CHAPTER I

Introduction

The theory of stochastic processes has developed in recent times into one of the most important branches of science. While the theoretical ranges of the problems connected with it have given the mathematician a fertile field of investigation, their results have found numerous applications, particularly in different questions of physical and technical statistics. The object of the present work is to apply the theory of stochastic processes to physical problems, in particular those involving the notions of the change of the statistical distribution in time or space and of the correlation between statistical variables.

Essentially there are two methods to deal with such 'dynamic' statistical problems. The first method, hereinafter referred to as Method I, is especially useful in the case of Markoff processes. It is the characteristic feature of a Markoff process that for each instant, the future development of the process depends only on its present state but not on its prehistory. Method I makes direct use of this property; the procedure consists in expressing the state of the system at \( t + \Delta t \) in terms of the state at \( t \) (where \( t \) is the parameter with respect to which the process develops) assuming \( t \) is continuous and passing to the limit as \( \Delta t \) tends to zero. If \( t \) were discrete assuming values \( t_1, t_2, \ldots, t_k, \ldots \) we express the state of the system at \( t_k \) in terms of the state of the system at \( t_{k-1} \).
The second method, hereinafter referred to as Method II, is based upon an entirely different mode of approach. In its original form, for continuous $t$, it yielded integral equations (integral with respect to $t$) and this has been described in detail in Chapter VII. This method, in the case of processes which are Markovian with respect to $t$, is equivalent to dealing with the stochastic process from the following point of view.

At the outset it must be stated that the method is too general (in fact it is applicable even to some non-Markovian processes, an elementary example of which is given in Chapter IX) to admit of a very simple complete formulation but to illustrate its use we proceed thus. Giving the most general interpretation to the state of a system, but assuming for the present that the system can occupy a discrete number of states $S_1, S_2, \ldots$ in any stochastic process we are interested in calculating the probability that the system is in state $S_j$ at $t$ given that it was in $S_i$ at $t = 0$. Instead of considering what happens in the interval between $t$ and $t + \Delta t$ (which is the procedure adopted in Method I) we ask: what changes do occur in the interval between $t = 0$ and $t = \Delta t$? Giving an equally general interpretation to the transition probability per unit $t$ we define $R_{KL}$ as the probability per unit $t$ that the system in a state $S_K$ jumps to the state $S_L$. Let us assume for the sake of simplicity, that $R_{KL}$ is independent of $t$ and the stochastic process is homogeneous in and Markovian with respect to $t$. We define $P(T_i \mid i; t)$ as the required
probability. In the first instant of 'time' \( \Delta t \) (\( t \) is a general parameter and the word 'time' is used merely for the sake of illustration) the system jumps from the state \( S_i \) to state \( S_K \) with probability \( R_{ik} \Delta t \) (\( \Delta t \) is an infinitesimal quantity). It is still in the state \( S_i \) with probability

\[
1 - \sum_{k, k \neq i} R_{ik} \Delta t
\]

Thus we can write

\[
\mathbb{P}(J|i; t) = \left[ 1 - \sum_{k, k \neq i} R_{ik} \Delta t \right] \mathbb{P}(J|i; t - \Delta t) + \sum_{k, k \neq i} \mathbb{P}(J|k; t - \Delta t) R_{ik} \Delta t
\]

yielding

\[
\frac{\partial \mathbb{P}(J|i; t)}{\partial t} = -\mathbb{P}(J|i; t) \sum_{k, k \neq i} R_{ik} + \sum_{k, k \neq i} \mathbb{P}(J|k; t) R_{ik}
\]

(1.1)

If the states do not form a discrete system but are continuous infinite, we get integrals instead of sums. In contrast to the above, Method I yields

\[
\mathbb{P}(J|i; t+\Delta t) = \left[ 1 - \sum_{k, k \neq J} R_{jk} \Delta t \right] \mathbb{P}(J|i; t) + \sum_{k, k \neq J} \mathbb{P}(K|i; t) R_{kj} \Delta t
\]
Making $\Delta t$ tend to zero we have

$$\frac{\partial P(J|i;t)}{\partial t} = -P(J|i;t) \sum_{K, K \neq J} R_{JK} + \sum_{K, K \neq J} P(K|i;t) R_{KJ} \quad (1.2)$$

The striking contrast in approach is obvious on a mere examination of the equations (1.1) and (1.2)*.

In the case when the transition probabilities are $t$ dependent, by the application of Method I we obtain the same equation (1.2) but we must apply Method II in its original form yielding an integral equation with respect to $t$ and this has been described later in Chapter VII.

* In the simple Markoff chain case, Method II is equivalent to the 'backward differential equation' of Kolmogoroff in contrast with the 'forward differential equation' obtained by Method I (cf. Arley, H. 'Stochastic processes and cosmic radiation', p.31 (1943) (Copenhagen) and W. Feller's 'Introduction to the theory of probability', p.390 (Wiley, 1950