CHAPTER-II

BASIC CONCEPTS

2.1. List of symbols

\( \lambda, \mu \) \quad \text{Lame's constants}

\( \overbar{\lambda}, \overbar{\mu} \) \quad \text{Lame's constants for cracked solid}

\( K \) \quad \text{bulk modulus}

\( \overbar{K} \) \quad \text{effective bulk modulus}

\( \rho \) \quad \text{density of generalized thermoelastic solid}

\( \overbar{\rho} \) \quad \text{density of cracked generalized thermoelastic solid}

\( K^* \) \quad \text{coefficient of thermal conductivity}

\( \overbar{K}^* \) \quad \text{thermal conductivity of cracked generalized thermoelastic solid}

\( \overbar{K}^* = \overbar{K}^*/\overbar{\rho} \)

\( C^* \) \quad \text{specific heat at constant strain}

\( \overbar{C}^* \) \quad \text{specific heat of cracked generalized thermoelastic solid}

\( \theta \) \quad \text{temperature variable}

\( \theta_0, t_0 \) \quad \text{initial uniform temperature}

\( t_0, t_1 \) \quad \text{relaxation times}
$\tilde{t}_0, \tilde{t}_1$ effective relaxation times

$u, U$ displacement vectors

$\nabla$ del operator

$u_k$ components of displacement vector $u$

$k$ wave number

$c$ apparent phase velocity on the surface

$\omega (= kc)$ angular frequency

$\delta_{kl}$ Kronecker delta

$\sigma_{ij}$ components of the force stress tensor

$\bar{\sigma}_{ij}$ components of the force stress tensor in cracked solid

$x_j$ components of the position vector

$\nu$ Poisson's ratio

$\bar{\nu}$ effective poisson ratio

$\nu_1$ thermocoupling coefficient

$\bar{\nu}_1$ coupling coefficient in cracked solid

$\bar{\nu}_l$ $\bar{\nu}_l/\bar{p}$

$D, \Omega$ saturation parameters
ε crack density
r radius of circular crack
d thickness of circular crack
d/ţ aspect ratio
βc crack porosity
ρs density of solid in the absence of cracks
ρf density of the fluid present in the cracks
ξ partial saturation parameter
i = √−1

the quantities with primes correspond to upper medium.

qi components of the heat flux vector

\( u_{i,i} = \frac{\partial u_i}{\partial x_i} \)

\( \dot{u}_{i,j} = \frac{\partial^2 u_i}{\partial x \partial t} \)

2.2. Governing equations of thermoelasticity with circular cracks

Taking a fixed rectangular Cartesian co-ordinate system with co-
ordinate axes \( x_i(i = 1, 2, 3) \). The basic equations of linear thermoelasticity
following Boley and Weiner (1967) are
(a) The balance of linear momentum gives the equation of motion without body forces

\[ \sigma_{ij,j} = \rho \ddot{u}_i \quad i, j = 1, 2, 3 \]  \hspace{1cm} (2.1)

(b) The local energy balance gives the energy equation

\[ -q_{i,i} = \rho C^* \theta + v_1 \theta_0 \dot{u}_{k,k} \]  \hspace{1cm} (2.2)

(c) The stress-strain temperature relation

\[ \sigma_{ij} = \lambda u_{k,k} \delta_{i,j} + \mu (u_{i,j} + u_{j,i}) - v_1 \theta \delta_{ij} \]  \hspace{1cm} (2.3)

(d) Fourier law of heat conduction

\[ q_i = -K^* \theta_{,i} \]  \hspace{1cm} (2.4)

Following Lord and Shulman (1967) the modified Fourier law with thermal relaxation is

\[ q_i + t_0 \dot{q}_i = -K^* \theta_{,i} \]  \hspace{1cm} (2.5)

From equation (2.1) and (2.3), the equation of motion becomes

\[ (\lambda + \mu) u_{k,ki} + \mu \nabla^2 u_i - v_1 \theta_{,i} = \rho \ddot{u}_i \]  \hspace{1cm} (2.6)

From equations (2.2) and (2.5), the equation of heat conduction becomes

\[ \rho C^*(\dot{\theta} + t_0 \ddot{\theta}) + v_1 \theta_0 (\dot{u}_{k,k} + t_0 \ddot{u}_{k,k}) = K^* \theta_{,ii} \]  \hspace{1cm} (2.7)

Equations (2.6) and (2.7) are the governing equations of generalized thermoelasticity in context of Lord-Shulman theory.
Following Green and Lindsay (1972) the equation (2.2) and (2.3) with two thermal relaxations can be written as

\[ -q_{i,i} = \rho C^* (\theta + t_0 \dot{\theta}) + v_l \theta_0 \dot{u}_{k,k} \]  \hspace{1cm} (2.8)

\[ \sigma_{ij} = \lambda \ u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) - v_l (\theta + t_1 \dot{\theta}) \delta_{ij} \]  \hspace{1cm} (2.9)

From (2.1) and (2.9), we have

\[ (\lambda + \mu) u_{k,ki} + \mu \nabla^2 u_i - v_l (\theta + t_1 \dot{\theta})_{,i} = \rho \ \ddot{u}_i \]  \hspace{1cm} (2.10)

From (2.4) and (2.8), we have

\[ \rho C^* (\dot{\theta} + t_0 \ddot{\theta}) + v_l \theta_0 \dot{u}_{k,k} = K^* \theta_{,ii} \]  \hspace{1cm} (2.11)

Equations (2.10) and (2.11) are governing equations of generalized thermoelasticity in context of Green-Lindsay theory.

The equations (2.6), (2.10) and (2.7), (2.11) can be written jointly in the following form

\[ (\lambda + \mu) u_{k,ki} + \mu \nabla^2 u_i - v_l (\theta + t_1 \dot{\theta})_{,i} = \rho \ \ddot{u}_i \]  \hspace{1cm} (2.12)

\[ \rho C^* (\dot{\theta} + t_0 \ddot{\theta}) + v_l \theta_0 (\dot{u}_{k,k} + \Delta t_0 \ \ddot{u}_{k,k}) = K^* \theta_{,ii} \]  \hspace{1cm} (2.13)

For the L-S(Lord-Shulman) theory, \( t_1 = 0, \Delta = 1 \) and for G-L(Green-Lindsay) theory \( t_1 > 0 \) and \( \Delta = 0 \). The thermal relaxations \( t_0 \) and \( t_1 \) satisfy the inequality \( t_1 \geq t_0 \geq 0 \) for the G-L theory only.
In absence of thermal relaxation times, equations (2.12) and (2.13) reduces to the thermoelastic field equations as obtained by Chadwick and Sneddon (1958) or Lockett (1958).

The field equations (2.12) and (2.13) of generalized thermoelastic solid with two relaxation times can also be written in following form

\[ c_1^2 \nabla (\nabla \cdot u) - c_2^2 \nabla \times (\nabla \times u) - \nabla_1 \nabla (\theta + t_1 \dot{\theta}) = \ddot{u} \tag{2.14} \]

\[ \rho C^*(\theta + t_0 \dot{\theta}) + v_1 \theta_0 (\ddot{u}_{ij} + \Delta t_0 \dddot{u}_{ij}) = K^* \nabla^2 \theta \tag{2.15} \]

where

\[ c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho} \]

and symbols have their usual meanings.

Now we consider a generalized thermoelastic solid containing a random distribution of circular cracks of radius \( r \) and thickness \( d \), with very small aspect ratio \( d/r \). Cracks are saturated by a fluid of bulk modulus \( K \).

Elastic constants are modified according to the regimes of connection between cracks, defined by O’Connell and Budiansky (1974). Modified elastic constants for saturated isolated cracks are as follows:

**Saturated isolated**: No fluid is able to flow out of (or into) any crack.

Changes in fluid pressure due to application to external stress will be
different in every crack. The elastic constants of the cracked solids are
given by Budiansky and O'Connell (1976) by using self consistent method.

\[
\frac{K}{K^*} = 1 - \left( \frac{16}{9} \right) \frac{(1 - \tilde{\nu}^2)}{(1 - 2\tilde{\nu})} D\varepsilon
\]

(2.16)

and

\[
\frac{\mu}{\mu^*} = 1 - \left( \frac{32}{45} \right) \frac{(1 - \tilde{\nu})}{(2 - \tilde{\nu})} \frac{D + 3}{2} \varepsilon
\]

(2.17)

where

\[
K = \lambda + \frac{2}{3} \mu , \quad \nu = \frac{\lambda}{2(\lambda + \mu)}
\]

The barred quantities represent corresponding elastic parameters for
saturated cracked solid. Following Sharma (1996), saturation parameter D is
expressed in terms of elastic constants and crack density \( \varepsilon \) as

\[
\varepsilon D(1 - \tilde{\nu}^2) = 45 \frac{(\nu - \tilde{\nu})}{(1 + 3\nu)} \{16(1 + 3\nu)\}
\]

\[
+ 2\varepsilon (1 - 2\nu)(1 - \tilde{\nu})^2 / (1 + 3\nu)(2 - \tilde{\nu})
\]

(2.18)

Effective Poisson’s ratio (\( \tilde{\nu} \)) is computed numerically from a fifth degree
equation (Sharma (1996)) for given values of \( \varepsilon, \nu \) and another saturation
parameter \( \Omega \), defined by

\[
\Omega = \alpha / c (K^*/K).
\]

(2.19)
For circular cracks (radius = r), volume of sample crack (thickness = d) is given by \( \frac{4}{3} \pi r^2 d \). Crack porosity \( (\beta_c) \) measures the fraction of volume occupied by cracks and is related to crack density \( (\varepsilon) \) by

\[
\beta_c = \frac{4\pi d \varepsilon}{3r}.
\]  
(2.20)

The density of cracked solid \( (\bar{\rho}) \) is modified as

\[
\bar{\rho} = (1 - \beta_c) \rho_s + \beta_c \rho_f.
\]  
(2.21)

Elastic constants for partially saturated cracked solid are obtained by replacing \( D \) with \( 1 + \xi + D\xi \). Density of such a solid is given by

\[
\bar{\rho} = (1 - \beta_c) \rho_s + \xi \beta_c \rho_f.
\]  
(2.22)

The thermal constants and thermo-coupling co-efficient are also changed due to modifications in elastic constants, density etc. Following Nayfeh and Nasser (1971), effective relaxation time for cracked generalized thermoelastic solid becomes

\[
\bar{t}_0 = \frac{3K^*}{C^*\bar{\rho} \bar{c}_i^2},
\]  
(2.23)

where \( \bar{c}_i^2 = \frac{(K + 4/3\mu)}{\bar{\rho}} \) is the velocity of modified P-wave in a cracked solid, \( K^*, C^* \) are the thermal conductivity and specific heat in presence of cracks which are approximated by Qin et al. (1998) using self-consistent
method. The other relaxation time $\tilde{t}_1$ is assumed to be of same order as that of $t_0$.

With the above modifications the constitutive and field equations for generalized thermoelasticity in presence of circular cracks becomes

\begin{align}
\overline{\sigma}_{ij} &= \left(\overline{K} - \frac{2}{3} \overline{\mu}\right) u_{k,k} \delta_{ij} + \overline{\mu}(u_{i,j} + u_{j,i}) - \overline{\nu}_i \theta_0 \dot{u}_{k,k} \quad (2.24) \\
\overline{c}_i^2 \nabla (\nabla \cdot u) - \overline{c}_2^2 \nabla \times (\nabla \times u) - \overline{\nu}_i \nabla (\theta + \tilde{t}_i \dot{\theta}) &= \ddot{u} \quad (2.25) \\
\overline{\rho} \overline{C}^*(\theta + \tilde{t}_0 \dot{\theta}) + \overline{\nu}_i \theta_0 (\dot{u}_{i,i} + \Delta \tilde{t}_0 \dot{u}_{i,i}) &= \overline{K}^* \nabla^2 \theta \quad (2.26)
\end{align}

where

\begin{align}
\overline{c}_i^2 &= \left(\overline{K} + \frac{4}{3} \overline{\mu}\right)/\overline{\rho}, \quad \overline{c}_2^2 = \frac{\overline{\mu}}{\overline{\rho}}
\end{align}

2.3. Solution of governing equations

To solve the equations (2.25) and (2.26), we decompose the displacement vectors as

$$u = \nabla \phi + \nabla \times \overline{\psi}, \quad \nabla \cdot \overline{\psi} = 0. \quad (2.27)$$

Using equation (2.27) in equation (2.25), we have

\begin{align}
\overline{c}_i^2 \nabla \{ \nabla \nabla \phi + \nabla \cdot (\nabla \times \overline{\psi}) \} - \overline{c}_2^2 \nabla \times \{ \nabla \times \nabla \phi + \nabla \times (\nabla \times \overline{\psi}) \} \\
- \overline{\nu}_i \nabla (\theta + \tilde{t}_i \dot{\theta}) = \nabla \dot{\phi} + \nabla \times \dot{\overline{\psi}}
\end{align}

or

\begin{align}
\overline{c}_i^2 \nabla \{ \nabla^2 \phi \} - \overline{c}_2^2 \nabla \times \{ \nabla (\nabla \overline{\psi}) - \nabla^2 \overline{\psi} \}
\end{align}
\[-\nabla_i \nabla (\theta + \dot{t}_i \dot{\theta}) = \nabla \ddot{\phi} + \nabla \times \ddot{\psi}\]

or

\[\bar{c}_i^2 \nabla (\nabla^2 \phi) - \bar{c}_2^2 \nabla \times \nabla (\nabla \cdot \psi) + \bar{c}_2^2 \nabla \times \nabla^2 \psi\]

\[-\nabla_i \nabla (\theta + \dot{t}_i \dot{\theta}) = \nabla \ddot{\phi} + \nabla \times \ddot{\psi}\]

or

\[\bar{c}_i^2 \nabla (\nabla^2 \phi) + \bar{c}_2^2 \nabla \times \nabla^2 \psi - \nabla_i \nabla (\theta + \dot{t}_i \dot{\theta}) = \nabla \ddot{\phi} + \nabla \times \ddot{\psi}\] (2.28)

Separating vectors and scalars, result in

\[\bar{c}_i^2 \nabla^2 \phi = \ddot{\phi} + \nabla_i (\theta + \dot{t}_i \dot{\theta})\] (2.29)

\[\bar{c}_2^2 \nabla^2 \psi = \ddot{\psi}\] (2.30)

From equation (2.29), we have

\[
\theta = \left( \bar{c}_i^2 \nabla^2 \phi - \ddot{\phi} \right) / \bar{\gamma}
\] (2.31)

where

\[\bar{\gamma} = \nabla_i \left[ 1 + t_i (\partial / \partial t) \right]\] (2.32)

Eliminating \(\theta\) from equations (2.31) and (2.26), we get

\[\nabla^4 \phi - \left[ \frac{\bar{C}}{K} \right]^2 \left( 1 + \bar{t}_0 \left( \frac{\partial}{\partial t} \right) \right) + \epsilon_i \left( 1 + \bar{t}_i \left( \frac{\partial}{\partial t} \right) \right) \left( 1 + \Delta \bar{t}_0 \left( \frac{\partial}{\partial t} \right) \right)
\]

\[+ \frac{1}{c_i^2} \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) \nabla^2 \phi + \frac{\bar{C}}{K} \frac{1}{c_i^2} \left( 1 + \bar{t}_0 \frac{\partial}{\partial t} \right) \frac{\partial^3 \phi}{\partial t^3} = 0\] (2.33)

where

\[\epsilon_i = \frac{\nabla_i^2 \theta_0}{(K + \frac{4}{3} \mu) \bar{C} \bar{\rho}}\] (2.34)
which is called thermoelastic coupling co-efficient. Chadwick and Sneddon (1958), Locket (1958), Lord and Shulman (1976), Nayfeh and Nasser (1971) have taken its value in range 0.03 to 0.073.

We assume the solution of equation (2.33) of the form

\[ \phi = f(z) \exp\{ik(ct - x)\} \quad (c > c_1) \quad (2.35) \]

With the help of equation (2.35), equation (2.33) reduces to

\[
\begin{align*}
(-ik)^4 f(z) + \frac{d^4 f(z)}{dz^4} + 2(-ik)^2 \frac{d^2 f(z)}{dz^2} & \exp\{ik(ct - x)\} \\
- \left[ \frac{C^*}{K^*} \left( \left( 1 + \tilde{r}_0 \frac{\partial}{\partial t} \right) + \varepsilon_1 \left( 1 + \tilde{r}_1 \frac{\partial}{\partial t} \right) \left( 1 + \Delta t_0 \frac{\partial}{\partial t} \right) \right) \right] \\
+ \frac{1}{c_i^2} \frac{\partial^2 f(z)}{\partial t^2} & \exp\{(-ik)^2 f(z) + \frac{d^2 f(z)}{dz^2} \exp\{ik(ct - x)\} \} \\
+ \frac{C^*}{K^* c_i^2} \left( 1 + \tilde{r}_0 \frac{\partial}{\partial t} \right) f(z) & \exp\{ik(ct - x)\} \] \quad (ick)^3 = 0
\end{align*}
\]

or

\[
\begin{align*}
k^4 f(z) + \frac{d^4 f(z)}{dz^4} - 2k^2 \frac{d^2 f(z)}{dz^2} & \exp\{ik(ct - x)\} \\
- \left[ \frac{C^*}{K^*} \left( \left( 1 + \tilde{r}_0 \frac{\partial}{\partial t} \right) + \varepsilon_1 \left( 1 + \tilde{r}_1 \frac{\partial}{\partial t} \right) \left( 1 + \Delta t_0 \frac{\partial}{\partial t} \right) \right) + \frac{1}{c_i^2} \frac{\partial}{\partial t} \right] \\
(ick) \left( (-k^2) f(z) + \frac{d^2 f(z)}{dz^2} \right) & \exp\{ik(ct - x)\} \\
+ \frac{C^*}{K^* c_i^2} \left( 1 + \tilde{r}_0 \frac{\partial}{\partial t} \right) f(z) & \exp\{ik(ct - x)\} \] \quad (-ic^3k^3) = 0
\end{align*}
\]
or
\[
\left[ k^4 f(z) + \frac{d^4 f(z)}{dz^4} - 2k^2 \frac{d^2 f(z)}{dz^2} \right] \left[ \exp \{ ik(c t - x) \} \right]
\]
\[- \left[ \frac{C^*}{K^*} \left\{ \left( 1 + \tilde{t}_0 \frac{\partial}{\partial t} \right) + \varepsilon_1 \left( 1 + \tilde{t}_1 \frac{\partial}{\partial t} \right) \right\} \left( 1 + \Delta \tilde{t}_0 \frac{\partial}{\partial t} \right) \right] + \frac{1}{\c_i^2} \frac{\partial}{\partial t} \right] (ick) \left[ -k^2 f(z) + \frac{d^2 f(z)}{dz^2} \right] \left[ \exp \{ ik(c t - x) \} \right]
\]
\[
+ \frac{C^*}{K^*} \frac{1}{\c_i^2} (1 + \tilde{t}_0 (ick)) f(z) \left[ \exp \{ ik(c t - x) \} \right] (-ic^3 k^3) = 0
\]

or
\[
\left[ k^4 f(z) + \frac{d^4 f(z)}{dz^4} - 2k^2 \frac{d^2 f(z)}{dz^2} \right] \left[ \exp \{ ik(c t - x) \} \right]
\]
\[- \left[ \frac{C^*}{K^*} \left\{ (1 + i\tilde{t}_0 ck) + \varepsilon_1 (1 + i\tilde{t}_1 ck)(1 + i\Delta \tilde{t}_0 ck) \right\} + \frac{1}{\c_i^2} \right] (ick) \left[ -k^2 f(z) + \frac{d^2 f(z)}{dz^2} \right] \left[ \exp \{ ik(c t - x) \} \right]
\]
\[
+ \frac{C^*}{K^*} \frac{1}{\c_i^2} (1 + i\tilde{t}_0 ck) \left[ \exp \{ ik(c t - x) \} \right] (-ic^3 k^3) f(z) = 0
\]

or
\[
\frac{d^4 f(z)}{dz^4} + \frac{d^2 f(z)}{dz^2} \left[ -2k^2 - \frac{C^*}{K^*} \left\{ (1 + i\tilde{t}_0 ck) + \varepsilon_1 (1 + i\tilde{t}_1 ck)(1 + i\Delta \tilde{t}_0 ck) \right\} (ick) - \frac{1}{\c_i^2} (ick)^2 \right]
\]
\[
+ f(z) \left[ k^4 - \frac{C^*}{K^*} \left\{ (1 + i\tilde{t}_0 ck) + \varepsilon_1 (1 + i\tilde{t}_1 ck)(1 + i\Delta \tilde{t}_0 ck) \right\} (-ic^3) \right]
\]
\[-\frac{1}{c_i^2}(i\epsilon c)(-i\epsilon c^3) + \frac{C*}{K*} \frac{1}{c_i^2} (1 + i\epsilon_0 \epsilon c)(-i\epsilon c^3) \] = 0

\[
\frac{d^4f(z)}{dz^4} + \frac{d^2f(z)}{dz^2}\left[\frac{c^2k^2}{c_i^2} - 2k^2 - \epsilon c \frac{C*}{K*}(i - i_0 \epsilon c) + \epsilon_1(i - i_1 \epsilon c)(1 + i\Delta i_0 \epsilon c) - \frac{c^2}{c_i^2}(i - i_0 \epsilon c) \right]
\]

\[+ f(z)\left[k^4 - \frac{c^2}{c_i^2} k^4 - \frac{C*}{K*} \epsilon c^3 \{(i - i_0 \epsilon c) + \epsilon_1(i - i_1 \epsilon c)(1 + i\Delta i_0 \epsilon c) - \frac{c^2}{c_i^2}(i - i_0 \epsilon c)\}\right] = 0
\]

or

\[
\frac{d^4f(z)}{dz^4} + \frac{d^2f(z)}{dz^2}\left[k^2\left(\frac{c^2}{c_i^2} - 2\right) - \epsilon c \frac{C*}{K*}(i - i_0 \epsilon c)\right] + f(z)\left[k^4\left(1 - \frac{c^2}{c_i^2}\right) - \epsilon c^3\left(\frac{C*}{K*}\right) \right]
\]

\[\{(i - i_0 \epsilon c) + \epsilon_1(i - i_1 \epsilon c)(1 + i\Delta i_0 \epsilon c) - \frac{c^2}{c_i^2}(i - i_0 \epsilon c)\}\] = 0

Above equation can be written as

\[
\frac{d^4f(z)}{dz^4} + \frac{d^2f(z)}{dz^2} + A \frac{d^2f(z)}{dz^2} + B f(z) = 0 \quad (2.36)
\]

where

\[A = k^2\left(\frac{c^2}{c_i^2} - 2\right) - \epsilon c \frac{C*}{K*}(i - i_0 \epsilon c) + \epsilon_1(i - i_1 \epsilon c)(1 + i\Delta i_0 \epsilon c)\] \quad (2.37)

\[B = k^4\left(1 - \frac{c^2}{c_i^2}\right) + k^3\left(\frac{C*}{K*}\right)\{(i - i_0 \epsilon c) + \epsilon_1(i - i_1 \epsilon c)(1 + i\Delta i_0 \epsilon c)\}
\]

\[- \frac{c^2}{c_i^2}(i - i_0 \epsilon c)\] \quad (2.38)
The solution of equation (2.36) is of the form

\[ f(z) = B_1 \exp(m_1 z) + B_2 \exp(-m_1 z) + B_3 \exp(m_2 z) + B_4 \exp(-m_2 z) \]  \tag{2.39}

where

\[ m_1 = \frac{\sqrt{(A^2 - 4B) - A}}{2} \]  \tag{2.40}

\[ m_2 = \frac{\sqrt{-(A^2 - 4B) + A}}{2} \]  \tag{2.39}

corresponding to the thermal and modified P waves respectively and B_1, B_2, B_3, B_4 are arbitrary constants.

Equation (2.28) can be written as

\[ c^2 V^2 \psi = \ddot{\psi}, \quad \psi = (-\ddot{\psi}) \]  \tag{2.41}

We assume the solution of equation (2.40) of the form

\[ \psi = g(z) \exp\{ik(ct - x)\} \quad (c > c_2^2) \]  \tag{2.42}

With the help of (2.42), equation (2.41) reduces to

\[ \frac{d^2 g(z)}{dz^2} + k^2 \left( \frac{c^2}{c_2^2} - 1 \right) g(z) = 0 \]  \tag{2.43}

The solution of (2.43) is

\[ g = (B_3 \exp(m_3 z) + B_6 \exp(-m_3 z)) \exp\{ik(ct - x)\} \]  \tag{2.44}

where

\[ m_3 = \pm k(1 - \frac{c^2}{c_2^2})^{1/2} \]  \tag{2.45}