APPENDIX III

A discrete model having the properties of a continuum of stochastic variables

We can build up a system of particles occupying a discrete number of states (finite or enumerably infinite) which presents features analogous to the continuous system provided that we assume that in every state there can be 1 or 0 particles. In other words we have a sequence of stochastic variables

\[ n_1, n_2, \ldots, n_i, \ldots \]

each of which can assume the values of 0 or 1. In such a case we can define the probability that \( n_i \) assumes the value 1 as \( P_i \). We can define a joint probability of degree two that \( n_i \) assumes the value 1 and \( n_j \) assume the value 1 as \( P_{ij} \). We can also in the same way define a joint probability of degree \( m \) that \( m \) variables \( n_i, n_j, \ldots \) each assume the value 1, represented by \( (m) P_{ij} \ldots \). By simple arguments it follows that if \( V \) is the sum of stochastic variables, i.e.

\[ V = \sum_i n_i \]

the \( \gamma \)th moment of \( V \) is given by

\[ \overline{V^\gamma} = \sum_S \left\{ C_S^\gamma \sum_i \sum_j \cdots \sum_{(S)} P_{ij} \cdots \right\} \quad (c.1) \]

In particular

\[ \overline{V} = \sum_i (1) P_i \quad \overline{V^2} = \sum_i (1) P_i^2 + \sum_i \sum_j (2) P_{ij} \quad (c.2) \]
It is quite clear that if these stochastic variables \( \{n_i\} \) are independent and the number of such variables tends to infinity keeping \( J \) finite, then \( J \) approaches a Poisson distribution.

Recently Koopman considered such a sequence if it possesses Markovian properties. We shall call the sequence Markovian if the probability that \( n_J \) assumes the value 0 or 1, given that \( n_i \) has assumed the value 0 or 1 \((i < J)\) does not depend upon the additional information about what value \( n_K \) has assumed \((K < I)\). Accordingly we define

\[
\begin{align*}
\alpha_J &= p(n_J = 1 | n_{J-1} = 1) \\
\beta_J &= p(n_J = 1 | n_{J-1} = 0)
\end{align*}
\]

where \( \alpha_J \) and \( \beta_J \) are the probabilities that \( n_J \) assumes the value 1 given that \( n_{J-1} \) has assumed the value 1 or 0, respectively. We further define \( P_J^K \) as the probability that \( n_J = 1 \) given that \( n_K = 1 \) \((K < J)\), and \( \Pi_J^K \) as the probability that \( n_J = 1 \) given that \( n_K = 0 \) \((K < J)\).

Following Koopman, \( P_J^K \) and \( \Pi_J^K \) satisfy the classical probability equations

\[
\begin{align*}
P_J^K &= P_{J-1}^K \alpha_J + (1 - P_{J-1}^K)\beta_J, \\
P_{K+1}^K &= \alpha_K + (1 - \alpha_K)\beta_K
\end{align*}
\]

\[
\begin{align*}
\Pi_J^K &= \Pi_{J-1}^K \alpha_J + (1 - \Pi_{J-1}^K)\beta_J, \\
\Pi_{K+1}^K &= \beta_K + (1 - \beta_K)\alpha_K
\end{align*}
\]

It is quite easy to prove that the joint probability of degree \( n \) that \( n \) members of the sequence \( n_J, n_L \ldots \) are
each equal to 1 given that \( n_k = 1 \) \((k < j < l \ldots)\) defined by

\[
(n) P^k_j \ell
\]

can be uniquely determined in terms of the joint probability of degree one

\[
(n) P^k_j = P^k_j P^j_\ell \ldots
\]

(a product of \( n \) such terms)

Note that here we have denoted \( P^k_j \) as probability of degree one which we originally called \((1)^j P_j\).

Now let us assume that the \( \alpha_i \)'s and \( \beta_i \)'s are independent of \( i \). This immediately yields the result

\[
P^1_K = c^{K-1} + \beta \left( \frac{1 - c^{K-1}}{1 - c} \right), \ c = \alpha - \beta
\]

(c.6)

Also

\[
P^j_K = P^j_{K-j+1}
\]

(c.7)

Using the above result we obtain

\[
\nu = \sum_{i=2}^{\infty} p^i_1 = \left\{ \frac{l + \alpha}{\ell - \alpha} \right\}
\]

(c.8)

the condition for the convergence of the sum being \( n \beta \to \ell \) as \( n \to \infty \), \( \beta \to 0 \). Note \( c \to \alpha \).

Also

\[
\nu^2 = \frac{\ell \alpha + \ell + \alpha}{(1 - \alpha)^2} + \nu^2
\]

(c.9)
This result agrees with that of Koooman who by a different method obtained the generating function \( \mathcal{Y} \) as

\[
\phi(u) = \left[ 1 - \frac{1-u}{1-\gamma u} \right] \exp \left\{ \frac{-(\bar{m}(1-u)(1-\gamma)}{1-\gamma u} \right\}
\]

Koooman's \( \bar{m} = \nu_0 + 1 \), \( \nu \equiv \gamma \) i.e. \( \bar{m} \) includes the value of the initial stochastic variable.