CHAPTER - I

INTRODUCTION

The basic concepts and fundamentals of the study that have been developed and applied so far are discussed here. The fundamentals of non-homogeneous elastic media, different types of wave propagation, classification of materials, fundamental of linear theory & basic laws, constitutive laws, orthogonal curvilinear co-ordinate system and special problems are discussed here.

Many monographs are available on the study matter, however, inclusion of some additions and improvements in these fundamentals for non-homogeneous elastic media will enhance the advantage of the discussion for investigation of wave propagation.

1.1 Fundamentals of Linear Theory

1.1.1 Basic Definitions

(i) **Particle:** - A particle denotes an infinitesimal volumetric element or a small part of a material continuum.

(ii) **Point:** - Point is used to represent a location of particles in fixed space.

(iii) **Configuration:** - The identification of the particles of the continuum with the points of the space, it occupies at time \( t \) by reference to a suitable set of coordinate axes is said to specify the configuration of the continuum at that instant.

(iv) **Deformation:** - It is the change in the shape of the continuum between some initial configuration (un-deformed state) and a subsequent configuration (deformed state).

1.1.2 Motion

Suppose, in the initial configuration \( (t = 0) \) a representative particle of the continuum consisting of material volume \( V \) and surface \( S \) occupies a point \( P_0 \) in space and has the position vector,
\[
\vec{X} = X_1 \hat{I}_1 + X_2 \hat{I}_2 + X_3 \hat{I}_3 = X_k \hat{I}_k
\] (1.1.1)

with respect to the rectangular cartesian axes \(OX_1X_2X_3\), where \(\hat{I}_i\) is the unit vector in \(X_i\)-direction. Here, the coordinates \((X_1X_2X_3)\) are called the material coordinates. In the deformed configuration \((t = t)\) the particle originally at \(P_0\) is located at the point \(P\) consisting of material volume \(V\) and surface \(S\) having the position vector

\[
\vec{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 = x_i \hat{e}_i
\] (1.1.2)

with respect to the rectangular cartesian axes \(ox_1x_2x_3\), where \(\hat{e}_i\) is the unit vector in \(x_i\)-direction. Here, the coordinates \((x_1x_2x_3)\) are called the spatial coordinates.

The relative orientation of the material axes and the spatial axes is given by direction cosines \(\alpha_{kk}\) and \(\alpha_{kk}\) as

\[
\hat{e}_k \cdot \hat{I}_k = \hat{I}_k \cdot \hat{e}_k = \alpha_{kk} = \alpha_{kk}
\] (1.1.3)

In fig. 1.1(a), the vector \(\vec{u}\) joining the points \(P_0\) and \(P\) is known as the displacement vector. The vector \(\vec{b}\) in fig. 1.1(a) serves to locate the origin o w.r.t. O and from the geometry of the figure,

\[
\vec{u} = \vec{b} + \vec{x} - \vec{X}.
\] (1.1.4)

In continuum mechanics, it is possible to consider the material and spatial coordinate system superimposed with \(\vec{b} \equiv 0\), so that

\[
\vec{u}' = \vec{x} - \vec{X}
\] (1.1.5)

and the general expression in cartesian components is given by

\[
u_k = x_k - X_k
\] (1.1.6)
During deformation, the particles of the continuum move in space and this motion may be expressed as

\[ x_i = x_i(X_1, X_2, X_3, t), \quad i = 1, 2, 3, \]  \hspace{1cm} (1.1.7)

which give the present location \( x_i \) of the particle that occupied the point \( (X_1, X_2, X_3) \) at time \( t = 0 \). This deformation of motion is known as the Lagrangian formulation or material representation.
Alternatively, the motion in Eulerian formulation or spatial representation is given by

\[ X_i = X_i(x_1, x_2, x_3, t), \quad i = 1,2,3, \]  \hspace{1cm} (1.1.8)

which provides a tracing to its original position of the particle that now occupies the point \((x_1x_2x_3)\). The two mappings given by (1.1.7) and (1.1.8) are the unique inverses of each other if these mappings are continuous one to one with continuous partial derivatives and the necessary and sufficient condition for the inverse functions to exist is that the Jacobian \( J = \left| \frac{\partial X_i}{\partial x_j} \right| \) should not vanish.

### 1.1.3 Linearised Theory

In figure 1.1(b), the initial and final configurations of a continuum are referred to the superposed rectangular cartesian coordinate axes \(OX_1X_2X_3\) and \(\alpha x_1x_2x_3\). The neighboring particles occupying points \(P_0\) and \(Q_0\) before deformation move to points \(P\) and \(Q\) respectively in the final configuration.

The square of the differential element of length between \(P_0\) and \(Q_0\) is

\[ (d\bar{X})^2 = d\bar{x}_i d\bar{x}_j = dX_i dX_j = \delta_{ij} dx_i dx_j \]  \hspace{1cm} (1.1.9)

\[ = \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} dx_i dx_j, \]  \hspace{1cm} (1.1.10)

Also,

\[ (dx)^2 = dx_i dx_j = \delta_{ij} dx_i dx_j \]  \hspace{1cm} (1.1.11)

\[ = \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} dx_i dx_j, \]  \hspace{1cm} (1.1.12)

where \( \delta_{ij} = 0, \) if \( i \neq j \) and \( 1 \) if \( i = j \).

The difference \( (dx)^2 - (d\bar{X})^2 \) for two neighboring particles of a continuum is used as the measure of deformation that takes place in the neighborhood of the particles between the initial and final configuration. A rigid displacement is said to occur if this difference is identically zero for all neighboring particles of a continuum.
Now,\[
(dx)^2 - (dX)^2 = \left( \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} - \delta_{ij} \right) dX_i dX_j
\]
\[= 2L_{ij} dX_i dX_j,
\]
where\[
L_{ij} = \frac{1}{2} \left( \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} - \delta_{ij} \right)
\]
is the Lagrangian finite strain tensor.

Figure 1.1 (b)
Also,

\[ (dx)^2 - (dX)^2 = \left( \delta_{ij} - \frac{\partial X_k \partial X_k}{\partial x_i \partial x_j} \right) dx_i dx_j \]

\[ = 2E_{ij} dx_i dx_j , \tag{1.1.15} \]

where

\[ E_{ij} = \frac{1}{2} \left( \delta_{ij} - \frac{\partial X_k \partial X_k}{\partial x_i \partial x_j} \right) \tag{1.1.16} \]

is the Eulerian finite strain tensor.

Now, from equation (1.1.6), the material gradient is given by

\[ \frac{\partial u_i}{\partial x_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij} \tag{1.1.17} \]

and the spatial gradient is given by

\[ \frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j} , \tag{1.1.18} \]

Using (1.1.17) and (1.1.18) in equations (1.1.14) and (1.1.16) respectively, it is obtained as

\[ L_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k \partial u_k}{\partial x_i \partial x_j} \right) \tag{1.1.19} \]

and

\[ E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k \partial u_k}{\partial x_i \partial x_j} \right) . \tag{1.1.20} \]

If the displacement gradient components \( \partial u_i / \partial X_j \) are infinitesimal then their products can be neglected, so (1.1.19) reduces to

\[ l_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) . \tag{1.1.21} \]

Similarly, for \( \partial u_i / \partial x_j \ll 1 \), relation (1.1.20) reduces to
Thus, if both the displacement gradients and the displacements are infinitesimal, then there is very little difference in the material and spatial coordinates of a continuum particle which can be neglected and so, only lower case symbols i.e. spatial description is used in the following work.

1.2 Deformation/ Strain – Displacement Relations

Any change in the relative positions of elements of a medium are called deformations. For one dimensional case, deformation is the extension/contraction divided by the original length of the element.

\[
\frac{\partial u}{\partial x} \text{ can be taken as the deformation in spatial coordinates and is known as unit of elongation. The strain components using Taylor’s Theorem, for linear theory are obtained as}
\]

\[
e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3, \tag{1.2.1}
\]

where \(e_{ij}\) are symmetrical tensor components with \(e_{12}, e_{23}, e_{31}\) representing shear strain components, \(e_{11}, e_{22}, e_{33}\) representing normal strain components and \(u_{ij}\) denotes partial derivative of \(u_i\) w.r.t \(x_j\).

Also,

\[
\theta = e_{11} + e_{22} + e_{33} = u_{ij} \tag{1.2.2}
\]

which represents the expansion of a unit volume due to strain produced in the medium and is known as cubical dilatation or dilatation and

\[
\sigma = -\frac{e_{22}}{e_{11}} = -\frac{e_{33}}{e_{11}} \tag{1.2.3}
\]

i.e. \(\sigma\) denotes the ratio of the contraction of linear elements perpendicular to the axis of the rod to the longitudinal extension of the rod and is called Poisson ratio.
1.3 Equations of Compatibility

The six independent relations, given by (1.1.22) i.e.
\[ e_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) \]
which can be viewed as a system of six partial differential equations for determining the three displacement components \( u_1 \), if strain components are known. But this system is over-determined and will not, in general, possess a solution for an arbitrary choice of the strain components \( e_{ij} \). Thus, the strain components must satisfy some other conditions in order that the displacement components \( u_1 \) are to be single valued and continuous. The necessary and sufficient conditions for such a displacement field are given by

\[ e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0, \quad i,j,k,l = 1,2,3. \]  \hspace{1cm} (1.3.1)

These are 81 equations in terms of strain components. But some of these equations are identically satisfied and some are mere repetitions due to symmetry in the indices \( ij \) and \( kl \). Hence out of these 81 equations, only six are distinct. These six are written in explicit form as

\[
\begin{align*}
    e_{11,22} + e_{22,11} &= 2e_{12,12} \\
    e_{22,33} + e_{33,22} &= 2e_{23,23} \\
    e_{33,11} + e_{11,33} &= 2e_{31,31} \\
    (-e_{23,1} + e_{31,2} + e_{12,3})_{,1} &= e_{11,23} \\
    (e_{23,1} + e_{31,2} - e_{12,3})_{,3} &= e_{33,12}.
\end{align*}
\]  \hspace{1cm} (1.3.2)

For plane strain parallel to the \( x_1x_2 \) plane, the six equations in (1.3.2) reduce to only single equation

\[ e_{11,22} + e_{22,11} = 2e_{12,12}. \]  \hspace{1cm} (1.3.3)
1.4 Time – Rates of Change

1.4.1 Velocity

The velocity of a material particle is the time rate of change of displacement $u_i$ and is given by

$$v_i = \frac{\partial u_i}{\partial t} = u_{i,t} \quad (1.4.1)$$

1.4.2 Acceleration

The time rate change of velocity $v_i$ is called the acceleration $a_i$ and is given by

$$a_i = \frac{\partial v_i}{\partial t} = v_{i,t} = u_{i,tt} \quad (1.4.2)$$

1.5 Stress Vector/Stress Tensor

The stress vector is defined as the force per unit area that a portion of a body exerts on an adjacent portion across an imaginary surface $S$ that separates them.

Therefore, stress vector

$$\vec{t} = \frac{\text{Force}}{\text{Area}} = \frac{\vec{F}}{A}.$$ 

If we shrink the surface area to a point, then surface force $= \text{stress vector}$. If the area element is normal to $Ox_1$-axis then the corresponding stress vector $\vec{t}_1$ will have three components denoted by $\tau_{11}, \tau_{12}, \tau_{13}$ along the axes. Hence the state of stress at a point is completely determined if the nine components of stress tensor at that point are known. $\tau_{11}, \tau_{22}, \tau_{33}$ are normal components of stress and $\tau_{12}, \tau_{21}, \tau_{32}, \tau_{23}, \tau_{31}, \tau_{13}$ are shear or tangential components with $\tau_{ij} = \tau_{ji}$. The nine scalar quantities are the components of a stress tensor.
1.6 Classification of Materials

The various materials are classified according to the two physical concepts associated with the deformation of the body when some work is to be done. First the work done by external forces on the body can be stored in the material and this storage of the work is expressed by the strain energy function ‘W’ per unit volume in the material and secondly the work done can be dissipated irreversibly which is expressed as rate of dissipation ‘D’ per unit volume per unit time. Depending upon these two concepts the various materials can be classified as follows.

1.6.1 Rigid Materials

A rigid body is an ideal body such that the distance between every pair of particles i.e. molecular distance remains invariant under the action of forces howsoever large. A rigid body involves translation and rotation and thus $W = 0, D = 0$.

1.6.2 Deformable Materials

A body is said to be deformed when the relative positions of particles in a continuous body are changed. These materials further classified as:

- Elastic Materials
- Viscous Materials
- Viscoelastic Materials

1.6.2.1 Elastic Materials

A body is said to be ideally elastic if it has the property of recovering its original shape, when the external loads, causing deformations are withdrawn. In these types of materials, no energy is dissipated, since all the work is stored as the strain energy. Thus we have, $W \neq 0, D = 0$. These are of two types:

- Anisotropic elastic materials
- Isotropic elastic materials
According to generalized Hooke’s law:

\[ \tau_{ij} = C_{ijkl} e_{kl} , \quad i,j,k,l = 1,2,3, \]  

(1.6.1)

where \( \tau_{ij} \) and \( e_{kl} \) describe the stress and strain components, \( C_{ijkl} \) are the Elastic constants for homogenous materials and are known as elastic coefficients for non-homogeneous materials and are space dependent. However, due to the symmetry of both the stress and strain tensors, there are at-most 36 distinct elastic constants. In single indexed system, it can be written as

\[ \tau_{i} = C_{ij} e_{j} , \quad i,j = 1,2,3,4,5,6. \]  

(1.6.2)

1.6.2.1 Anisotropic Elastic Materials

A material which does not respond the same way in all directions i.e. in which properties of a material depend on the direction, for example wood. In the relation (1.6.2) elastic constants reduce to 21 whenever there exist a strain energy function

\[ w = \frac{1}{2} C_{ij} e_{i} e_{j} , \]  

(1.6.3)

with the property that

\[ \frac{\partial w}{\partial e_{i}} = \tau_{i}. \]  

(1.6.4)

These 21 independent elastic constants in the generalised Hooke’s law form an elastic anisotropic body.

If a material is elastically symmetric with respect to a plane, then the independent elastic constants reduce to 13 in the generalised Hooke’s law and if the material is symmetric with respect to three mutually perpendicular planes then the independent elastic constants are further reduced to 9.
1.6.2.1.2 Isotropic Elastic Materials

A material in which elastic properties are independent of the orientation of co-ordinate axes is known as isotropic elastic material. Generalized Hooke’s Law for an isotropic medium in terms of constants \( \lambda \) and \( \mu \) is

\[
\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},
\]

where \( \lambda \) and \( \mu \) are known as Lame’s constants for homogeneous materials and are functions of space for non-homogeneous elastic materials, i.e. \( \lambda = \lambda(x_1, x_2, x_3) \), \( \mu = \mu(x_1, x_2, x_3) \), \( \delta_{ij} \) is the Kronecker’s delta whose value is 0 if \( i \neq j \) and 1 if \( i = j \).

1.6.2.2 Viscous Materials

A material having property of resistance to flow is known as viscous material. When the external forces are removed, all the changes in the viscous material remain irrecoverable. The work done by external forces on a viscous material is entirely dissipated in the form of heat and there is no storage of work as strain energy. So, we can say that in an ideally Viscous material \( W = 0, D \neq 0 \).

According to Newton’s hypothesis, the stress components are related to the rate of strain components as

\[
\tau_{ij} = -p\delta_{ij} + 2\mu d_{ij},
\]

where, \( p \) is the pressure, \( d_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) \) is the rate of strain components, \( \mu \) is viscosity coefficient.

Viscosity is a measure of its resistance to (flow) gradual deformation by shear stress or tensile stress. It is defined as

\[
\text{viscosity} = \frac{\text{Shearing Stress} (f/A)}{\text{Velocity Gradient}(dv/dx_3)}.
\]

A fluid that has no resistance to shear stress is called ideal fluid or in-viscid fluid. A liquid whose viscosity is less than that of water is known as mobile liquid while greater than water is called viscous liquid.
Velocity gradient is the difference in velocity between the layers of the fluid and is given by $\frac{dv}{dx_3}$, where $dv$ is velocity difference and $dx_3$ is the distance between the layers. Also, velocity gradient is the rate of change of velocity of propagation with distance normal to the direction of flow.

1.6.2.3 Viscoelastic Materials

Viscoelastic materials exhibit both viscous and elastic properties when undergoing deformation. Some part of work done by external forces on these materials is dissipated and another part of work is stored in the materials as strain energy. Thus, such materials undergo partial recovery after removing external loads. We can say that $W \neq 0, D \neq 0$ on deformation in Viscoelastic materials. But the ratio of work dissipated to the work stored may be different for different Viscoelastic materials.

1.7 Elastic Constants

An elastic modulus is the mathematical description of an object or substance’s ability to be deformed elastically when a force is applied to it.

Elastic Modulus = Stress / Strain

It can be of following three types:

(i) **Young’s modulus (E):** It is the ratio of tensile or longitudinal stress to tensile strain of a material body. It describes tensile elasticity or the tendency of an object to deform along an axis or simply known as elastic modulus.

Thus

\[
E = \frac{\tau_{ij}}{e_{ij}}, \quad i = j. \tag{1.7.1}
\]

(ii) **Shear modulus (G or μ):** It is also known as modulus of rigidity, which describes an object’s tendency to shear (the deformation of shape at constant volume) when acted upon by opposing forces. Thus,
\[ \mu = \frac{\tau_{ij}}{e_{ij}}, \quad i \neq j, \quad (1.7.2) \]

which represents the ratio of the shearing stress \( \tau_{ij} \) to the shearing strain \( e_{ij} \).

(iii) **Bulk modulus** (\( \mathcal{K} \)): The ratio of volumetric stress to volumetric strain of a material body is known as bulk modulus. It describes volumetric elasticity or the tendency of an object to deform in all directions when uniformly loaded in all directions.

**Relations between elastic constants**

\[
\begin{align*}
\mathcal{K} &= \frac{E}{3(1 - 2\sigma)}, \\
\mu &= \frac{E}{2(1 + \sigma)}, \\
\lambda &= \frac{\sigma E}{(1 + \sigma)(1 - 2\sigma)}, \\
\sigma &= \frac{3\mathcal{K} - E}{6\mathcal{K}}, \\
E &= 3\mathcal{K}(1 - 2\sigma),
\end{align*}
\]

where \( \sigma \) is Poisson’s ratio.

**1.8 Universal Laws**

There are the basic physical concepts valid for all bodies subjected to mechanical loading.

**1.8.1 Principle of Mass Balance**

Matter can neither be created nor be destroyed. If \( \rho(x_i) \) is the measure of mass density for the non homogeneous body, then total mass \( M \) of volume \( V \) is given by

\[ M = \int_V \rho(x_i) dV. \]
Thus
\[
\frac{d}{dt} \int_{V} \rho(x_i) dV = 0 \quad \text{as } M \text{ is constant.} \quad (1.8.1)
\]

This is the law of conservation of mass. The principle of mass balance gives the mass balance equation or the equation of continuity.

### 1.8.2 Principle of Momentum Balance

The time rate of change of momentum of a body is equal to the resultant of surface and body forces i.e.

\[
\frac{d}{dt} \int_{V} \rho(x_i) v_i(x_i, t) dV = F_i , \quad (1.8.2)
\]

where

\[
F_i = \int_{S} \tau_{ij} n_j(x_i) dS + \int_{V} \rho(x_i) f_i dV ,
\]

\(\tau_{ij} n_j(x_i)\) is the surface force per unit area of the surface ‘S’ and \(f_i\) is the body force per unit volume of the body. If no external body and surface forces act then its linear momentum is conserved. It gives the momentum balance equation or equation of motion.

### 1.8.3 Principle of Moment of Momentum Balance

The time rate of change of moment of momentum is equal to the resultant moment of all forces and couples acting on the body i.e.

\[
\frac{d}{dt} \int_{V} \rho(x_i) p_i v_i dV = \int_{S} p_i \tau_{ij}(\vec{n}) dS + \int_{V} \rho(x_i) p_i f_i dV . \quad (1.8.3)
\]

Here, the moments are taken about the origin and \(p_i\) is the displacement vector from origin.

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1.8.4 Principle of Energy Balance

The time rate of change of energy (kinetic energy and internal energy) of a body is equal to rate at which the energy is supplied to it in any form like thermal, electromagnetic, chemical or by doing any work on the body i.e.,

$$\frac{d}{dt}(E) = W + \sum_{i} U_i \quad (i = 1,2,3, \ldots \ldots), \quad (1.8.4)$$

where $W$ is the work done on the body, $\sum_i U_i$ is the resultant sum of all other energies that enter and leave the body per unit time, $E$ is the sum of kinetic energy and internal energy. So

$$E = \frac{1}{2} \int_{V} \rho(x_i) v_i v_j dV + \int_{V} \rho(x_i) dV, \quad (1.8.5)$$

$$W = \int_{S} \tau_{ij} n_{j}(x_i) u_i dS + \int_{V} \rho(x_i) f_i u_i dV, \quad (1.8.6)$$

If no energy is supplied to the body, then its energy is conserved and this gives principle of conservation of energy.

1.9 Waves

Waves originate due to forced action of applied force in a portion of a deformable continuum. Wave disturbance transfers energy from one point to another due to their interaction pulling and pushing. In this process, the resistance offered to the deformation by the consistency of the medium, as well as to the motion by inertia must be overcome. Mechanical wave is a wave that needs a medium to travel. These can be produced only when media possess elasticity (deformability) and inertia. Mechanical waves transport energy only and mass is not transported. The energy is transferred in the direction of wave or applied force. A mechanical wave requires an initial energy input in the form of load. Once this initial energy is added, the wave travels through the medium until all its energy is utilized (consumed). All the real materials are of course deformable and possess mass and so all real materials transmit mechanical waves. But, in the case of rigid
medium, the waves are not produced and there is only translation/rotation of the medium as a whole, or conglomerates of molecules.

There are two types of waves – Mechanical waves and Electromagnetic waves. Mechanical waves propagate through a medium at a wave speed which depends on the elastic and inertial properties of the medium. As the medium is deformed, so after this the deformation reverses itself owing to the restoring forces resulting from its deformation e.g. sound waves propagate through air molecules interacting with their neighbors. Electromagnetic waves do not require a medium. They consist of periodic oscillations of electrical and magnetic fields generated by charged particles and can therefore travel through vacuum.

A wave propagates either as transverse or longitudinal disturbance, depending on the direction of oscillation of particles of the medium about their mean positions with respect to direction of wave. Longitudinal waves occur when the oscillations of particles are parallel to the direction of propagation. The P-waves (Primary waves) in an earthquake are examples of longitudinal waves. The P-waves travel with faster velocity and are the first to arrive from the source of disturbance. Transverse waves occur when a disturbance creates oscillations in a plane perpendicular to the direction of propagation (the direction of energy transfer). The S-waves (Shear or Secondary waves) in seismic disturbances are examples of transverse waves. S-waves propagate with a velocity less than that of P-waves. Mechanical waves can be of both types transverse as well as longitudinal. All electromagnetic waves are transverse.

Deformations produced are generally related to the applied loads and therefore to the stress tensor. These relations are called stress-strain relations or the constitutive equations of the material. By removing the load, material regains its original shape i.e. equilibrium state or natural state. This property is known as elasticity. It is the ability of the material to return to its original shape after removing the applied load. Elasticity can be explained by the Hooke’s law. In other words, amount of compression or stretching is directly proportional to the applied force. However, until certain point under applied force material regains its original shape. This certain point is known as the elastic limit of that material. It is specific
for each material. If the external forces are not below these limits, the materials under external forces get permanently deformed or breakdown.

Among the most important aspects of wave motion are the reflection and transmission. When a wave encounters an interface separating the two media with different properties, a part of the disturbance is reflected back in first medium known as reflected wave and a part is transmitted known as transmitted wave into the second medium. It is a case of discontinuity of the media properties.

![Figure 1.9(a)](image)

In air, sound travels by compressions and rarefactions of air molecules in the direction of travel. However, in solids molecules can support vibrations in other directions also, hence different types of elastic wave modes are possible in solids which are given below:

- The Longitudinal waves (parallel to the direction of wave/ disturbance)
- The Transverse or Shear waves (perpendicular to the direction of wave)
- Surface (Rayleigh) waves (in elliptical orbit or symmetrical mode)
- Plate (Lamb) waves (component perpendicular to surface i.e. extensional wave)
- Plate (Love) waves (parallel to a plane layer, perpendicular to wave direction)
- Stoneley (Leaky-Rayleigh) waves (guided waves along interface)

Longitudinal and transverse waves are already discussed. Surface waves are the combination of longitudinal and transverse waves to form an
elliptical/circular orbit motion. The major axis of the ellipse is perpendicular to the solid surface. Surface (Rayleigh) waves are useful as they are very sensitive to the surface defects. Plate waves are similar to surface waves with the only difference is that they can be generated in a few wavelengths thick materials. Lamb waves will travel several meters in steel and so are useful to scan wires, tubes etc. The two most common modes of Lamb waves are symmetrical and asymmetrical. Symmetrical Lamb wave mode is also called extensional mode because the wave is ‘stretching and compressing’ the surface in the wave motion direction. This mode of wave motion is produced when the exciting force is parallel to the surface. Asymmetrical Lamb wave mode is known as flexural mode as in this mode large portion of wave motion moves in a normal direction to the plate.

### 1.9.1 Elastic Waves

Elastic waves occur in a medium in which, when particles are displaced, a restoring force proportional to the displacement acts on the particles to restore them to their equilibrium positions. These waves propagate through the medium without causing permanent deformation or fracture of any portion of the medium. For example, sound propagates through the air as elastic waves and water waves propagate on the surface of a pond as elastic waves. There are mainly two types of elastic waves which are as under:

- Body Waves
- Surface Waves

#### 1.9.1.1 Body Waves

A seismic (earthquake) wave that travels through the interior of the earth rather than across its surface. These waves have usually smaller amplitudes and shorter wavelengths than surface waves and so travel at higher speeds. Body waves arrive before the surface waves during a seismic disturbance.

Body waves are of two types:-

- Primary waves (Longitudinal Waves)
- Secondary waves (Shear Waves)
1.9.1.1.1 Primary Waves

It is a first type of body wave in which particles vibrate parallel to the direction of wave propagation.

![Particle Motion Diagram](image)

Figure 1.9(b)

These waves are also called P-waves or Pressure waves. P-waves push and pull the medium particles through which these waves travel as sound waves push and pull the air molecules. These waves are the fastest travelling waves and are therefore the earlier to be recorded. Hence these waves are known as Primary waves. They can travel through all states of matter i.e. solids, liquids and gases. P-waves are also known as compressional waves because of the pulling and pushing they produce. These are longitudinal in nature. In air they take the form of sound waves. In longitudinal waves, deformation is a combination of uniform compression or extension and pure shear. The speed of P-waves in a homogeneous isotropic medium is given by

\[ v_p = c_L = \sqrt{\frac{K + \frac{4}{3} \mu}{\rho}} = \sqrt{\frac{\lambda + 2 \mu}{\rho}}, \]

where \( K \) is the bulk modulus (the modulus of incompressibility), \( \mu \) is the shear modulus (modulus of rigidity, sometimes denoted as G and also called the second Lame’s parameter), \( \rho \) is the density of the medium through which the wave travels, and \( \lambda \) is the first Lame’s parameter. Of these, density shows the least variation, so the velocity is mostly controlled by \( K \) and \( \mu \). The elastic moduli P-wave modulus, \( M \), is defined as \( M = K + 4\mu/3 \) and thereby speed can be expressed:
In Fig 1.9(c), the propagation of P waves in x-direction is shown.

\[ v_p = c_L = \sqrt{\frac{M}{\rho}}. \]

1.9.1.1.2 Secondary Waves

The second type of body wave is the S-wave or secondary wave or shear wave which is the delayed wave one feels in an earthquake. In S-waves particles vibrate perpendicular to the direction of propagation and so are transverse in nature. As transverse waves, S-waves exhibit properties such as polarization and birefringence. S-waves polarized in the horizontal plane are classified as SH-waves and polarized in the vertical plane are classified as SV-waves.
S-waves are slower than P-waves and speeds are typically around 60% of that of P-waves in any given material. S-waves can travel only through solids, as fluids do not support shear stresses. It is this property of S-waves that led seismologists to conclude that the earth’s outer core is a liquid. The velocity of S-waves in a homogeneous isotropic medium is given by

\[ v_s = c_T = \sqrt{\frac{\mu}{\rho}}. \]

In Fig 1.9(e), the propagation of S waves in x-direction is shown.
1.9.1.2 Surface Waves

It is a wave that travels across the surface of the Earth. Surface waves have larger amplitudes and longer wavelengths than body waves and so they travel more slowly than body waves do. In other sense, surface waves can propagate at the boundary between a solid half-space and a vacuum, liquid or gas. Though surface waves arrive later than body waves, these are the waves that are responsible and prominent for the damage and destruction associated with earthquakes, due to large amplitudes that create unstability. The damage and the strength of the surface waves reduce in deeper earthquakes. Examples of surface waves are the waves at the surface of water and air or ripples in the sand at the interface with water or air. These types of surface waves are the combination of P and S-waves.

There are two kinds of surface waves:

- Love waves
- Rayleigh waves

1.9.1.2.1 Love Waves

Love waves are transverse in nature i.e. particles move perpendicular to the direction of wave propagation and also more precisely particles move with a side to side motion perpendicular to the main propagation of the wave like SH waves guided by an elastic layer, which is in welded contact with an elastic half-space on one side while bordering a vacuum on other side. In seismology, Love waves are surface seismic waves that cause horizontal shifting of the earth during an earthquake, due to side to side motion.

Love waves, named after A.E.H. Love, a British Mathematician, who worked out the mathematical model for this kind of wave in 1911. These are the fastest among the surface waves but travel with a lesser velocity than P or S waves. The amplitude in the motion of these waves decreases with depth and since Love waves travel on the earth’s surface, the strength of the wave decreases exponentially with the depth of an earthquake. Love waves cause the rocks, they pass through, to change in shape. It was often thought that the animals like cats and dogs could predict an earthquake before it happen. But the real fact is that they are
more sensitive to ground vibrations than humans and so are able to detect the subtler waves that precede Love waves.

In Fig 1.9(f), the propagation of Love waves in x-direction is shown.

![Figure 1.9(f)](image)

1.9.1.2.2 Rayleigh Waves

Rayleigh waves include both longitudinal and transverse motion. A Rayleigh wave rolls along the ground just like a wave rolls across a lake or an ocean. It raises the ground up and down normal to the direction of propagation and moves it side to side in the direction of wave propagation. These waves are a type of surface waves that travel near the surface of solids.

Rayleigh waves are named after Lord Rayleigh and were predicted in 1885. In isotropic solids, these waves cause the surface particles to move in elliptical paths with the major axis perpendicular to the solid surface. The amplitude of this motion decreases with depth. In water waves, all particles travel in clockwise direction. However, particles in Rayleigh waves travel in counter clockwise at surface while at a depth travel in clockwise direction. Like primary waves, Rayleigh waves are of alternating compressional and extensional nature. These waves travel slower than Love waves. These are frequently used in non-destructive testing for detecting defects. When guided in layers they are referred as Lamb waves.
In Fig 1.9(g), the propagation of Rayleigh waves in x-direction is shown.

![Figure 1.9(g)](image)

### 1.10 Orthogonal Curvilinear Coordinates

A set of coordinates $\alpha_1 = \alpha_1(x,y,z), \alpha_2 = \alpha_2(x,y,z), \alpha_3 = \alpha_3(x,y,z)$ are called curvilinear coordinates of a point $P(x,y,z)$. The coordinate axes are determined by the tangents to the coordinate curves at the point $P$. The directions of these coordinate axes depend on the chosen point $P$ of space.

If at every point $P(x,y,z)$, the coordinate axes are mutually perpendicular, then $\alpha_1, \alpha_2, \alpha_3$ are called orthogonal curvilinear coordinate of $P$.

#### Equation of Motion

$$
\sum_{j=1}^{3} \left[ h_j \frac{\partial}{\partial \alpha_j} \left( \frac{\sqrt{g}}{h_i h_j} \tau_{ij} \right) + 2 \frac{\tau_{ij}}{h_i h_j} \frac{\partial h_i}{\partial \alpha_j} - \frac{\tau_{ji}}{h_i h_j} \frac{\partial h_j}{\partial \alpha_i} \right] + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (1.10.1)
$$

where

$\alpha_i =$ orthogonal curvilinear coordinates (mutually perpendicular surfaces),

$\tau_{jj} =$ normal stress components,

$\tau_{ij}(i \neq j) =$ shearing stress components,
\[ \bar{g}_i = \frac{\partial \bar{r}}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i} (x \hat{i} + y \hat{j} + z \hat{k}) \]

= base vector at P along the coordinate curve \( \alpha_i \),

\[ h_i = |\bar{g}_i|, \]

\[ \sqrt{g} = \det(\bar{g}_1 \bar{g}_2 \bar{g}_3). \]

\[ ds^2 = \sum_{i=1}^{3} h_i^2 d^2 \alpha_i, \]

\( F_i \) = body force per unit volume in the direction of the \( \alpha_i \) coordinate,

\( u_i \) = displacements in the direction of curvilinear coordinates.

Strain Displacement Relations

\[ e_{ii} = \frac{1}{h_i} \frac{\partial u_i}{\partial \alpha_i} + \sum_{j=1}^{3} \frac{1}{h_i h_j} \frac{\partial h_i}{\partial \alpha_j} u_{ij}, \]

\[ e_{ij} = \frac{1}{2} \left( \frac{h_i}{h_j} \frac{\partial}{\partial \alpha_j} \left( \frac{u_i}{h_i} \right) + \frac{h_j}{h_i} \frac{\partial}{\partial \alpha_i} \left( \frac{u_i}{h_i} \right) \right), \]

where,

\( e_{ii} \) = normal strains,

\( e_{ij} (i \neq j) \) = shearing strains.

1.10.1 Rectangular Coordinate System

In the three dimensional rectangular coordinate system, three axes (x-axis, y-axis, z-axis) are to be taken as mutually perpendicular. Take

\( \alpha_1 = x, \alpha_2 = y, \alpha_3 = z \) = planes,

\[ h_1 = h_2 = h_3 = 1, \sqrt{g} = 1, \]

\[ ds^2 = dx^2 + dy^2 + dz^2. \]
Equations of Motion

\[ \tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z} + F_x = \rho \, u_{x,tt}, \]
\[ \tau_{xy,x} + \tau_{yy,y} + \tau_{yz,z} + F_y = \rho \, u_{y,tt}, \]
\[ \tau_{xz,x} + \tau_{yz,y} + \tau_{zz,z} + F_z = \rho \, u_{z,tt}, \]

where \( F_x, F_y, F_z \) are the components of body force. In compact form, these equations can be written as

\[ \tau_{ij,j} + F_i = \rho \, u_{i,tt}, \quad (1.10.4) \]

where \( F_i \)’s are the body forces and \( \tau_{ij,j} \) are obtained from surface forces.

The forces which are proportional to the mass contained in the volume element, such as gravitational forces are called body forces and the forces which act on the surface of the volume elements are known as surface forces.

The equations of equilibrium for an isotropic elastic material are given by

\[ \tau_{ii,j} = -F_i \quad (1.10.5) \]

Strain-Displacement Relations

Coordinates measured in terms of displacements:

\[ \delta x = u_x, \quad \delta y = u_y, \quad \delta z = u_z. \]

The strain equations are

\[ e_{xx} = u_{x,x}, \quad e_{yy} = u_{y,y}, \quad e_{zz} = u_{z,z}, \]
\[ e_{xy} = \frac{1}{2} (u_{x,y} + u_{y,x}), \quad e_{yz} = \frac{1}{2} (u_{y,z} + u_{z,y}), \quad e_{zx} = \frac{1}{2} (u_{z,x} + u_{x,z}) \]

or

\[ e_{ij} = \frac{1}{2} (u_{ij} + u_{ji}), \quad (1.10.6) \]

where \( u_x, u_y, u_z \) are displacement components in the directions of \( x, y, z \)-axes respectively.
1.10.2 Cylindrical Polar Coordinate System

Here
\[ \alpha_1 = r = \text{circular cylinders}, \]
\[ \alpha_2 = \theta = \text{planes through the oz axis}, \]
\[ \alpha_3 = z = \text{planes parallel to the xoy plane}, \]
\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \]
from which \( h_1 = 1, \quad h_2 = r, \quad h_3 = 1, \quad \sqrt{\gamma} = r, \)
\[ ds^2 = dr^2 + r^2 d\theta^2 + dz^2. \]

Equations of Motion

\[
\tau_{rr,r} + \frac{\tau_{r\theta,\theta}}{r} + \tau_{rz,z} + \frac{(\tau_{rr} - \tau_{\theta\theta})}{r} + F_r = \rho u_{r,tt}, \\
\tau_{r\theta,r} + \frac{\tau_{\theta\theta,\theta}}{r} + \tau_{\theta z,z} + 2 \frac{\tau_{r\theta}}{r} + F_\theta = \rho u_{\theta,tt}, \\
\tau_{rz,r} + \frac{\tau_{\theta z,\theta}}{r} + \tau_{zz,z} + \frac{\tau_{rz}}{r} + F_z = \rho u_{z,tt}.
\]

(1.10.7)

Strain-Displacement Relations

Coordinates measured in terms of displacements:
\[ \delta r = u_r, \quad \delta \theta = \frac{u_\theta}{r}, \quad \delta z = u_z. \]

The strain equations are
\[ e_{rr} = u_{r,r}, \quad e_{\theta\theta} = \frac{(u_r + u_{\theta,\theta})}{r}, \]
\[ e_{zz} = u_{z,z}, \quad 2 e_{r\theta} = u_{r,\theta} + ru_{\theta,r} - u_\theta, \]
\[ 2 e_{rz} = u_{r,z} + u_{z,r}, \quad 2 e_{z\theta} = u_{\theta,z} + \frac{u_{z,\theta}}{r}, \]

(1.10.8)

where \( u_r, u_\theta, u_z \) are the displacement components in the increasing directions of \( r, \theta, z. \)
1.10.3 Spherical Polar Co-Ordinate System

Here,
\[ \alpha_1 = r = \text{spheres}, \]
\[ \alpha_2 = \theta = \text{cones with vertex at the point } o, \]
\[ \alpha_3 = \phi = \text{planes through the oz axis}, \]
\[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta, \]
from which
\[ h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin \theta, \quad \sqrt{g} = r^2 \sin \theta, \]
\[ ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \]

Equations of Motion

If \( \tau_{\theta \theta}, \tau_{\phi \phi}, \tau_{r r}, \tau_{r \phi}, \tau_{r \theta} \) and \( \tau_{\phi \theta} \) are the stress components and \( F_r, F_\theta, F_\phi \) are the body forces then the equations are given by

\[
\begin{align*}
\tau_{rr,r} + \frac{\tau_{r\theta,\theta}}{r} + \frac{\tau_{r\phi,\phi}}{r \sin \theta} + \frac{2\tau_{rr} - \tau_{\theta \theta} - \tau_{\phi \phi} + \tau_{r \theta} \cot \theta}{r} + F_r &= \rho u_{r,tt}, \\
\tau_{r\theta,r} + \frac{\tau_{\theta \theta,\theta}}{r} + \frac{\tau_{\theta \phi,\phi}}{r \sin \theta} + \frac{\left(\tau_{\theta \theta} - \tau_{\phi \phi}\right) \cot \theta + 3 \tau_{r \theta}}{r} + F_\theta &= \rho u_{\theta,tt}, \\
\tau_{r\phi,r} + \frac{\tau_{\phi \theta,\theta}}{r} + \frac{\tau_{\phi \phi,\phi}}{r \sin \theta} + \frac{2 \tau_{\theta \phi} \cot \theta + 3 \tau_{r \phi}}{r} + F_\phi &= \rho u_{\phi,tt},
\end{align*}
\]

Strain-Displacement Relations

Coordinates measured in terms of displacements:

\[ \delta r = u_r, \quad \delta \theta = u_\theta/r, \quad \delta \phi = u_\phi/r \sin \theta. \]

The strain equations are
\[ \begin{align*}
\mathbf{e}_{rr} &= u_{r,r}, \\
\mathbf{e}_{\theta\theta} &= \left(\frac{u_r + u_{\theta,\theta}}{r}\right), \\
2\mathbf{e}_{\phi\phi} &= \left(\frac{u_r + u_{\theta,\phi} \cot \theta + \frac{u_{\phi,\phi}}{\sin \theta}}{r}\right), \\
2\mathbf{e}_{\phi r} &= \left(\frac{ru_{r,\phi} + ru_{\phi,r} - u_{\theta}}{r}\right), \\
2\mathbf{e}_{\phi \theta} &= \left(u_{\phi,\theta} - \cot \theta u_{\phi} + \frac{u_{\theta,\phi}}{\sin \theta}\right).
\end{align*} \tag{1.10.10} \]

1.11 Wave Equation

Consider an infinite medium. The body forces $\mathbf{F}$ acting on this medium and the displacement field $\mathbf{u}$ are represented as

\[ \mathbf{F} = \nabla \Phi + \text{rot} \mathbf{\Psi}, \tag{1.11.1} \]

\[ \mathbf{u} = \nabla \Phi + \text{rot} \mathbf{\Psi}, \tag{1.11.2} \]

where $\Phi$ and $\phi$ are scalar functions of the coordinates $(x, y, z)$ and the time $t$. $\mathbf{\Psi}$ and $\mathbf{\Psi}$ are vector functions of the coordinates $(x, y, z)$ and the time $t$.

From (1.11.2), it follows that

\[ \text{div} \mathbf{u} = \nabla^2 \phi. \tag{1.11.3} \]

Also, using generalized form of Hooke’s Law, $\tau_{ij} = \lambda \delta_{ij} + 2\mu e_{ij}$, the equations of motion (1.10.4) in a homogeneous isotropic elastic media can be represented as

\[ \begin{align*}
(\lambda + \mu)\theta_{x,x} + \mu \nabla^2 u_x + F_x &= \rho u_{x,tt}, \\
(\lambda + \mu)\theta_{y,y} + \mu \nabla^2 u_y + F_y &= \rho u_{y,tt}, \\
(\lambda + \mu)\theta_{z,z} + \mu \nabla^2 u_z + F_z &= \rho u_{z,tt}.
\end{align*} \tag{1.11.4} \]

Here

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator.

The system of equations (1.11.4) is equivalent to the differential equation in vector form
\[(\lambda + \mu) \nabla \text{div} \mathbf{u} + \mu \nabla^2 \mathbf{u} + \rho \left( \mathbf{F} - \frac{\partial^2 \mathbf{u}}{\partial t^2} \right) = 0. \tag{1.11.5}\]

Substituting expressions (1.11.1) and (1.11.2) in equation (1.11.5), taking into account (1.11.3) and interchanging the order of the differential operators, it is obtained as

\[
(\lambda + \mu) \nabla \nabla^2 \phi + \mu \nabla^2 \left( \nabla \phi + \text{rot} \vec{\psi} \right) + \rho \left( \nabla \Phi + \text{rot} \vec{\Psi} - \frac{\partial^2 \Phi}{\partial t^2} \left( \nabla \phi + \text{rot} \vec{\psi} \right) \right) = 0,
\]

\[
\nabla \left[ (\lambda + 2\mu) \nabla^2 \phi + \rho \phi - \rho \frac{\partial^2 \phi}{\partial t^2} \right] + \text{rot} \left[ \mu \nabla^2 \vec{\psi} + \rho \vec{\psi} - \rho \frac{\partial^2 \vec{\psi}}{\partial t^2} \right] = 0,
\]

\[
\nabla \left[ c_1^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} + \phi \right] + \text{rot} \left[ c_1^2 \nabla^2 \vec{\psi} - \frac{\partial^2 \vec{\psi}}{\partial t^2} + \vec{\psi} \right] = 0, \tag{1.11.6}
\]

where

\[
c_1^2 = \frac{(\lambda + 2\mu)}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}. \tag{1.11.7}
\]

Equation (1.11.6) is satisfied by assuming,

\[
c_1^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -\phi, \tag{1.11.8}
\]

\[
c_2^2 \nabla^2 \vec{\psi} - \frac{\partial^2 \vec{\psi}}{\partial t^2} = -\vec{\psi}. \tag{1.11.9}
\]

Thus, the vector field \( \mathbf{u} \) defined by (1.11.2) is the solution of equation (1.11.5) if the functions \( \phi \) and \( \vec{\psi} \) satisfy (1.11.8) and (1.11.9); the function \( \phi \) is called the longitudinal potential and \( \vec{\psi} \) is the transverse potential.
Some important consequences about these equations are mentioned below

a) Let $\vec{\psi} \equiv 0$ and the initial conditions be $\vec{\psi} = 0$ when $t = t_0$. The resulting equation for the determination of $\vec{\psi}$ is then the homogeneous equation (1.11.9) with zero initial conditions. This means that $\vec{\psi}$ is always zero. It follows from equation (1.11.2) that $\vec{u} = \nabla \phi$ and $\text{rot} \ \vec{u} = 0$.

This shows that a wave described by the function $\phi$ involves no rotation of the particle of the medium i.e. each of them has a motion of translation. Such waves are called longitudinal waves.

b) Let $\Phi \equiv 0$ and the initial conditions be $\phi = 0$ when $t = t_0$. Then, the resulting equation for the determination of $\phi$ is the homogeneous equation (1.11.8) with zero initial condition. It follows that $\phi \equiv 0$ and $\vec{u} = \text{rot} \ \vec{\psi}$.

This shows that in this field the dilatation is zero i.e. $\text{div} \ \vec{u} = 0$. Such waves are called transverse or shear waves.

### 1.12 Classification of Second Order Partial Differential Equations

The general linear partial differential equation of the second order in two independent variables is of the form

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + f(x,y,u,p,q) = 0$$

i.e.

$$A(x,y)p + B(x,y)s + C(x,y)t + f(x,y,u,p,q) = 0,$$

where $p = u_x$, $q = u_y$.

Such a partial differential equation is said to be

(i) **Elliptic**, if $B^2 - 4AC < 0$

(ii) **Hyperbolic**, if $B^2 - 4AC > 0$

(iii) **Parabolic**, if $B^2 - 4AC = 0$

Consider the following partial differential equations
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{(Laplace Equation)} \tag{1.12.1}
\]
\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}, \quad \text{(Wave Equation)} \tag{1.12.2}
\]
\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t}, \quad \text{(Heat Conduction Equation)} \tag{1.12.3}
\]

where \((x, y)\) are space coordinates and \(t\) is time coordinate.

In (1.12.1), \(A = 1, B = 0, C = 1\), so \(B^2 - 4AC = -4 < 0\).
Hence, Laplace equation is of elliptic type.

In (1.12.2), \(A = 1, B = 0, C = -1/c^2\), so \(B^2 - 4AC = 4/c^2 > 0\).
Hence, Wave equation is of hyperbolic type.

In (1.12.3), \(A = 1, B = 0, C = 0\), so \(B^2 - 4AC = 0\).
Hence, Heat conduction equation is of parabolic type.

### 1.13 Classification of Problems

#### 1.13.1 One Dimensional Problems

Some of the characteristic functions of wave motion in a continuum can be determined by an analysis in one dimensional geometry with one spatial variable. One length co-ordinate and time are sufficient to describe one dimensional problems. All particles move along parallel lines in one–dimensional longitudinal motion.

#### 1.13.1.1 Longitudinal Strain

Longitudinal strain is the ratio of change in length to the original length. In this, out of all displacement components only the longitudinal displacement component \(u\) is non zero and so there is only one non zero strain component \(e_{xx} = u_{,x}\).
1.13.1.2 Longitudinal Stress

In longitudinal stress $\tau_{xx} = \tau(x, t)$ is the only non-zero stress component which is a function of $x$ and $t$ only. It is a stress that tends to change the length of a body. The transversal normal stresses $\tau_{yy}$ and $\tau_{zz}$ will be zero in this case. The equation of motion in the absence of body force is given by

$$\tau_x = \rho(x) u_{,tt} \quad (\text{for non-homogeneous medium}). \tag{1.13.1}$$

1.13.1.3 Shearing Stress

Shearing stress acts to change the angles/shape in a body. It also tends to deform originally rectangular objects into parallelograms. In this, deformation takes place perpendicular to a given axis ($x$-axis say) rather than parallel to it i.e. displacement components are $u = 0$ and $v = v(x, t), w = w(x, t)$ and the stress components are given by

$$\tau_{xy} = \mu v_x \quad \text{and} \quad \tau_{zx} = \mu w_x. \tag{1.13.2}$$

1.13.2 Two Dimensional Problems

There are two general types of plane problems having great practical importance in many problems of elasticity, viz. plane strain and plane stress.

1.13.2.1 Plane Strain

A body is said to be in the state of plane strain or plane deformation parallel to the $xy$-plane if the strain normal to the $xy$-plane $e_{zz}$ and the shear strains $e_{xz}$ and $e_{yz}$ are assumed to be zero. We can also say that

$$\begin{align*}
u & = u(x, y, t), \\
v & = v(x, y, t), \\
w & = 0. \tag{1.13.3}
\end{align*}$$
Then the strain components reduce to:

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = 0, \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \varepsilon_{yz} = 0, \quad \varepsilon_{xz} = 0.
\end{align*}
\] (1.13.4)

Because of the presence of the component \(\tau_{zz}\), a state of plane stress is not achieved. For a body in plane strain, there are five unknowns only viz., \(u, v, \tau_{xx}, \tau_{yy}, \tau_{xy}\).

In plane strain, the presence of prismatic body of infinite length with straight axis i.e. in which the dimension of considered body is very large in one direction (say \(z\)-axis) as compared to other \(x\) and \(y\) dimensions is considered. The loads are applied only perpendicular to \(z\)-axis and are uniformly distributed along the axis of the body. Some applications of this type of presentation involve in the analysis of dams, tunnels and other geotechnical works.

**1.13.2.2 Plane Stress**

A body is said to be in the state of plane stress parallel to \(xy\)-plane, if the normal stress \(\tau_{zz}\) and the shear stresses \(\tau_{xz}, \tau_{yz}\) perpendicular to the \(xy\)-plane are assumed to be zero. The stress – strain relations for plane stress become

\[
\begin{align*}
\tau_{xx} &= \lambda(e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{xx}, \quad \tau_{xy} = 2\mu e_{xy} \\
\tau_{yy} &= \lambda(e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{yy}, \quad \tau_{xz} = 0, \\
\tau_{zz} &= 0 = \lambda(e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{zz}, \quad \tau_{yz} = 0.
\end{align*}
\] (1.13.5)

The geometry of the body in a plane stress problem is like that of a thin plate with one dimension very small than the others. The loads are uniformly distributed over the thickness, parallel to the plane of the plate. This geometry is in contrast with the geometry of the plane strain problem. The non- zero stress components remain constant over the thickness of the plate and are functions of \(x\) and \(y\) only. The plane stress condition is the simplest form of behavior for continuum structures.
1.13.3 Special Problems

Here two special types of problems (thin rods and half-space) are discussed in which three dimensional formulation reduce to boundary value problems and field variables depend upon only one spatial co-ordinate and time.

1.13.3.1 Thin Rods

In thin rod problems, x-axis is to be taken along the axis of the rod and stresses $\tau_{xx}, \tau_{xy}, \tau_{xz}$ are assumed to be non-zero whereas $\tau_{zz} = \tau_{yy} = \tau_{yz} = 0$. Here all stress, strain and displacement components are functions of $x$ and $t$ only.

1.13.3.2 Half-space

In this case, x-axis is to be taken perpendicular to the plane face and the strain $e_{xx}$ is assumed to be non-zero whereas $e_{yy} = e_{zz} = 0$. All components are function of $x$ and $t$ only. The governing equations for both the cases are similar apart from the constants in the equations.

1.14 Methods of Solution

1.14.1 General Methods

Partial differential equations (model theory) occur quite frequently in the wave propagation models. Many methods are available for the solutions of these types of equations like separation of variables; steady state harmonic conditions method and Integral transform methods.

1.14.1.1 Separation of Variables

It is a common technique to obtain a solution of the wave equation. It is so called because the equation is rearranged in such a manner that all terms involving the dependent variable appear on one side of the equation and all terms involving the independent variable appear on the other side.
1.14.1.2 Steady State Harmonic Conditions Method

It is a special case of separation of variables. In this method, it is not the condition that two isotropic mechanical characteristic functions to be related by a real constant and they remain independent.

1.14.1.3 Laplace Transform

It is one of the integral transform techniques. Here the kernel is exponential form i.e. \( k(s, t) = e^{-st} \), so

\[
f(s) = \int_{0}^{\infty} f(t)e^{-st}dt,
\]

\( \text{where } s \text{'s'} \text{ is the Laplace transform parameter.} \)

Inverse Laplace transform is given by

\[
f(t) = \frac{1}{2\pi i} \int_{y-i\infty}^{y+i\infty} f(s)e^{st}ds,
\]

\( \text{where the path of integration is any line parallel to the imaginary axis and to the right of all the singularities of } f(s). \)

The complex integral is usually found by the use of Residue Theorem of complex variables given by

\[
\int_{\gamma} f(s)e^{st}ds = 2\pi i \sum \text{(residues)}.
\]

Sometimes, it is difficult to use inversion of Laplace transformed solution for all times. So solutions for small times are derived by the use of Abel’s theorem

\[
\lim_{t \to 0} f(t) = \lim_{s \to \infty} f(s),
\]

\( \text{where the small values of time correspond to large values of Laplace transform parameter.} \)
1.14.2 Sine Function Method

This method is used for the solution of non linear partial differential equations of the form

\[ F(u, u_t, u_x, u_{xx}, u_{xxt}, \ldots \ldots ) = 0, \]  

where \( u(x, t) \) is the solution of non linear partial differential equation (1.14.4).

In this method, the following transformations are applied.

\[ u(x, t) = f(\xi), \xi = x - ct, \quad \text{(equation of wave front)}. \]  

With the help of equation (1.14.5), equation (1.14.4) transforms in the form

\[ G(f, f', f'', f''', \ldots \ldots \ldots ) = 0. \]  

The solution of equation (1.14.6) can be expressed in the form

\[ f(\xi) = \lambda \sin^\alpha (\mu \xi), \quad |\xi| \leq \frac{\pi}{\mu}, \]  

where \( \lambda, \alpha, \mu \) are unknown parameters which are to be determined.

Substituting equation (1.14.7) in equation (1.14.6), a trigonometric equation in terms of \( \sin^\alpha (\mu \xi) \) is obtained. To determine the parameters first balance the exponents of each pair of sine. Then collecting all terms with the same power in \( \sin^\alpha (\mu \xi) \) and then put their coefficients equal to zero to get a system of algebraic equations among the unknowns \( \lambda, \alpha \) and \( \mu \). The problem reduces to a system of algebraic equations that can be solved to obtain all the unknown parameters.

1.14.3 Green Function Method

This technique is very useful tool to solve inhomogeneous differential equations subject to certain boundary conditions. It serves analogous role in partial differential equations as do Fourier series in solution of ordinary differential equations. In this technique, first of all a function \( G(x,t) \) known as Green’s
function is constructed as a solution for a point source of non-homogeneous B.V.P. of the form

\[ L y(x) = f(x) \]  \hspace{1cm} (1.14.8)

in the interval \( a \leq x \leq b \) with \( f(x) \) a specified function known as source function. The boundary conditions being specified at \( x = a \) and \( x = b \). Here \( L \) is a differential operator given by

\[ L = \frac{d}{dx} \left\{ p(x) \frac{d}{dx} \right\} + q(x) \]  \hspace{1cm} (1.14.9)

The constructed Green’s function is such that

(i) \( G(x, t) \) satisfies the differential equation

\[ L G(x, t) = -\delta(x - t). \]

Where \( \delta(x - t) \) is the Dirac–delta function, its value is zero except at \( x = t \).

(ii) \( G(x, t) \) satisfies the given boundary conditions at end points of interval i.e. at \( x = a \) and \( x = b \). Then the solution of given boundary value equation (1.14.8) will be given by

\[ y(x) = -\int_{a}^{b} G(x, t)f(t)dt. \]  \hspace{1cm} (1.14.10)

This method has been applied in the present work.

1.14.4 Approximate Analytic Procedures

There are methods which are useful for the solutions of the problems in an approximate manner as

(i) W.K.B.J. (Wentzol – Krammers – Brilloum - Jeffreys) is applicable to the second order linear differential equations.

(ii) Rayleigh – Ritz method is useful for investigating approximate solutions of problems expressed in various form.

(iii) Fourier series method holds for solving physically interesting partial differential equations. In this, given function \( f(x) \) is represented as a
convergent series in the elementary trigonometric functions defined for 
\(-\pi \leq x \leq \pi\) as

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx], \quad (1.14.11)
\]

where \(a_0, a_n\) and \(b_n\) are Fourier coefficients given by:

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\]

The notation of a Fourier series to complex coefficients is given by

\[
f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx}, \quad (1.14.12)
\]

where

\[
A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.
\]

Fourier Transform is a generalization of the complex Fourier series in the limit as \(L \to \infty\). Replace the discrete \(A_n\) with the continuous \(F(k)dk\) while letting \(n/L \to k\). Then changing the sum to an integral, equation becomes

\[
f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i k x} \, dk. \quad (1.14.13)
\]

Here

\[
F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx \quad (1.14.14)
\]

is called forward Fourier transform and
(1.14.15)

\[ f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i kx} \, dx \]

is called inverse Fourier transform.

This transformation has been used in the present work.

### 1.15 Survey of Literature and Motivation

In the earlier times, many researchers have shown a lot of interest on wave propagation in non-homogeneous Engineering material. Early attempt in this field was made by Victorov (1958). Singh (1967) studied the SH-waves propagation in a laterally and vertically heterogeneous layered media. For non-homogeneous isotropic materials properties have been taken into various forms: Exponential variation by Rostovtsev & Khanevkskaia (1973); Rao & Goda (1978): Integral Power Law variation by Mukhopadyaya (1979); Chandra (1980); Narian (1980) and Binominal by Singh and Dhaliwal (1980); Backus (1962), Chow (1971), Crampin (1981) and Nayfeh (1991) have discussed the wave propagation problems in non-homogeneous anisotropic media. Some authors also have studied different fields of wave propagation like: simulating seismic wave propagation in 3D elastic media using staggered-grid finite differences method by Graves (1996); velocity dispersion of guided plate-waves by Chimenti (1997); propagation of SH-Waves in layered half space by Romeo (1997); propagation of transient SH-waves in a cylindrical anisotropic solid by Watanabe and Payton (1997); propagation of time harmonic circumferential waves in two dimensional infinite long hollow cylinder by Valle et al. (1999); thermo elastic Lamb waves in a plate bordered with layer of in-viscid liquid by Sharma and Pathania (2003); propagation of Lamb waves in homogeneous isotropic plate by Sharma and Pal (2004); thermo elastic wave propagation in isotropic cylindrical curved plates by Sharma and Pathania (2005); elastic wave propagation in sinusodially corrugated waveguides by Banerjee and Kundu (2006); wave propagation in elastic and poroelastic media in the frequency domain by the boundary element method by Ferro and Mansur (2006); effect of rotation on Rayleigh waves in piezothermoelastic half-space by Sharma and walia (2007); experimental study of surface wave propagation in strongly heterogeneous
media by Aggelis and Shiotani (2007); influences of rotation, magnetic field, initial stress and gravity on Rayleigh waves in a homogeneous orthotropic elastic half-space by Abd-Alla et al. (2010).

The study of wave propagation in non-homogeneous elastic rod has been carried out by many researchers. Rosenfeld and Miklowitz (1965), Rosenfeld and Keller (1973), Vasudeva and Bhaskara (1978), Gupta (1978), Chaudhuri and Sen (1983) have discussed wave propagation in elastic rod with variable cross sectional area. Datta (1956), Sur (1961), Lindholm and Doshi (1965) have studied the problem of rod with constant density and variable elastic parameters. Datta (1956) has taken elastic parameters as linear functions in the space coordinate, x, measured along the rod. But Sur (1961) has discussed with variation as exponential in x and elastic parameters variation as proportional to $x^n$ has been taken by Lindholm and Doshi (1965). In all these studies, the solution has been obtained in terms of Bessel function. Kishtaiah and Shukla (1983) have investigated the effect of wave propagation in a non-homogeneous rod with variable density and elastic parameters as sine functions of x. Here, extending the work of Kishtaiah and Shukla (1983), effect of wave propagation in a non-homogeneous rod has been observed by assuming cosine variation of elastic parameters and density along the length of the rod using Legendre’s functions.

Now in recent times, more emphasis on: Dispersion equation of Love wave propagation in homogeneous layer lying over laterally heterogeneous half-space by DE (1985); the effect of point source and heterogeneity on the propagation of SH-waves by Chattopadhyay et al. (2010); effect of point source and self reinforcement on magneto-elastic shear waves by Chattopadhyay et al. (2011) and others. To simplify the mathematics, generally half-space is taken as heterogeneous. But, the problems involving layer to be heterogeneous are more relevant since earth’s crust is relatively more heterogeneous than the mantle. So, to investigate Love wave propagation, considering layer to be heterogeneous is more realistic. Chatterjee (1971), Sidhu (1971), Gupta et al. (2013) have investigated the effect on Love wave propagation by considering both the layer and half-space as non-homogeneous. In the present work, wave propagation has been studied by considering heterogeneous sandwiched layer lying between homogeneous upper
half-space and non-homogeneous lower half-space using Green’s function technique.

Also anisotropic media has been used to study seismic waves by various authors like: propagation of Rayleigh and Love-waves in anisotropic media by Bouden and Datta (1990); wave propagation in arbitrary anisotropic laminates by Liu et al. (1990); Surface waves in fibre-reinforced anisotropic elastic media by Sengupta and Nath (2001); elastic waves propagated in circumferential direction in anisotropic cylindrical curved plate by using Fourier series by Towfighi et al. (2002); higher order asymptotic homogenization and wave propagation in periodic composite materials by Andrianov et al. (2008); torsional waves in self-reinforced medium by Chattopadhyay et al. (2009); study of wave motion in an anisotropic fiber reinforced thermo-elastic solid by Kumar and Gupta (2010); torsional wave in pre-stressed fiber reinforced media by Kakar and Kakar (2012). In all these studies, fiber reinforced direction has been taken in x-direction only. In the present work, both x and z axes has been taken as directions of fiber reinforcement separately and then particular cases of seismic wave has been discussed.

Here, the wave propagation discontinuities in mathematical models are given in the form of second order partial differential equations in space variable (x, y, z) and time variable t. Integral transforms like Fourier transforms, Laplace transforms are used to solve this model. Functions as harmonics by substitution are used to reduce P.D.E. to O.D.E., which further can be solved by the method of variable of separation, homogeneous and non homogeneous, exact differential equation, differential equation with constant, coefficients, Cauchy's differential equation, differential equation of Lagrange's form, variation of parameter, undetermined coefficients and many more. The solution of differential equations by numerical methods like Gauss Seidel method, Finite element method, Jacobi method, Finite difference method, W.K.B.J method, Frobenius method series solution is found out and special functions method is used if exact solution of differential equation can't be find out.

In the survey of literature and research work, following types of problems on non-homogeneity are observed.
1. When density is constant and mechanical properties are space dependent \((x, y, z)\), then wave speed is also space dependent.

2. When both density and mechanical properties are space dependent \((x, y, z)\) but;

   I. Wave speed is constant i.e. the ratio of elastic coefficient \((\mu)\) to density \((\rho)\) is taken as constant, so in this case the non homogeneity of both density and mechanical properties affects the wave amplitude only.

   II. Wave speed is not constant i.e. space dependent which means the ratio of elastic coefficient to density is not constant.

3. When density, elastic coefficients and area of cross-section are space dependent \((x, y, z)\).

   In the present work, the characteristics of engineering materials by wave propagation in non-homogeneous elastic media, where both the density and mechanical properties are dependent on space coordinates are considered and following problems have been discussed:

   1. Effect of wave propagation in non-homogeneous elastic rod with variable characteristics.

   2. Effect of impulsive line source on the propagation of SH-Wave in non-homogeneous isotropic media.

   3. Surface waves in fibre-reinforced anisotropic solid elastic media under the influence of gravity.

   4. Wave propagation in compressed material with reinforcement in preferred direction subjected to gravity and initial compression.

   The applications of these studies can be helpful in Seismology, Earthquake and Civil Engineering, Acoustics, Applied Physics, Applied Mathematics, for detecting impurities/defects in composite materials, for investigating the structure of the interior of the earth and also in industrial materials such as textile and paper industries.