth and the sixth, have similar structure, it turns out to be convenient to club these terms as have discussed before. In Figure 4.2, we have plotted the value of $k^6$ times the different contributions to the bi-spectrum, $G_1 + G_3$, $G_2$, $G_4 + G_7$ and $G_5 + G_6$, as a function of $\kappa$ when the integrals have been carried out from $k/(a H)$ of $10^2$, $10^3$ and $10^4$ for a mode which leaves the Hubble radius around 53 e-folds before the end of inflation. The figure clearly indicates $\kappa = 0.1$ to be a highly suitable value. A larger $\kappa$ leads to a sharper cut-off reducing the value of
Figure 4.2: The quantities $k^6$ times the absolute values of $G_1 + G_3$ (in green), $G_2$ (in red), $G_4 + G_7$ (in blue) and $G_5 + G_6$ (in purple) have been plotted as a function of the cut-off parameter $\kappa$ for a given mode in the case of the conventional, quadratic inflationary potential. Note that these values have been arrived at with a fixed upper limit [viz. corresponding to $k/(aH) = 10^{-5}$] for the integrals involved. The solid, dashed and the dotted lines correspond to integrating from $k/(aH)$ of $10^2$, $10^3$ and $10^4$, respectively. It is clear that the results converge for $\kappa = 0.1$, which suggests it to be an optimal value. While evaluating the bi-spectrum for the other models, we shall choose to work a $\kappa$ of 0.1 and impose the initial conditions as well as carry out the integrals from $k/(aH)$ of $10^2$ (barring the case of the axion monodromy model, as we have discussed in the text). An additional point that is worth noticing is the fact the term $G_4 + G_7$ seems to be hardly dependent of the cut-off parameter $\kappa$. This can possibly be attributed to the dependence of $G_4$ on $\epsilon_2'$ which can be rather small during slow roll, thereby effectively acting as a cut off.

The integrals. One could work with a smaller $\kappa$, in which case, the figure suggests that, one would also need to necessarily integrate from deeper inside the Hubble radius. In Figure 4.3, after fixing $\kappa$ to be 0.1 and, with the initial conditions imposed at $k/(aH) = 10^2$, we have plotted the four contributions to the bi-spectrum for a mode that leaves the Hubble radius at 50 e-folds before the end of inflation as a function of the upper limit of the integrals. It is evident from the figure that the values of the integrals converge quickly once the mode leave the Hubble radius. For efficient numerical integration, as
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Figure 4.3: The quantities $k^6$ times the absolute values of $G_1 + G_5$, $G_2$, $G_4 + G_7$ and $G_5 + G_6$ have been plotted (with the same choice of colors as in the previous figure) as a function of the upper limit of the integrals involved for a given mode in the case of the quadratic potential. Evidently, the integrals converge fairly rapidly to their final values once the mode leaves the Hubble radius. The independence of the results on the upper limit support the conclusions that we had earlier arrived at analytically in the last subsection, \textit{viz.} that the super-Hubble contributions to the bi-spectrum are entirely negligible.

in the case of the power spectrum, we have chosen the super-Hubble limit to correspond to $k/(aH) = 10^{-5}$. We have repeated similar tests for the other models of our interest too. These tests confirm the conclusions that we have arrived at above, indicating the robustness of the numerical methods and procedures that we have adopted.

4.3.4 Comparison with the analytical results in the cases of power law inflation and the Starobinsky model

Before we go on to consider the bi-spectra generated in the inflationary models of our interest, we shall compare the numerical results we obtain with the analytical results that can be arrived at in two cases in the equilateral limit. The first case that we shall consider is power law inflation wherein, as we shall soon outline, the spectral shape of the non-
zero contributions to the bi-spectrum can be easily calculated. The second example that we shall discuss is the Starobinsky model described by the potential (4.3) wherein, under certain conditions, the complete scalar bi-spectrum can be evaluated in the equilateral limit (see Ref. [54]; in this context, also see Refs. [55]).

Let us first consider the case of power law inflation described by the scale factor (4.36) with $\gamma \leq -2$. In such a case, as we have seen, $\epsilon_1$ is a constant and, hence, $\epsilon_2$ and $\epsilon_3$, which involve derivatives of $\epsilon_1$, reduce to zero. Since the contributions due to the fourth and the seventh terms, viz. $G_4(k)$ and $G_7(k)$, depend on $\epsilon_2'$ and $\epsilon_2$, respectively [cf. Eqs. (4.11d) and (4.12)], these terms vanish identically in power law inflation. Note that the modes $v_k$ given by Eq. (4.37) depend only on the combination $k \eta$. Moreover, as $\epsilon_1$ is a constant in power law inflation, we have $f_k \propto v_k/a$. Under these conditions, with a simple rescaling of the variable of integration in the expressions (4.11a), (4.11b), (4.11c), (4.11e) and (4.11f), it is straightforward to show that, in the equilateral limit we are focusing on, the quantities $G_1, G_2, G_3, G_5$ and $G_6$, all depend on the wavenumber as $k^{\gamma+1/2}$. Then, upon making use of the asymptotic form of the modes $f_k$, it is easy to illustrate that the corresponding contributions to the bi-spectrum, viz. $G_1 + G_3$, $G_2$ and $G_5 + G_6$, all behave as $k^{2(\gamma+1)}$. Since the power spectrum in power law inflation is known to have the form $k^{2(\gamma+2)}$ (see, for example, Refs. [105]), the expression (4.34) for $f_{\text{NL}}^\text{eq}$ then immediately suggests that the quantity will be strictly scale invariant for all $\gamma$. In fact, apart from these results, it is also simple to establish the following relation between the different contributions: $G_5 + G_6 = -(3 \epsilon_1/16) (G_1 + G_3)$, a result, which, in fact, also holds in slow roll inflation [54]. In other words, in power law inflation, it is possible to arrive at the spectral dependence of the non-zero contributions to the bi-spectrum without having to explicitly calculate the integrals involved. Further, one can establish that the non-Gaussianity parameter $f_{\text{NL}}^\text{eq}$ is exactly scale independent for any value of $\gamma$. While these arguments does not help us in determining the amplitude of the various contributions to the bi-spectrum or the non-Gaussianity parameter, their spectral shape and the relative magnitude of the above-mentioned terms provide crucial analytical results to crosscheck our numerical code. In Figure 4.4, we have plotted the different non-zero contributions to the bi-spectrum computed using our numerical code and the spectral dependence we have arrived at above analytically for two different values of $\gamma$ in the case of power law inflation. We have also indicated the relative magnitude of the first and the third and the fifth and the sixth terms arrived at numerically. Lastly, we have also illustrated the scale independent behaviour of the non-Gaussianity parameter $f_{\text{NL}}^\text{eq}$ for both the values of $\gamma$. It is clear from the figure that the numerical results agree well with the results and conclusions that we arrived at.
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Figure 4.4: The quantities $k^6$ times the absolute values of the non-zero contributions in the power law case, viz. $G_1 + G_3$, $G_2$ and $G_5 + G_6$, obtained numerically, have been plotted on the left for two different values of $\gamma$ ($\gamma = -2.02$ on top and $\gamma = -2.25$ below), as solid curves with the same choice of colors to represent the different quantities as in the previous two figures. Note that we have followed the same color scheme to represent the differential quantities as in the previous two figures. The dots on these curves are the spectral shape arrived at from the analytical arguments, with amplitudes chosen to match the numerical results at a specific wavenumber. The dots of a different color on the solid purple curves represents $G_5 + G_6$ obtained from its relation to $G_1 + G_3$ discussed in the text. The plots on the right are the non-Gaussianity parameter $f_{\text{NL}}$ associated with the different contributions, arrived at using the numerical code. Note that, as indicated by the analytical arguments, the quantity $f_{\text{NL}}$ corresponding to all the contributions turns out to be strictly scale invariant for both values of $\gamma$.

Let us now turn to the Starobinsky model. As we have discussed earlier, in this case, the change in the slope causes a brief period of fast roll which leads to sharp features in the scalar power spectrum (as we had illustrated in Figure 4.1). It was known that, for certain range of parameters, one could evaluate the scalar power spectrum analytically...
in the Starobinsky model, which matches the actual, numerically computed spectrum exceptionally well [71, 54]. Interestingly, it has been recently shown that, in the equilateral limit, the model allows the analytic evaluation of the scalar bi-spectrum too (see Ref. [54]; in this context, also see Refs. [55]). In Figure 4.5, we have plotted the numerical as well as the analytical results for the functions $G_1 + G_3$, $G_2$, $G_4 + G_7$, and $G_5 + G_6$ for the Starobinsky model. We have plotted for parameters of the model for which the analytical results are considered to be a good approximation [54]. It is evident from the figure that the numerical results match the analytical ones very well. Importantly, the agreement proves to be excellent in the case of the dominant contribution $G_4 + G_7$. A couple of points concerning concerning the numerical results in the case of the Starobinsky model (both in Figure 4.1 wherein we have plotted the power spectrum as well as in Figure 4.5 above containing the bi-spectrum) require some clarification. The derivatives of the potential (4.3) evidently contain discontinuity. These discontinuities need to be smoothened in order for the problem to be numerically tractable. The spectra and the bi-spectra in the Starobinsky model we have illustrated have been computed with a suitable smoothing of the discontinuity, while at the same time retaining a sufficient level of sharpness so that they closely correspond to the analytical results that have been arrived at [54, 55].
Figure 4.5: The quantities $k^6$ times the absolute values of $G_1 + G_3$ (in green), $G_2$ (in red), $G_4 + G_7$ (in blue) and $G_5 + G_6$ (in purple) have been plotted as a function of $k/k_0$ for the Starobinsky model. These plots correspond to the following values of the model parameters: $V_0 = 2.36 \times 10^{-12} M_{pl}^4$, $A_+ = 3.35 \times 10^{-14} M_{pl}^3$, $A_- = 7.26 \times 10^{-15} M_{pl}^3$ and $\phi_0 = 0.707 M_{pl}$. Note that $k_0$ is the wavenumber which leaves the Hubble radius when the scalar field crosses the break in the potential at $\phi_0$. The solid curves represent the analytical expressions that have been obtained recently [54, 55], while the dashed curves denote the numerical results computed using our Fortran code. We should mention that we have also arrived at these results independently using a Mathematica [108] code. We find that the numerical results match the analytical results exceptionally well in the case of the crucial, dominant contribution to the $f_{NL}$, viz. due to $G_4 + G_7$. 
4.4 Results in the equilateral limit

We shall now discuss the bi-spectra arrived at numerically in the various models of our interest. In Figure 4.6, we have plotted the various contributions, viz. $G_1 + G_3$, $G_2$, $G_4 + G_7$ and $G_5 + G_6$ (in the equilateral limit) for the punctuated inflationary scenario driven by the potential (4.1), the quadratic potential (2.1) with the step (2.5), and the axion monodromy model (3.2) which contains oscillations in the inflaton potential. These plots and the one in previous figure clearly point to the fact that it is the combination $G_4 + G_7$ that contributes the most to the scalar bi-spectrum in these cases [53].

In Figure 4.7, we have plotted the quantity $f^\text{eq}_{NL}$ due to the dominant contribution that arises in the various models that we have considered. It is clear from this figure that, while in certain cases $f^\text{eq}_{NL}$ can prove to be a good discriminator, it cannot help in others, and its ability to discriminate depends strongly on the differences in the background dynamics. For instance, the evolution of the first two slow roll parameters are very similar when a step is introduced in either the quadratic potential or a small field model [85]. Hence, it is not surprising that the $f^\text{eq}_{NL}$ behaves in a similar fashion in both these models. Whereas, $f^\text{eq}_{NL}$ proves to be substantially different in punctuated inflation and the Starobinsky model. Recall that, in the Starobinsky model, the first slow roll parameter remains small throughout the evolution. In contrast, it grows above unity for a very short period (leading to a brief interruption of the accelerated expansion) in the punctuated inflationary scenario. It is this departure from inflation that leads to a sharp drop in the power spectrum and a correspondingly sharp rise in the parameter $f^\text{eq}_{NL}$ in punctuated inflation. In fact, this occurs for modes that leave the Hubble radius just before inflation is interrupted [83]. However, note that, $f^\text{eq}_{NL}$ grows with $k$ at large wavenumbers in the Starobinsky model. This can be attributed to the fact that $\epsilon_2$, which determines the contribution due to the fourth term, diverges due to the discontinuity in the second derivative of the potential [55]. Similarly, we find that $f^\text{eq}_{NL}$ is rather large in the axion monodromy model in contrast to the case wherein the conventional quadratic potential is modulated by an oscillatory term. The large value of $f^\text{eq}_{NL}$ that arises in the monodromy model can be attributed to the resonant behavior encountered in the model [98, 53, 101]. In fact, we have also evaluated the $f^\text{eq}_{NL}$ for the case of quadratic potential modulated by sinusoidal oscillations, which too leads to continuing, periodic features in the scalar power spectrum as we had seen in the last chapter. However, we find that the $f^\text{eq}_{NL}$ in such a case proves to be rather small (of the order $10^{-2}$ or so).
Figure 4.6: The set of quantities $k^6 |G_n(k)|$ plotted as in the previous figure with the same choice of colors to represent the different $G_n(k)$. The figures on top, in the middle and at the bottom correspond to punctuated inflation, the quadratic potential with a step and the axion monodromy model, respectively, and they have been plotted for values of the parameters that lead to the best fit to the WMAP data [77, 85, 101]. In the middle figure, the dashed lines correspond to the quadratic potential when the step is not present.
Figure 4.7: A plot of $f_{\text{NL}}^{\text{eq}}$ corresponding to the various models that we have considered. The figure at the top contains the absolute value of $f_{\text{NL}}^{\text{eq}}$, plotted on a logarithmic scale (for convenience in illustrating the extremely large values that arise), in the Starobinsky model and the punctuated inflationary scenario. The inset highlights the growth in $f_{\text{NL}}^{\text{eq}}$ at large wavenumbers in the case of the Starobinsky model, in conformity with the conclusions that have also been arrived at analytically [55]. The figure in the middle contains $f_{\text{NL}}^{\text{eq}}$ for the cases wherein a step has been introduced in a quadratic potential and a small field model. The figure at the bottom corresponds to that of the axion monodromy model. As we had mentioned before, these sets of models lead to scalar power spectra with certain common characteristics. Needless to say, while $f_{\text{NL}}^{\text{eq}}$ is considerably different in the first and the last sets of models, it is almost the same in the case of models with a step. These similarities and differences can be attributed completely to the background dynamics.
4.5 Discussion

In this chapter, we have been interested in examining the power of the non-Gaussianity parameter $f_{\text{NL}}$ to discriminate between various single field inflationary models involving the canonical scalar field. With this goal in mind, using a new numerical code which can efficiently calculate the bi-spectrum for any triangular configuration, we have evaluated the quantity $f_{\text{NL}}^{\text{eq}}$ in a slew of models that generate features in the scalar perturbation spectrum [109]. We find that the amplitude of $f_{\text{NL}}^{\text{eq}}$ proves to be rather different when the dynamics of the background turns out reasonably different, which, in retrospect, need not be surprising at all. For instance, models such as the punctuated inflationary scenario and the Starobinsky model which lead to very sharp features in the power spectrum also lead to substantially large $f_{\text{NL}}$. Such possibilities can aid us discriminate between the models to some extent. We had focused on evaluating the quantity $f_{\text{NL}}$ in the equilateral limit. It will be worthwhile to compute the corresponding values in the other limits, such as the squeezed and the orthogonal limits as well. In particular, it will be interesting to examine if the so-called consistency relation between the non-Gaussianity parameter $f_{\text{NL}}$ between the equilateral and the squeezed limits is valid even in situations wherein extreme deviations from slow roll occur (in this context, see Refs. [53, 110, 111]).

We would like to conclude by highlighting one important point. Having computed the primordial bi-spectrum, the next logical step would be to compute the corresponding CMB bi-spectrum, an issue which we have not touched upon as it is beyond the scope of the current work. While tools seem to be available to evaluate the CMB bi-spectrum based on the first order brightness function, the contribution due to the brightness function at the second order remains to be understood satisfactorily (in this context, see Ref. [59] and the last reference in Ref. [58]). This seems to be an important aspect that is worth investigating closer.