Chapter 2

Generalized Modular Functions

2.1 Introduction

In this chapter we discuss some properties of exponents of q-product expansions of certain class of generalized modular functions (GMF) on the Hecke congruence subgroup $\Gamma_0(N)$. The results of this chapter are contained in a joint work with W. Kohnen [22].

Let f be a non-zero GMF. Then by a theorem of Eholzer and Skoruppa [9], each GMF f has a product expansion

$$f(z) = cq^h \prod_{n\geq 1}(1-q^n)^{c(n)} \qquad (0<|q|<\epsilon), \quad q=e^{2\pi i z}, \quad z\in\mathcal{H},$$

where $h \in \mathbb{Z}$ and c, $c(n)(n \geq 1)$ are uniquely determined complex numbers.

It was proved in [21] that for each square-free integer $N \geq 11$, one can find a GMF f on $\Gamma_0(N)$ such that f has no zeros on $\mathcal{H}$ and the q-exponents $c(n)$ $(n \geq 1)$ take infinitely many different values. This result and the proof given are actually easily seen to be valid for arbitrary integer $N \geq 11$. Kohnen proved in [19] that for any non-constant GMF f with empty divisor, the q-exponents $c(n)$ $(n \geq 1)$ take infinitely many different values. The first result of the chapter sharpens the above statement under certain conditions on f. Let $\text{div}(f)$ denote the divisor of f, i.e., the set of zeros and poles of f in $\mathcal{H}$ and at all cusps. Our second result shows that under the hypothesis that the divisor of f empty, $c(n)$ $(n \geq 1)$ change signs infinitely often, provided that $c(n)$ are real numbers.

2.2 Preliminaries

Definition 2.2.1. (Generalized modular function) A generalized modular function f on $\Gamma_0(N)$ is a holomorphic function on $\mathcal{H}$, meromorphic at cusps such that

$$f\left(\frac{az+b}{cz+d}\right) = \chi(\gamma) f(z),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, where $\chi : \Gamma_0(N) \to \mathbb{C}^\times$ is a homomorphism with $\chi(\gamma) = 1$ for γ parabolic with trace $= 2$.

M. I. Knopp and G. Mason [15] introduced the notion of a generalized modular function. We abbreviate GMF for a generalized modular function. For further details on GMF, we refer to [15].

Let $g = \sum_{n\geq 1} b(n) q^n$ be a cusp form on $\Gamma_0(N)$. Then the L-series associated to this cusp form is given by

$$L(g,s) = \sum_{n\geq 1} \frac{b(n)}{n^s}.$$

It is known that $L(g,s)$ can be analytically continued over the whole s-plane to an entire function.

2.3 Overview of Earlier Works

Theorem 2.3.1. (*Knopp-Mason, [15]*) *If f is a GMF on $\Gamma_0(N)$, then $g = \frac{1}{2\pi i}\frac{f'}{f}$ is a modular function of weight 2 with trivial character. If the GMF f doesn't have any zero or pole in $\mathcal{H}$ as well as at cusps, then g is a cusp form. Conversely, if g is a cusp form of weight 2 on $\Gamma_0(N)$, then there exists a GMF f on $\Gamma_0(N)$ such that $g = \frac{1}{2\pi i}\frac{f'}{f}$ and f is uniquely determined up to multiplication with non-zero scalars.*

Remark 2.3.1. Since $g = \frac{1}{2\pi i}\frac{f'}{f}$, the exponents of q-expansion of f and the Fourier coefficients of g are related. In fact, if $g(z) = \sum_{n\geq 1} b(n) q^n$ is a cusp form, then

$$b(n) = -\sum_{d|n} d c(d), \qquad (n \geq 1). \tag{2.3.1}$$

Regarding the properties of $c(n)$, we have a theorem of Kohnen and Martin, which is the following.

Theorem 2.3.2. (*Kohnen-Martin, [21]*) *For each square-free integer $N \geq 11$, there exists a GMF f on $\Gamma_0(N)$ with div(f) $= \emptyset$ such that the exponents $c(n)$ ($n \geq 1$) take infinitely many different values.*

Remark 2.3.2. The result and the proof given in [21] are actually easily seen to be valid for any integer $N \geq 11$ and to hold for any non-constant f, if one exploits the fact proved in [15] that GMFs f with $\mathrm{div}(f) = \emptyset$ correspond to cusp forms of weight 2 by taking logarithmic derivatives.

More generally, Kohnen [17] has proved the following result for any GMF with empty divisor.

Theorem 2.3.3. (*Kohnen, [17]*) *For any non-constant GMF f on $\Gamma_0(N)$ with div(f) $=$ $\emptyset$, $c(n)$ take infinitely many different values.*

2.4 Main Results

To state our results, we define certain operators on a GMF. Let f be a GMF on $\Gamma_0(N)$ and $M|N$. Assume that $\mathcal{R} = \{\gamma_1, \ldots, \gamma_r\}$ is a set of representatives for $\Gamma_0(N)$ modulo $\Gamma_0(M)$, then we define a "norm" of f w.r.t. $\mathcal{R}$ by

$$\mathcal{N}^M_{\mathcal{R},N}(f) := \prod_{\nu=1}^{r} f|_0\gamma_\nu. \tag{2.4.1}$$

Lemma 2.4.1. *Let f be a GMF on $\Gamma_0(N)$ and $M|N$. If we have two sets of representatives $\mathcal{R}$ and $\mathcal{R}'$ for $\Gamma_0(N)\backslash\Gamma_0(M)$, then $\mathcal{N}^M_{\mathcal{R},N}(f)$ and $\mathcal{N}^M_{\mathcal{R}',N}(f)$ differ by a non-zero scalar. Further, for any set of representatives $\mathcal{R}$, $\mathcal{N}^M_{\mathcal{R},N}(f)$ is a GMF.*

Proof. Let $\mathcal{R} = \{\gamma_1, \ldots, \gamma_r\}$ and $\mathcal{R}' = \{\alpha_1, \ldots, \alpha_r\}$ be two sets of representatives for $\Gamma_0(N)\backslash\Gamma_0(M)$. Then for each γ_i there exists a unique α_j such that $\gamma_i{\alpha_j}^{-1} \in \Gamma_0(N)$. Since f is a GMF on $\Gamma_0(N)$, we get

$$f|_0(\gamma_i{\alpha_j}^{-1}) = \chi(\gamma_i{\alpha_j}^{-1})f,$$

where χ is the character associated to the GMF f. Thus,

$$f|_0\gamma_i = \chi(\gamma_i{\alpha_j}^{-1})f|_0\alpha_j,$$

this implies

$$\prod_{i=1}^{r} f|_0\gamma_i = \prod_{j=1}^{r} \chi(\gamma_i{\alpha_j}^{-1})f|_0\alpha_j$$

$$\Rightarrow \mathcal{N}^M_{\mathcal{R},N}(f) = (\prod_{j=1}^{r} \chi(\gamma_i {\alpha_j}^{-1}))\mathcal{N}^M_{\mathcal{R}',N}(f).$$

This proves the first assertion. The second assertion follows from the first, since if $\{\gamma_1, \ldots, \gamma_r\}$ is a set of representatives for $\Gamma_0(N)\backslash\Gamma_0(M)$, then for any $\gamma \in \Gamma_0(M)$, $\{\gamma_1\gamma, \ldots, \gamma_r\gamma\}$ is another set of representatives for $\Gamma_0(N)\backslash\Gamma_0(M)$. □

The "trace" operator on the space of cusp forms $S_k(N)$ is defined as

$$F|Tr^M_N = \sum_{\nu=1}^{r} F|_k\gamma_\nu. \tag{2.4.2}$$

The trace operator maps $S_k(N)$ to $S_k(M)$.

Remark 2.4.1. The norm and trace operators have a relation. Let $g \in S_2(N)$ be the cusp form corresponding to a GMF f, i.e., $g = \frac{1}{2\pi i}\frac{f'}{f}$. Then, we have

$$\frac{1}{2\pi i}\frac{\mathcal{N}^M_{\mathcal{R},N}(f)'}{\mathcal{N}^M_{\mathcal{R},N}(f)} = \frac{1}{2\pi i}\sum_{\nu=1}^{r}\frac{(f|_0\gamma_\nu)'}{(f|_0\gamma_\nu)} = \sum_{\nu=1}^{r}\frac{1}{2\pi i}\left(\frac{f'}{f}\right)|_2\gamma_\nu = \sum_{\nu=1}^{r} g|_2\gamma_\nu = g|Tr^M_N. \tag{2.4.3}$$

With respect to the trace operator, one has a nice characterization of the space of newforms $S_k^{new}(N)$ obtained by A. P. Ogg. Below we give a version from [26].

Theorem 2.4.2. *(Li, [26]) If $F \in S_k(N)$, then $F \in S_k^{new}(N)$ iff for all primes $p|N$, we have $F|Tr_N^{N/p} = 0 = (F|_kH_N)|Tr_N^{N/p}$, where H_N is the Fricke involution defined by, $z \mapsto \frac{-1}{Nz}$.*

Remark 2.4.2. Let f be a GMF on $\Gamma_0(N)$. Since $H_N\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c/N \\ bN & a \end{pmatrix} H_N$, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, it follows that $f|_0H_N$ is also a GMF on $\Gamma_0(N)$.

We now state the main results of this chapter.

Theorem 2.4.3. *Let f be a non-constant GMF on $\Gamma_0(N)$ with $div(f) = \emptyset$ and assume that f has algebraic Fourier coefficients. Suppose further that $\mathcal{N}^M_{\mathcal{R},N}(f)$ and $\mathcal{N}^M_{\mathcal{R},N}(f|_0H_N)$ are constants for every divisor $M|N$, $M \neq N$. Then the exponents $c(p)$ (p prime) take infinitely many different values.*

Theorem 2.4.4. *Let f be a non-constant GMF on $\Gamma_0(N)$ with $div(f) = \emptyset$ and suppose that the $c(n)\,(n \geq 1)$ are real. Then the $c(n)\,(n \geq 1)$ change signs infinitely often.*

2.5 Proofs

2.5.1 Proof of Theorem 2.4.3

By assumption f has algebraic Fourier coefficients. Therefore, the same is true for $g = \frac{1}{2\pi i}\frac{f'}{f}$. Then, by bounded denominator argument, there exists an integer $A \in \mathbb{N}$ such that the Fourier coefficients of Ag are algebraic integers. Hence replacing f by f^A we may assume without loss of generality that g has integral algebraic Fourier coefficients. By hypothesis, $\mathcal{N}^M_{\mathcal{R},N}(f)$ and $\mathcal{N}^M_{\mathcal{R},N}(f|_0 H_N)$ are constants for every divisor $M|N$, $M \neq N$. This implies that $g|Tr^M_N$ and $(g|_2 H_N)|Tr^M_N$ are zero for each $M|N$, $M \neq N$. Hence, by Theorem 2.4.2, we conclude that $g \in S_2^{new}(N)$. Let us write the Fourier expansion of g as

$$g(z) = \sum_{n\geq 1} b(n)q^n.$$

Since $g = \frac{1}{2\pi i}\frac{f'}{f}$, we get the following relation.

$$b(n) = -\sum_{d|n} dc(d), \qquad (n \geq 1).$$

In particular, for each prime p, we have the relation

$$b(p) = -c(1) - pc(p).$$

Now assume on contrary that $c(p)$ take finitely many different values. Then using the Deligne's bound

$$b(p) \ll_g \sqrt{p}$$

for the Fourier coefficients $b(p)$ of g, where the constant depends only on g, we get

$$-c(1) - pc(p) \ll_g \sqrt{p}.$$

Thus, for sufficiently large prime p, the above relation implies that $c(p) = 0$. Hence, for sufficiently large prime p, we get the following relation.

$$b(p) = -c(1) = b(1). \tag{2.5.1}$$

On the other hand, since $0 \neq g \in S_2^{new}(N)$ has integral algebraic Fourier coefficients, by a result of Ono and Skinner (Lemma on p. 459, [39]), there exists a positive proportion of primes p with $b(p) \neq \alpha$, for any fixed algebraic integer α. This gives a contradiction to (2.5.1). This proves our theorem.

2.5.2 Proof of Theorem 2.4.4

To prove this theorem, we first recall Landau's theorem on Dirichlet series with non-negative coefficients. See (Theorem 11.13, [1]) for more details.

Theorem 2.5.1. *(Landau) Let $h(s)$ be represented in the half-plane $\sigma = Re(s) > c$ by the Dirichlet series*

$$h(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where c is finite, and assume that $a(n) \geq 0$ for all $n \geq n_0$, for some n_0. If the Dirichlet series has a finite abscissa of convergence σ_c, then $h(s)$ has a singularity on the real axis at the point $s = \sigma_c$.

Now we recall a Lemma of Kohnen.

Lemma 2.5.2. (*Kohnen, [20]*) *Suppose that $0 \neq F \in S_k(N)$. Then the abscissa of absolute convergence of the L-series $L(F, s)$ associated to F is $\frac{k+1}{2}$.*

We are now ready to prove Theorem 2.4.4. Since f is not constant, $g \neq 0$. The identity between the exponents $c(n)$ of the product expansion of f and the Fourier coefficients $b(n)$ of g can be rewritten as an identity between Dirichlet series

$$L(g, s) = -\zeta(s) \sum_{n \geq 1} \frac{c(n)}{n^{s-1}}, \qquad \sigma = Re(s) > \frac{3}{2}. \tag{2.5.2}$$

Let us assume on the contrary that $c(n) \geq 0$ for almost all n. Since $L(g, s)$ extends to an entire function and $\zeta(s)$ is holomorphic for $\sigma > 1$, by Theorem 2.5.1, the series

$$\sum_{n \geq 1} \frac{c(n)}{n^{s-1}}$$

converges in the range $\sigma > 1$. Further, almost all of its coefficients are non-negative by hypothesis. Therefore the convergence in this range must be absolute. From (2.5.2) we therefore conclude that $L(g, s)$ is absolutely convergent for $\sigma > 1$. This is a contradiction, since the abscissa of absolute convergence of $L(g, s)$ is exactly $3/2$ by Lemma 2.5.2. This completes the proof.