Chapter 3

Newforms of Half-integral Weight

3.1 Introduction

In 1973, G. Shimura introduced the theory of modular forms of half-integral weight and obtained a correspondence between modular forms of half-integral weight and integral weight. After the works of G. Shimura and S. Niwa, T. Shintani gave a correspondence in the reverse direction. In [49, 50], J. -L. Waldspurger proved a remarkable result in showing that the square of the Fourier coefficients of a half-integral Hecke eigenform f is proportional to the special value (at the center) of the L- function of the corresponding Hecke eigenform F of integral weight twisted with a quadratic character, where f maps to F by the Shimura correspondence. In 1980's, W. Kohnen characterized certain subspace, called the Kohnen plus space w.r.t. certain operators and further extended the Shimura correspondence to the plus space. Kohnen also studied the analogous Atkin-Lehner theory of newforms in the plus space on $\Gamma_0(4N)$, where N is odd and square-free and obtained the Shintani lifting which is adjoint to the Shimura-Kohnen map. As a consequence, he obtained the explicit form of the Waldspurger theorem for the newforms in the plus space. Analogous theory of newforms for the full space on $\Gamma_0(4N)$, N odd and square-free was obtained by M. Manickam, B. Ramakrishnan and T. C. Vasudevan in [31]. Manickam in his thesis obtained the theory of newforms for the full space on $\Gamma_0(8N)$, N odd square-free [27, 28].

Let $M = 2^a N$, N is an odd square-free positive integer, a is an integer, $0 \le a \le 2$. Let χ be a real Dirichlet character modulo M and χ is primitive modulo 8 if $a = 2$. By explicit computation of traces, in [45], M. Ueda proved that there exists a Hecke equivariant isomorphism between the spaces $S_{k+1/2}(4M, \chi)$ and $S_{2k}(2M)$. In this chapter, which is a joint work with M. Manickam and B. Ramakrishnan, we set up the theory of newforms for both the Kohnen plus space and the full space on these spaces $S_{k+1/2}(2^{a+2}N,\chi)$, for $0 \leq a \leq 2$ and χ primitive modulo 8 if $a = 2$. As mentioned above, the theory of newforms is known for the spaces when $a = 0$ [17, 18, 31]. If $a = 1$, the theory is known for the full space by the work of Manickam [27, 28] and for the Kohnen plus space by a recent work of M. Ueda and S. Yamana [48].

Using Ueda's isomorphism as a main tool, we establish the theory of newforms for the remaining cases for both the Kohnen plus space and the full space. Using the explicit dimension formulas, we observe that the Shimura maps $\mathscr{S}_{t,16N}$ (see §3.2 for the definition) map the space $S_{k+1/2}(16N)$ into $S_{2k}(4N)$ and thereby we deduce that there is no newform space on $S_{k+1/2}(16N)$. However, we establish the theory of newforms for $S_{k+1/2}(16N,\chi)$, which is compatible with the integral weight theory when χ is a primitive character modulo 8.

3.2 Newforms on the plus space $S^+_{k+1/2}(8N)$

From now onwards, let us assume that N is an odd positive integer. In [17, 18], Kohnen introduced a subspace of $S_{k+1/2}(4N)$ (referred to as the Kohnen plus space) as follows:

$$S^+_{k+1/2}(4N) = \left\{ f \in S_{k+1/2}(4N) \mid a_f(n) = 0 \text{ if } (-1)^k n \equiv 2,3 \pmod 4 \right\}. \tag{3.2.1}$$

In [48], Ueda and Yamana considered the same for $S_{k+1/2}(8N)$ and they defined the plus space by

$$S^+_{k+1/2}(8N) = \left\{ f \in S_{k+1/2}(8N) \mid a_f(n) = 0 \text{ if } (-1)^k n \equiv 2,3 \pmod 4 \right\}. \tag{3.2.2}$$

In this section, we develop the theory of newforms in the Kohnen plus space $S^+_{k+1/2}(8N)$, where N is an (odd) square-free natural number.

3.2.1 W-operators

For $p|2N$, let W_p denote the Atkin-Lehner W-operator on $S_{2k}(2N)$. For $p = 2$, we define the analogous Atkin-Lehner W-operators $W(4)$ on $S_{k+1/2}(4N)$ and $W(8)$ on $S_{k+1/2}(8N,\chi)$ as follows:

$$W(4) = \left(\begin{pmatrix} 4a & b \\ 4Nc & 4 \end{pmatrix},\ 2^{1/2}\, e^{\frac{i\pi}{4}} (Nc\tau + 1)^{1/2} \right),$$

where a, b, c are integers satisfying $4a - Nbc = 1$ and $b \equiv 1 \pmod 4$.

$$W(8) = \left(\begin{pmatrix} 8x & y \\ 8Nw & 8 \end{pmatrix},\ 8^{1/4}\, e^{i\pi/4} (Nw\tau + 1)^{1/2} \right), \tag{3.2.3}$$

where x, y, w are integers such that $y \equiv 1 \pmod 8$, $8x - Nwy = 1$. We also let

$$W_*(4) = \left(\begin{pmatrix} 4u & v \\ 4Nr & 8 \end{pmatrix}, \ 2^{1/2}\ e^{\frac{i\pi}{4}}(Nr\tau + 2)^{1/2} \right),$$

where r, u, v are integers satisfying $8u - Nrv = 1$ and $v \equiv 1 \pmod 8$.

Remark 3.2.1. The W-operators defined above are independent of the choice of the integers $a, b, c, x, y, w, r, u, v$. Since $W_*(4)B(2) = W(8)$ on $S_{k+1/2}(8N, \chi)$, by applying the operator $U(2)$ we see that $W_*(4) = W(8)U(2)$ on $S_{k+1/2}(8N, \chi)$. Also, note that $W_*(4) = W(4)$ on $S_{k+1/2}(4N)$ (see [27, 28] for details). The operator $W(8)$ maps $S_{k+1/2}(8N, \chi)$ into $S_{k+1/2}(8N, \chi\chi_8)$ and $W(8)^2 = I$ on $S_{k+1/2}(8N, \chi)$, where χ is the principal character or $\chi = \chi_8$ (we denote by χ_8, the real quadratic character modulo 8 defined by $\chi_8(n) = \left(\frac{2}{n}\right)$, the extended Jacobi symbol) and I denotes the identity operator.

We now obtain a characterization of $S^+_{k+1/2}(4N)$ and $S^{new}_{k+1/2}(4N)$ (N odd square-free) in terms of the operator $U(2)W(8)$.

Proposition 3.2.1. *Let N be an odd square-free integer.*
(i) If $f \in S_{k+1/2}(4N)$, then

$$f|U(2)W(8) = \chi_8(2k+1)\ 2^{k/2-1/4} f \tag{3.2.4}$$

if and only if $f \in S^+_{k+1/2}(4N)$ and
(ii)

$$f|U(2)W(8) = -\chi_8(2k+1)\ 2^{k-1} f \tag{3.2.5}$$

if and only if $f \in S^{new}_{k+1/2}(4N)$.

Proof. For (i) we refer to [27, 28]. Let us now prove (ii). Let $f \in S^{new}_{k+1/2}(4N)$. Then

$$f|U(4)W(4) = -\chi_8(2k+1)2^{k-1}f.$$

Therefore, using the fact that $f|W(8)^2 = f$, $f|W_*(4) = f|W(8)U(2)$, we have

$$\begin{aligned} f|U(4)W(4) &= -\chi_8(2k+1)2^{k-1}f, \\ f|U(2)W(8)W(8)U(2) &= -\chi_8(2k+1)2^{k-1}f|W(4), \\ f|U(2)W(8)W_*(4) &= -\chi_8(2k+1)2^{k-1}f|W(4). \end{aligned}$$

Now applying $W_*(4)^{-1}$ both sides and using the fact that $W(4)W_*(4)^{-1} = I$ on $S_{k+1/2}(8N)$, we get

$$f|U(2)W(8) = -\chi_8(2k+1)2^{k-1}f.$$

Retracing the above steps, we get the converse. □

3.2.2 Certain operators

In this section we give some general facts and so we assume that $M \geq 1$ is any integer. Let $\xi = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4} \right)$ and $\xi' = \left(\begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, e^{-\pi i/4} \right)$. Then

$$\xi + \xi' : S_{k+1/2}(4M) \longrightarrow S_{k+1/2}(8M), \tag{3.2.6}$$

and on formal Fourier series $\sum a_n q^n$, it transforms as

$$\sum a_n q^n | (\xi + \xi') = \chi_8(2k+1)\sqrt{2} \left(\sum_{(-1)^k n \equiv 0,1 \pmod 4} a_n q^n - \sum_{(-1)^k n \equiv 2,3 \pmod 4} a_n q^n \right). \tag{3.2.7}$$

This observation proves the following lemma.

Lemma 3.2.2. *Let $L_+ := \frac{1}{2}\left(\frac{\chi_8(2k+1)}{\sqrt{2}} (\xi + \xi') + I \right)$, where I is the identity operator. Then L_+ maps the space $S_{k+1/2}(4M)$ into the space $S_{k+1/2}(8M)$. It acts as identity on the plus space $S^+_{k+1/2}(4M)$:*

$$S^+_{k+1/2}(4M) | L_+ = S^+_{k+1/2}(4M).$$

Moreover, if $f = \sum_{n \geq 1} a_f(n) q^n \in S_{k+1/2}(4M)$, then $f|L_+$ has the following Fourier expansion.

$$f | L_+ = \sum_{(-1)^k n \equiv 0,1 \pmod 4} a_f(n) q^n.$$

Proposition 3.2.3. *We have the following assertions.*

(i) The operator L_+ is injective on $S^{new}_{k+1/2}(4M)$, where M is square-free. Moreover, if M is odd and square-free, then L_+ is injective on $S_{k+1/2}(4M)$.

(ii) If $f \in S_{k+1/2}(4M)$, then $f|L_+ T(p^2) = f|T(p^2) L_+$, for any prime p satisfying $(p, 2M) = 1$.

(iii) If $f, g \in S_{k+1/2}(4M)$, then $\langle f|L_+, g \rangle = \langle f, g|L_+ \rangle$.

Proof. Let $F \in S^{new}_{2k}(2M)$ be a normalized newform. Then the theory of newforms on $S_{k+1/2}(4M)$ gives a unique (upto a constant) non-zero newform $f \in S^{new}_{k+1/2}(4M)$, which corresponds to F under the Shimura correspondence. Hence, using Waldspurger's formula together with the fact that $L(F, \chi_D, k) \neq 0$ for some fundamental discriminant

D, $(-1)^k D > 0$ and $(D, 2M) = 1$, we get $a_f(|D|) \neq 0$. This proves the injectivity of L_+ on $S^{new}_{k+1/2}(4M)$. Let M be odd and square-free. Using the facts that

$$L_+ U(4) = U(4) \quad \text{on} \quad S_{k+1/2}(4M)$$

and $U(4)$ is an automorphism of $S_{k+1/2}(4M)$, we get the injectivity of L_+ on $S_{k+1/2}(4M)$. This proves (i). A direct computation gives (*ii*). Using the definition of L_+, we have

$$\langle f|\xi, g\rangle = \langle f, g|\xi'\rangle \quad \text{and} \quad \langle f|\xi', g\rangle = \langle f, g|\xi\rangle, \quad \text{where} \quad f, g \in S_{k+1/2}(4M),$$

from which it follows that $\langle f|L_+, g\rangle = \langle f, g|L_+\rangle$. This completes the proof. □

Shimura-Kohnen map on $S^+_{k+1/2}(8M)$.

Let t be a square-free integer with $(-1)^k t > 0$ and let D $(= t$ or $4t$ according as t is 1 or $2, 3$ modulo 4) be the corresponding fundamental discriminant. The t-th Shimura map on $S_{k+1/2}(4M)$ is defined as

$$f|\mathscr{S}_{t,4M}(\tau) = \sum_{n\geq 1} \left(\sum_{\substack{d|n \\ (d,2M)=1}} \left(\frac{4t}{d}\right) d^{k-1} a_f(|t|n^2/d^2) \right) q^n. \tag{3.2.8}$$

When M is odd, we define the D-th Shimura-Kohnen map $\mathscr{S}^+_{D,M}$ on $S^+_{k+1/2}(8M)$ by

$$f|\mathscr{S}^+_{D,8M}(\tau) = f|\mathscr{S}_{t,8M}|U(2)(\tau), \quad f \in S^+_{k+1/2}(8M). \tag{3.2.9}$$

Since $U(2) : S_{2k}(4M) \to S_{2k}(2M)$, it follows immediately (using the mapping property of the Shimura map $\mathscr{S}_{t,8M}$) that $\mathscr{S}^+_{D,8M}$ maps $S^+_{k+1/2}(8M)$ into $S_{2k}(2M)$.

3.2.3 The plus space $S^+_{k+1/2}(8N)$

In this section, we assume that $N \geq 1$ is an odd square-free integer. Define the space of oldforms in $S^+_{k+1/2}(8N)$ as follows.

$$S^{+,old}_{k+1/2}(8N) = S^{old}_{k+1/2}(4N)|L_+, \tag{3.2.10}$$

and define the space of newforms in $S^+_{k+1/2}(8N)$ as

$$S^{+,new}_{k+1/2}(8N) = S^{new}_{k+1/2}(4N)|L_+.$$

Using Proposition 3.2.3, we get $S^{+,old}_{k+1/2}(8N)$ and $S^{+,new}_{k+1/2}(8N)$ are orthogonal with respect to the Petersson inner product. Hence, we define the Kohnen plus space on $S_{k+1/2}(8N)$ by

$$S^+_{k+1/2}(8N) := S_{k+1/2}(4N)|L_+. \tag{3.2.11}$$

Theorem 3.2.4. *(i) We have the orthogonal direct sum decomposition:*

$$S^{+}_{k+1/2}(8N) = S^{+,new}_{k+1/2}(8N) \oplus S^{+,old}_{k+1/2}(8N),$$

where

$$S^{+,old}_{k+1/2}(8N) = \bigoplus_{\substack{rd|2N \\ d<2N}} S^{+,new}_{k+1/2}(4d)|U(r^2)_{+}, \tag{3.2.12}$$

with the sums over various d's are orthogonal.

(ii) "Multiplicity 1" theorem holds on $S^{+,new}_{k+1/2}(8N)$.

(iii)

$$U(4) : S^{+,new}_{k+1/2}(8N) \longrightarrow S^{new}_{k+1/2}(4N)$$

is an isomorphism.

(iv) (Waldspurger formula): We have

$$\frac{|a_f(4|D|)|^2}{\langle f|U(4), f|U(4)\rangle} = 2^{\nu(N)-1}\frac{(k-1)!}{\pi^k}|D|^{k-1/2}\frac{L(F,\chi_D,k)}{\langle F,F\rangle}, \tag{3.2.13}$$

where D is a fundamental discriminant with $(D,2N)=1$, $\nu(N)$ is the number of prime factors of N and $L(F,\chi_D,k)$ is the special value of the L-function of F twisted with the quadratic character χ_D at $s=k$.

Proof. As in [31], we have

$$S^{old}_{k+1/2}(4N) = \bigoplus_{\substack{rd|N \\ d<N}} S^{new}_{k+1/2}(4d)|U(r^2) \bigoplus_{\substack{d|N \\ rd|2N}} S^{+,new}_{k+1/2}(4d)|U(r^2). \tag{3.2.14}$$

Applying L_+ on both the sides and using Proposition 3.2.3, we have

$$S^{+,old}_{k+1/2}(8N) = \bigoplus_{\substack{rd|N \\ d<N}} S^{+,new}_{k+1/2}(8d)|U(r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)L_{+}.$$

Further, the nature of the operator $L+$ and the multiplicity one theorem on $S^{new}_{k+1/2}(4N)$ give the validity of the multiplicity one result on $S^{+,new}_{k+1/2}(8N)$.
If $f \in S^{+,new}_{k+1/2}(8N)$, then $f|U(4) \in S_{k+1/2}(4N)$. In addition, if f is a Hecke eigenform, then $f|U(4)$ is also a Hecke eigenform which is not in the plus space. Therefore, by the theory of newforms on $S_{k+1/2}(4N)$, $f|U(4) \in S^{new}_{k+1/2}(4N)$. Thus, $U(4) : S^{+,new}_{k+1/2}(8N) \to S^{new}_{k+1/2}(4N)$. Also for any $g \in S_{k+1/2}(4N)$, we know that $g|L_+U(4) = g|U(4)$. From this we get (iii).

We now derive the Waldspurger formula. Let F be a normalized newform in $S_{2k}^{new}(2N)$. Then by the work of [31], there exits a newform $g \in S_{k+1/2}^{new}(4N)$ which corresponds to F under the Shimura correspondence. Let $f = g|L_+$. Then $f|U(4) = g$. Therefore, the Waldspurger formula obtained in [32, Corollary 1], and the fact that $g = f|U(4)$ give the required formula (iv). □

3.3 Remark about a result of Ueda and Yamana

In this section, we make an observation about Proposition 4 of [48]. It says that the operator $\tilde{Y}(8)$ (defined on page 4 of [48]) maps the space $S_{k+1/2}(8N)$ into $S_{k+1/2}^{+}(8N)$ and further, $4^{-1}\chi_8(2k+1)\tilde{Y}(8)$ is an involution on $S_{k+1/2}^{+}(8N)$. In the following, we first show that $\tilde{Y}(8) = \epsilon\ 2^{-3k/2+9/4}U(8)W(8)$ on $S_{k+1/2}(8N)$, where ϵ is a fourth root of unity. Next, we show that $U(8)W(8)$ does not preserve the space $S_{k+1/2}^{+,new}(8N)$. As a consequence, Proposition 4 of [48] can not be true.

We begin with the following lemma, which follows from a straightforward computation.

Lemma 3.3.1.

$$\tilde{Y}(8) = \epsilon 2^{-3k/2+9/4}U(8)W(8) \quad on \quad S_{k+1/2}(8N).$$

We now prove that $U(8)W(8)$ is not a constant times the identity function on $S_{k+1/2}^{+,new}(8N)$. Let $f \in S_{k+1/2}^{+,new}(8N)$ and assume that $f|U(8)W(8) = \lambda f$ for some constant λ. Let $g = f|U(4)$. Then $g \in S_{k+1/2}^{new}(4N)$. Now by Proposition 3.2.1, we have

$$-\chi_8(2k+1)2^{k-1}g = g|U(2)W(8) = f|U(8)W(8) = \lambda f \in S_{k+1/2}^{+,new}(8N), \tag{3.3.1}$$

which is a contradiction.

3.4 Newforms on the plus space $S_{k+1/2}^{+}(16N)$

In this section we extend the results of the previous section to forms on $\Gamma_0(16N)$ (N is odd and square-free.) We need the following orthogonal decomposition of $S_{k+1/2}(8N)$ (in [27] and [28] a slightly different decomposition was given).

$$S_{k+1/2}(8N) = S_{k+1/2}^{new}(8N) \oplus S_{k+1/2}^{old}(8N), \tag{3.4.1}$$

where the space of oldforms $S^{old}_{k+1/2}(8N)$ has the following decomposition.

$$\begin{aligned} S^{old}_{k+1/2}(8N) = & \bigoplus_{\substack{d<N \\ rd|N}} S^{new}_{k+1/2}(8d)|U(r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(8d)|U(r^2) \bigoplus_{rd|N} S^{new}_{k+1/2}(4d)|U(r^2) \\ & \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)L_+. \end{aligned} \tag{3.4.2}$$

We need to show only that for a fixed divisor $d|N$, the sum $S^{+,new}_{k+1/2}(4d)+S^{+,new}_{k+1/2}(4d)|U(4)+S^{+,new}_{k+1/2}(4d)|U(4)L_+$ is direct. For some constants α, β, γ and a newform $f \in S^{+,new}_{k+1/2}(4d)$, if we have

$$\alpha f + \beta f|U(4) + \gamma f|U(4)L_+ = 0,$$

then applying the operator $U(4)$, we get

$$\alpha f|U(4) = -(\beta+\gamma) f|U(16),$$

from which we conclude that $\alpha = 0$. Since the sum $S^{+,new}_{k+1/2}(4d)|U(4) \oplus S^{+,new}_{k+1/2}(4d)|U(4)L_+$ is a direct sum, it follows that $\beta = \gamma = 0$. This proves the required direct sum.

We define the space $S^{+,new}_{k+1/2}(16N)$ as follows.

$$S^{+,new}_{k+1/2}(16N) = S^{new}_{k+1/2}(8N)|L_+. \tag{3.4.3}$$

We also define $S^{+,old}_{k+1/2}(16N)$ as follows.

$$\begin{aligned} S^{+,old}_{k+1/2}(16N) = & \sum_{\substack{rd|N \\ d<N}} S^{new}_{k+1/2}(8d)|U(r^2)L_+ + \sum_{rd|N} S^{+,new}_{k+1/2}(8d)|U(r^2) \\ & + \sum_{rd|N} S^{+,new}_{k+1/2}(8d)|U(4r^2)B(4) + \sum_{rd|N} S^{+,new}_{k+1/2}(4d)|U(r^2) \\ & + \sum_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)B(4) + \sum_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)L_+ \end{aligned} \tag{3.4.4}$$

Since L_+ and $B(4)$ commute with the Hecke operators $T(p^2)$ for $(p, 2N) = 1$, each of the distinct eigenspaces of the decomposition of $S^{+,old}_{k+1/2}(16N)$ is orthogonal to $S^{+,new}_{k+1/2}(16N)$. Thus, the following orthogonal direct sum defines the plus space:

$$S^{+}_{k+1/2}(16N) = S^{+,new}_{k+1/2}(16N) \oplus S^{+,old}_{k+1/2}(16N). \tag{3.4.5}$$

Theorem 3.4.1. *Let t be a square-free integer with $(-1)^k t > 0$. Then the t-th Shimura map $\mathscr{S}_{t,16N}$ maps the space $S^{+}_{k+1/2}(16N)$ into the space $S_{2k}(4N)$.*

Proof. Let us first show that $\mathscr{S}_{t,16N}$ maps the newforms space $S^{+,new}_{k+1/2}(16N)$ into the newforms space $S^{new}_{2k}(4N)$. Since $U(4)$ is zero on $S^{+,new}_{k+1/2}(16N)$, the plus space property shows that $\mathscr{S}_{t,16N}$ is non-zero on $S^{+,new}_{k+1/2}(16N)$ only when $t \equiv 1 \pmod 4$. It is known that $\mathscr{S}_{t,16N}$ maps $S^{+,new}_{k+1/2}(16N)$ into the space $S_{2k}(8N)$. By the theory of newforms, and the definition of $S^{+,new}_{k+1/2}(16N)$, the image is contained in the old class generated by $S^{new}_{2k}(4N)$. So, the image belongs to $S^{new}_{2k}(4N) \oplus S^{new}_{2k}(4N)|B(2)$. But $f|U(4) = 0$ if $f \in S^{+,new}_{k+1/2}(16N)$ implies that $f|\mathscr{S}_{t,16N}U(2) = 0$. Therefore, for a newform $F \in S^{new}_{2k}(4N)$, we have $(\alpha F + \beta F|B(2))|U(2) = 0$, which implies that $\beta = 0$. Hence, $f|\mathscr{S}_{t,16N} = F \in S^{new}_{2k}(4N)$. This proves that $\mathscr{S}_{t,16N}$ maps $S^{+,new}_{k+1/2}(16N)$ into the space $S^{new}_{2k}(4N)$. Next we consider the case of oldforms. We have

$$\begin{aligned} f|U(r^2)|\mathscr{S}_{t,16N} &= f|\mathscr{S}_{t,16N}|U(r), \ r|N \\ f|U(4)B(4)|\mathscr{S}_{t,16N} &= f|\mathscr{S}_{t,16N}|U(2)B(2), \quad f \in S^{+,old}_{k+1/2}(16N). \end{aligned} \tag{3.4.6}$$

Let $rd|N$, $d < N$. Since $U(4) = 0$ on $S^{new}_{k+1/2}(8d)$, $S^{new}_{k+1/2}(8d)|L_+$ contains forms whose n-th Fourier coefficient is non-zero only when $(-1)^k n \equiv 1 \pmod 4$. This implies that

$$S^{new}_{k+1/2}(8d)|U(r^2)L_+|\mathscr{S}_{t,16N} = S^{new}_{k+1/2}(8d)|U(r^2)|\mathscr{S}_{t,16N} \in S_{2k}(4N),$$

when $t \equiv 1 \pmod 4$, and when $t \equiv 2, 3 \pmod 4$, the image is zero. Next, when $rd|N$ and $f \in S^{+,new}_{k+1/2}(8d)$, using (3.4.6), it follows that both $f|U(r^2)|\mathscr{S}_{t,16N}$ and $f|U(4r^2)B(4)|\mathscr{S}_{t,16N}$ belong to the space $S_{2k}(4N)$. By a similar reasoning, we see that $f|U(r^2)|\mathscr{S}_{t,16N}$, $f|U(4r^2)B(4)|\mathscr{S}_{t,16N}$ and $f|U(4r^2)L_+|\mathscr{S}_{t,16N}$ belong to the space $S_{2k}(4N)$, where $rd|N$ and $f \in S^{+,new}_{k+1/2}(4d)$. This completes the proof. □

Remark 3.4.1. If $t \equiv 1 \pmod 4$ with $(-1)^k t > 0$, from the above theorem, we observe that the Shimura-Kohnen map $\mathscr{S}^{+}_{t,16N}$ on the plus space $S^{+}_{k+1/2}(16N)$ is the same as the Shimura map $\mathscr{S}_{t,16N}$.

The main results of this section is presented in the following theorem.

Theorem 3.4.2. *(i) The spaces $S^{+,new}_{k+1/2}(16N)$ and $S^{new}_{2k}(4N)$ are isomorphic under a linear combination of Shimura maps. In particular, 'multiplicity 1' theorem holds good in the space of newforms $S^{+,new}_{k+1/2}(16N)$.*

(ii) One has

$$\begin{aligned} S^{+,old}_{k+1/2}(16N) = & \bigoplus_{\substack{rd|N \\ d<N}} S^{new}_{k+1/2}(8d)|U(r^2)L_+ \bigoplus_{rd|N} S^{+,new}_{k+1/2}(8d)|U(r^2) \\ & \bigoplus_{rd|N} S^{+,new}_{k+1/2}(8d)|U(4r^2)B(4) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(r^2) \\ & \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)B(4) \bigoplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(4r^2)L_+, \end{aligned} \tag{3.4.7}$$

where the sums over various d's are orthogonal.

(iii) There is a linear combination of Shimura maps which is an isomorphism between the spaces $S^{+}_{k+1/2}(16N)$ and $S_{2k}(4N)$, mapping the space of newforms to newforms and the space of oldforms to oldforms.

Proof. We have already shown that for a square-free $t \equiv 1 \pmod 4$, $(-1)^k t > 0$, $\mathscr{S}_{t,16N}$ maps the newforms space $S^{+,new}_{k+1/2}(16N)$ into the newforms space $S^{new}_{2k}(4N)$. Moreover, as L_+ is injective on $S^{new}_{k+1/2}(8N)$, by the definition of $S^{+,new}_{k+1/2}(16N)$ and from the theory of newforms in $S_{k+1/2}(8N)$, it follows that the spaces $S^{+,new}_{k+1/2}(16N)$ and $S^{new}_{k+1/2}(4N)$ have equal dimensions. Therefore, it follows that there is a linear combination of the Shimura maps $\mathscr{S}_{t,16N}$ which is an isomorphism between these spaces. This proves (i).
We now prove that the decomposition is direct. We use inductive method to obtain this decomposition. Note that the direct sums over various d's are orthogonal as they belong to different eigenspaces. We observe that there exists a square-free t, $(t, 2N) = 1$ with $t \equiv 1 \pmod 4$ and $(-1)^k t > 0$ such that $a_f(|t|) \neq 0$, for a newform $f \in S^{+,new}_{k+1/2}(4d)$, where $d|N$. Such property is also true in $S^{new}_{k+1/2}(4d)$ and $S^{new}_{k+1/2}(8d)$, for $d|N$. Since the operator L_+ lifts the spaces $S^{new}_{k+1/2}(4d)$ and $S^{new}_{k+1/2}(8d)$ into $S^{+,new}_{k+1/2}(8d)$ and $S^{+,new}_{k+1/2}(16d)$ respectively, we have the validity of the said property for newforms in $S^{+,new}_{k+1/2}(8d)$ and $S^{+,new}_{k+1/2}(16d)$, $d|N$. Using this, we obtain the following facts.

$$S^{+,new}_{k+1/2}(4d) \cap S^{+,new}_{k+1/2}(4d)|U(4)B(4) = \{0\}, \qquad d|N, \tag{3.4.8}$$

$$S^{+,new}_{k+1/2}(8d) \cap S^{+,new}_{k+1/2}(8d)|U(4)B(4) = \{0\}, \qquad d|N. \tag{3.4.9}$$

Now we prove that

$$S^{+,new}_{k+1/2}(4d)|U(4)L_+ \cap S^{+,new}_{k+1/2}(4d)|U(4)B(4) = \{0\}, \qquad d|N \tag{3.4.10}$$

Let $f \in S^{+,new}_{k+1/2}(4d)$ be a newform such that $f|U(4)L_+ = \mathbb{C}f|U(4)B(4)$. Applying $U(4)$ both sides, we get $f|U(16) = f|U(4)$. Then by applying the Shimura map, it will lead to

$\mathbb{C}F|U(4) = F|U(2)$, a contradiction, where $F \in S_{2k}(4d)$ is the corresponding newform under the himura map. Now using the Shimura map $S_{t,16N}$ and the theory of newforms of integral weight, we obtain the following fact.

$$S^{+,new}_{k+1/2}(4d)|U(r_1^2) \cap S^{+,new}_{k+1/2}(4d)|U(r_2^2) = \{0\}, \qquad d|4N, r_1, r_2|N, r_1 \neq r_2. \tag{3.4.11}$$

Then the direct sums follows easily using induction on the number of prime factors of N. The fact that the sum

$$S^{+,new}_{k+1/2}(4d) + S^{+,new}_{k+1/2}(4d)|U(4)B(4) + S^{+,new}_{k+1/2}(4d)|U(4)L_+,$$

for each $d|N$ is a direct sum follows in a similar way as was done at the beginning of this section. This completes the proof. □

3.5 Newform theory on $S_{k+1/2}(16N)$

In this section, we extend the theory of newforms to the space $S_{k+1/2}(16N)$, where N is an odd square-free positive integer. Note that the space $S_{k+1/2}(8N)$ is isomorphic to $S_{k+1/2}(8N, (\frac{2}{\cdot}))$, isomorphism given by the operator $W(8)$. We now define the space of oldforms in $S_{k+1/2}(16N)$.

$$\begin{aligned} S^{old}_{k+1/2}(16N) = &\sum_{rd|N} \left(S^{+,new}_{k+1/2}(4d) + S^{+,new}_{k+1/2}(4d)|U(4) + S^{+,new}_{k+1/2}(4d)|U(4)L_+\right)|U(r^2) \\ &+ \sum_{rd|N} \left(S^{+,new}_{k+1/2}(4d)|U(8)B(2) + S^{+,new}_{k+1/2}(4d)|B(4) + S^{+,new}_{k+1/2}(4d)|U(4)B(4)\right)|U(r^2) \\ &+ \sum_{rd|N} \left(S^{new}_{k+1/2}(4d) + S^{new}_{k+1/2}(4d)|B(4) + S^{new}_{k+1/2}(4d)|U(2)B(2)\right)|U(r^2) \\ &+ \sum_{rd|N} \left(S^{+,new}_{k+1/2}(8d) + S^{new}_{k+1/2}(8d) + S^{+,new}_{k+1/2}(16d)\right)|U(r^2). \end{aligned} \tag{3.5.1}$$

We shall show that the above is a direct sum. Clearly, by using the theory of newforms on $S^{+}_{k+1/2}(4d)$, $S_{k+1/2}(4d)$, $S^{+}_{k+1/2}(8d)$, $S_{k+1/2}(8d)$ and $S^{+}_{k+1/2}(16d)$, where $d|N$, the eigenclasses generated by the three newform spaces $S^{+,new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(4d)$ and $S^{new}_{k+1/2}(8d)$ form a direct sum. Note that the eigenclasses of $S^{+,new}_{k+1/2}(8d)$ and $S^{+,new}_{k+1/2}(16d)$ belong to the eigenclasses of $S^{new}_{k+1/2}(4d)$ and $S^{new}_{k+1/2}(8d)$ respectively. So, it is enough to show the direct sum within each of the eigenclasses.

In the following cases, let d be a positive divisor of N such that $rd|N$.

Case (i): We first consider the eigenclass $S^{+,new}_{k+1/2}(4d)$. Let $f \in S^{+,new}_{k+1/2}(4d)$ be an eigenform such that

$$\alpha_1 f + \alpha_2 f|U(4) + \alpha_3 f|U(4)L_+ + \alpha_4 f|U(8)B(2) + \alpha_5 f|B(4) + \alpha_6 f|U(4)B(4) = 0, \quad (3.5.2)$$

where $\alpha_i \in \mathbb{C}$. Applying $U(4)$ to the above equation, we get $\alpha_5 = 0$. Then comparing the n^{th} Fourier coefficients, where $(-1)^k n \equiv 3 \pmod 4$, we get $\alpha_2 = 0$. Now comparing the n^{th} coefficients, where $n \equiv 2 \pmod 4$, we get $\alpha_4 = 0$. Thus the above equation reduces to the following equation

$$\alpha_1 f + \alpha_3 f|U(4)L_+ + \alpha_6 f|U(4)B(4) = 0. \quad (3.5.3)$$

Applying $U(4)$ to the above equation, we get $(\alpha_1 f - \alpha_6 f)|U(4) = -\alpha_3 f|U(16)$. This implies that $\alpha_3 = 0$. Thus, we get $\alpha_1 f + \alpha_6 f|U(4)B(4) = 0$. Now comparing the n^{th} coefficients, where $(-1)^k n \equiv 1 \pmod 4$, we get $\alpha_1 = 0 = \alpha_6$. This proves the direct sum within the eigenclass $S^{+,new}_{k+1/2}(4d)$.

Case (ii): We consider the eigenclass $S^{new}_{k+1/2}(4d)$. By comparing the Fourier coefficients and using the fact that there always exists n such that $a_f(n) \neq 0$ with $(-1)^k n \equiv 3 \pmod 4$, where $f = \sum_{n\geq 1} a_f(n) q^n \in S^{new}_{k+1/2}(4d)$, we get the following direct sum.

$$S^{new}_{k+1/2}(4d)|U(r^2) \oplus S^{new}_{k+1/2}(4d)|B(4)U(r^2) \oplus S^{new}_{k+1/2}(4d)|U(2)B(2)U(r^2) \oplus S^{+,new}_{k+1/2}(8d)|U(r^2).$$

Case (iii): Since $S^{new}_{k+1/2}(8d)|L_+ = S^{+,new}_{k+1/2}(16d)$, we see that

$$S^{new}_{k+1/2}(8d)|U(r^2) \cap S^{+,new}_{k+1/2}(16d)|U(r^2) = \{0\}.$$

This completes the proof of the direct sum decomposition of $S^{old}_{k+1/2}(16N)$.

Since the spaces $S^{+,new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(4d)$, $S^{+,new}_{k+1/2}(8d)$, $S^{new}_{k+1/2}(8d)$ and $S^{+,new}_{k+1/2}(16d)$ are isomorphic (under the Shimura correspondence) to the spaces $S^{new}_{2k}(d)$, $S^{new}_{2k}(2d)$, $S^{new}_{k+1/2}(2d)$, $S^{new}_{2k}(4d)$ and $S^{new}_{2k}(4d)$ respectively, we see that

$$\begin{aligned}\dim S^{old}_{k+1/2}(16N) &= \sum_{rd|N} \left(6\dim S^{new}_{2k}(d) + 4\dim S^{new}_{2k}(2d) + 2\dim S^{new}_{2k}(4d)\right)\\ &= 2\sum_{rd|N} \left(3\dim S^{new}_{2k}(d) + 2\dim S^{new}_{2k}(2d) + \dim S^{new}_{2k}(4d)\right) \quad (3.5.4)\\ &= 2\dim S_{2k}(4N).\end{aligned}$$

Let us now compute the dimensions of the spaces $S_{2k}(4N)$ and $S_{k+1/2}(16N)$. Using [33], we have

$$\dim S_{2k}(4N) = \frac{2k-1}{12} 4N \prod_{p|2N}\left(1+\frac{1}{p}\right) - \frac{3}{2}2^{\nu(N)} = \frac{(2k-1)}{2}\prod_{p|N}(p+1) - 3\cdot 2^{\nu(N)-1}, \quad (3.5.5)$$

where $\nu(N)$ is the number of prime factors of N. Now, using [7], we get

$$\begin{aligned}\dim S_{k+1/2}(16N) &= \frac{2k-1}{24}16N\prod_{p|2N}\left(1+\frac{1}{p}\right) - \frac{\zeta(k,16N,1)}{2}\prod_{p|N}\lambda(r_p,s_p,p)\\ &= (2k-1)\prod_{p|N}(p+1) - 3\cdot 2^{\nu(N)}.\end{aligned} \tag{3.5.6}$$

(In the above we have used the dimension formula as given in [38][Theorem 1.56, p.16]. Note that $\zeta(k,16N,1) = \lambda(r_2,s_2,2) = 6$ and $\lambda(r_p,s_p,p) = 2$, for $p|N$.) Equations (3.5.5) and (3.5.6) imply that $\dim S_{k+1/2}(16N) = 2\dim S_{2k}(4N)$. However, from (3.5.4), $\dim S^{old}_{k+1/2}(16N) = 2\dim S_{2k}(4N)$. Therefore, it follows that $S^{new}_{k+1/2}(16N) = \{0\}$.

Summarizing, we have proved the following theorem.

Theorem 3.5.1. *(i) For N odd and square-free, the space $S_{k+1/2}(16N)$ contains no newforms in it and the whole space is obtained upon duplicating forms in $S_{k+1/2}(8N)$.*

(ii) The space $S_{k+1/2}(16N)$ has the following decomposition:

$$\begin{aligned}&S_{k+1/2}(16N)\\ &= \oplus_{rd|N}\Big(S^{+,new}_{k+1/2}(4d)\oplus S^{+,new}_{k+1/2}(4d)|U(4)\oplus S^{+,new}_{k+1/2}(4d)|U(4)L_+\oplus S^{+,new}_{k+1/2}(4d)|U(8)B(2)\\ &\qquad \oplus S^{+,new}_{k+1/2}(4d)|B(4)\oplus S^{+,new}_{k+1/2}(4d)|U(4)B(4)\Big)\,|U(r^2)\\ &\oplus\oplus_{rd|N}\Big(S^{new}_{k+1/2}(4d)\oplus S^{new}_{k+1/2}(4d)|B(4)\oplus S^{new}_{k+1/2}(4d)|U(2)B(2)\oplus S^{+,new}_{k+1/2}(8d)\Big)\,|U(r^2)\\ &\qquad \oplus\oplus_{rd|N}\Big(S^{new}_{k+1/2}(8d)\oplus S^{+,new}_{k+1/2}(16d)\Big)\,|U(r^2).\end{aligned} \tag{3.5.7}$$

(iii) For square-free integer t with $(-1)^k t > 0$, the Shimura map $\mathscr{S}_{t,16N}$ has the following mapping property:

$$\mathscr{S}_{t,16N} : S_{k+1/2}(16N) \longrightarrow S_{2k}(4N).$$

There is a linear combination of Shimura maps which is an isomorphism between the spaces $S_{k+1/2}(16N)$ and $S_{2k}(4N)$

3.6 Newform theory on $S_{k+1/2}(16N,\chi_8)$

In this section, we study the theory of newforms on $S_{k+1/2}(16N,\chi_8)$, where χ_8 is the even quadratic character modulo 8, and N is an odd square-free positive integer. Define the space of oldforms in $S_{k+1/2}(16N,\chi_8)$ as follows.

$$\begin{aligned}S^{old}_{k+1/2}(16N,\chi_8) = &\sum_{rd|N}\left(S^{+,new}_{k+1/2}(4d)|B(2) + S^{+,new}_{k+1/2}(4d)|U(2)\right)U(r^2)\\ &+\sum_{rd|N}\left(S^{+,new}_{k+1/2}(4d)|U(8) + S^{+,new}_{k+1/2}(4d)|U(8)W(8)B(2)\right)U(r^2)\\ &+\sum_{rd|N}\left(S^{new}_{k+1/2}(4d)|U(2) + S^{new}_{k+1/2}(4d)|B(2)\right)U(r^2)\\ &+\sum_{rd|N} S^{+,new}_{k+1/2}(8d)|B(2)U(r^2) + \sum_{rd|N} S^{new}_{k+1/2}(8d)|B(2)U(r^2)\\ &+\sum_{rd|N} S^{new}_{k+1/2}(8d)|W(8)U(r^2) + \sum_{rd|N, d<N} S^{new}_{k+1/2}(16d,\chi_8)|U(r^2).\end{aligned} \tag{3.6.1}$$

We shall show that the above is a direct sum. As before, we only have to show the direct sum within each of the eigenclasses.

Case (i): We consider the eigenclass of $S^{+,new}_{k+1/2}(4d)$. Let $f \in S^{+,new}_{k+1/2}(4d)$ be an eigenform and suppose that

$$\alpha_1 f|B(2) + \alpha_2 f|U(2) + \alpha_3 f|U(8) + \alpha_4 f|U(8)W(8)B(2) = 0. \tag{3.6.2}$$

Comparing the n^{th} Fourier coefficients when $(-1)^k n \equiv 3 \pmod 4$, we see that $\alpha_3 = 0$. Now applying the operator $U(2)W_*(4)$ to the above equation and using the facts that $W(8)W_*(4) = U(2)$ on $S_{k+1/2}(8d)$, $W_*(4) = W(4)$ on $S_{k+1/2}(4d)$, $U(4) = 2^k W(4)$ on $S^+_{k+1/2}(4d)$ and $W(4)^2$ is the identity operator on $S_{k+1/2}(4d)$, we conclude that $\alpha_2 = 0$. Thus, the above equation reduces to

$$2^{-k}\alpha_1 f|U(4) + \alpha_4 f|U(16) = 0.$$

The above equality is true only when $\alpha_1 = 0 = \alpha_4$.

Case (ii): We consider the eigenclass of $S^{new}_{k+1/2}(4d)$. Let $f = \sum_{n\geq 1} a_f(n)q^n \in S^{new}_{k+1/2}(4d)$ be an eigenform such that

$$\alpha_1 f|U(2) + \alpha_2 f|B(2) + \alpha_3 f|L_+B(2) = 0. \tag{3.6.3}$$

Applying the operator $U(2)$ on both the sides, we get $\alpha_3 = 0$. Now comparing the odd Fourier coefficients, we see that $a_f(n) = 0$ for $n \equiv 2 \pmod 4$. This implies that f is in the plus space, giving a contradiction. Hence $\alpha_1 = 0$, which implies that $\alpha_2 = 0$.

Case (iii): We consider the eigenclass of $S^{new}_{k+1/2}(8d)$. By applying the operator $U(2)$ to both the spaces $S^{new}_{k+1/2}(8d)|B(2)$ and $S^{new}_{k+1/2}(8d)|W(8)$ and using the fact that $W(8)U(2) =$

$W_*(4)$ maps $S_{k+1/2}(8d)$ into $S_{k+1/2}(4d)$, we get $S^{new}_{k+1/2}(8d)|B(2) \cap S^{new}_{k+1/2}(8d)|W(8) = \{0\}$. We have thus shown that the following decomposition of the space of oldforms in $S_{k+1/2}(16N, \chi_8)$ is direct.

$$\begin{aligned} S^{old}_{k+1/2}(16N, \chi_8) = &\oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|B(2)U(r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(2r^2) \\ &\oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(8r^2) \oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(4d)|U(8)W(8)B(2)U(r^2) \\ &\oplus \oplus_{rd|N} S^{new}_{k+1/2}(4d)|U(2r^2) \oplus \oplus_{rd|N} S^{new}_{k+1/2}(4d)|B(2)U(r^2) \\ &\oplus \oplus_{rd|N} S^{+,new}_{k+1/2}(8d)|B(2)U(r^2) \oplus \oplus_{rd|N} S^{new}_{k+1/2}(8d)|B(2)U(r^2) \\ &\oplus \oplus_{rd|N} S^{new}_{k+1/2}(8d)|W(8)U(r^2) \oplus \oplus_{\substack{rd|N \\ d<N}} S^{new}_{k+1/2}(16d, \chi_8)|U(r^2). \end{aligned} \tag{3.6.4}$$

Define the space of newforms in $S_{k+1/2}(16N, \chi_8)$ to be the orthogonal complement (with respect to the Petersson scalar product) of $S^{old}_{k+1/2}(16N, \chi_8)$ in $S_{k+1/2}(16N, \chi_8)$. It is already known that the spaces $S^{+,new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(4d)$, $S^{new}_{k+1/2}(8d)$ are isomorphic (respectively) to $S^{new}_{2k}(d)$, $S^{new}_{2k}(2d)$, $S^{new}_{2k}(4d)$. Using induction on the number of prime factors of N, it follows that the space $S^{new}_{k+1/2}(16d, \chi_8)$ is isomorphic to $S^{new}_{2k}(8d)$, $d|N$ and $d < N$. Now, comparing the dimension of the space $S^{old}_{2k}(8N)$, we see that the spaces $S^{old}_{k+1/2}(16N, \chi_8)$ and $S^{old}_{2k}(8N)$ have equal dimension. In [45], Ueda has shown that the spaces $S_{k+1/2}(16N, \chi_8)$ and $S_{2k}(8N)$ are isomorphic when N is odd and square-free. Therefore, combining all these facts, it follows that the space $S^{new}_{k+1/2}(16N, \chi_8)$ is isomorphic to $S^{new}_{2k}(8N)$.

Theorem 3.6.1. *Let N be an odd square-free natural number and let χ_8 denote the even quadratic character modulo 8. Then the space $S_{k+1/2}(16N, \chi_8)$ can be decomposed as*

$$S_{k+1/2}(16N, \chi_8) = S^{new}_{k+1/2}(16N, \chi_8) \oplus S^{old}_{k+1/2}(16N, \chi_8),$$

where the above directsum is orthogonal with respect to the Petersson product and the decomposition of $S^{old}_{k+1/2}(16N, \chi_8)$ is given by (3.6.4). Moreover, the spaces $S^{new}_{k+1/2}(16N, \chi_8)$ and $S^{new}_{2k}(8N)$ are isomorphic under a linear combination of Shimura maps.