CHAPTER VII

Method II or the method of Regeneration Points

Until now, we have been dealing with the applications of Method I to physical problems. It must however be mentioned that the product density technique can be applied in conjunction with Method I or Method II provided that the densities are defined with respect to a variable, say, $E$ different from $t$ the variable with respect to which the stochastic process develops*. We shall in this chapter outline Janossy's application of Method II to the statistical problem of cosmic radiation which yielded integral equations with respect to $t$. We shall also deal with the application of this method to some simple multiplicative processes. This method had been used earlier by Palm, Bellman and Harris to processes where the difficulty relating to a continuous infinity of stochastic variable did not arise. The historical development of this method has been described in detail by Bartlett and Kendall who called it the method of regeneration points.

1. Outline of Janossy's treatment

Janossy, in order to avoid the complication of dealing with a continuous infinity of stochastic variables defined a function $\eta^{(\bar{t})}(\eta, E, E_0, t)$ representing the probability that

* In the next chapter we deal with a class of stochastic processes where the product densities have been defined with respect to $t$. 
there are \( \nu \) electrons above the energy \( E \) at \( t \), given that at \( t = 0 \) there was a primary of energy \( E_0 \). If the primary is an electron we put \( i = 1 \), if it is a photon \( i = 2 \). He visualised the stochastic process from the following point of view. Somewhere between \( 0 \) and \( t \), say between \( o \) and \( t \) the primary, if it is an electron may create a photon and the initial electron and the newly created photon will then become independent primaries of stochastic processes of 'duration' \( t - \tau \).

If at \( t = 0 \) we have a photon, it may create a pair of electrons between \( \tau \) and \( \tau + d\tau \) and these electrons become independent primaries of stochastic processes of duration \( t - \tau \). Viewing the development of the process thus, we find that \( \Pi_i(n, E, E_0, t) \) satisfies the integral equations (for \( i = 1, i = 2 \))

\[
\Pi_i^{(1)}(n, E, E_0, t) = \left[ \int_0^t e^{-\alpha_i \tau} d\tau \right] \chi
\]

\[
\Pi_i^{(2)}(n, E, E_0, t) = \left[ \int_0^t e^{-\alpha_2 \tau} d\tau \right] \chi
\]

where \( \alpha_1 = \int_{E_0}^{E_0 + \tau} R(E_0', E') dE' \), \( \alpha_2 = \int_{E_0 - \tau}^{E_0 - \tau} R(E_0', E') dE' \).

\( R(E_0, E') dE' \) represents the probability per unit \( t \) that an electron of energy \( E_0 \) drops to the interval lying between \( E' \) and \( E' + dE' \) radiating a photon of energy \( E_0 - E' \).
represents the probability per unit \( t \) that a photon of energy splits up into a pair of electrons one of which has an energy lying between \( E' \) and \( E' + dE' \). The equations are self-explanatory. For example in equation (7.1) \( e^{-\alpha_1 t} \) is the probability that no event occurs between 0 and \( \tau \) and \( e^{-\alpha_1 \tau} \) \( R(E_0, E') \) \( dE'd\tau \) is the probability that the first event defined by a transition of electron from \( E_0 \) to \( E' \) occurs. The first event may occur between 0 and \( t \), hence the integration over \( \tau \), or it may not occur at all with a probability \( e^{-\alpha_1 t} \), hence the term \( \delta(n-1) e^{-\alpha_1 t} \) where \( \delta(x) = 1 \) for \( x=0 \) and zero otherwise.

The energy transitions may occur over the whole range 0 to \( E_0 \), hence the integration over \( E' \). The summation \( \sum \) indicates that the new primaries at \( \tau \) must create \( n_1 \) and \( n_2 \) electrons respectively such that \( n_1 + n_2 = n \). Similar arguments apply to the equation for \( \Pi^{(2)} \).

Differentiating the equations with respect to \( t \) we have

\[
\frac{\partial \Pi^{(1)}(n, E, E_0, t)}{\partial t} = -\alpha_1 \Pi^{(1)}(n, E, E_0, t) + \int \left\{ \sum_{E_1, E_2} \Pi^{(1)}(n_1, E_1, E_2, t) \Pi^{(2)}(n_2, E, E_0 - E', t) \right\} x R(E_0, E') \ dE'
\]

\[
\frac{\partial \Pi^{(2)}(n, E, E_0, t)}{\partial t} = -\alpha_2 \Pi^{(2)}(n, E, E_0, t) + \int \left\{ \sum_{E_1, E_2} \Pi^{(1)}(n_1, E_1, E_2, t) \Pi^{(1)}(n_2, E, E_0 - E', t) \right\} x R(E_0, E') \ dE'
\]

This was the procedure adopted by Janossy. But if we follow the viewpoint of the author, (outlined in the introduction...
Chapter I) the above integro-differential equations could have
been obtained directly. To proceed to solve these equations
we use the method of generating functions. We also assume that
\( R(E, E') dE' \) and \( R'(E, E') dE' \) can be expressed as
functions of \( E'/E \) i.e. say \( R(\epsilon) e(\epsilon) \), and \( R'(\epsilon) d\epsilon \) respec-
tively.

Defining
\[
G(u, \epsilon, t) = \sum \Pi^{(1)}(\nu, \epsilon, t) u^\nu, \quad \epsilon = E/E_0
\]
\[
F(u, \epsilon, t) = \sum \Pi^{(2)}(\nu, \epsilon, t) u^\nu
\]
we obtain
\[
\frac{\partial G(u, \epsilon, t)}{\partial t} = -\alpha_1 G(u, \epsilon, t) + \int G(u, \epsilon'/\epsilon, t) F(u, \epsilon'/\epsilon, t) R(\epsilon') d\epsilon'
\]
\[
\frac{\partial F(u, \epsilon, t)}{\partial t} = -\alpha_2 F(u, \epsilon, t) + \int G(u, \epsilon'/\epsilon, t) G(u, \epsilon'/\epsilon', t) R'(\epsilon'/\epsilon') d\epsilon'
\]
These were the equations obtained by Janossy.

2. Simplified problem of Corneal Radiation

In the Janossy equations we shall make \( \epsilon \) tend to zero,
writing \( \Pi^{(1)}(\nu, \epsilon, t) \) for \( \epsilon = 0 \) as \( \Pi^{(1)}(\nu, t) \) and defining
\( G(u, t) \) and \( F(u, t) \) accordingly
\[
\frac{\partial \Pi^{(0)}(\nu, t)}{\partial t} = \left\{ -\Pi^{(1)}(\nu, t) + \sum \Pi^{(1)}(\nu_1, t) \Pi^{(2)}(\nu_2, t) \right\} x_t
\]
\[
\frac{\partial \Pi^{(2)}(n, t)}{\partial t} = \left\{-\Pi^{(2)}(n, t) + \sum_{n_1 + n_2 = n} \Pi^{(1)}(n_1, t) \Pi^{(1)}(n_2, t) \right\}\alpha_2
\] (7.8)

\[
\frac{\partial G_1}{\partial t} = (G_1 F - G_1) \alpha_1
\] (7.9)

\[
\frac{\partial F}{\partial t} = (G_2 F - F) \alpha_2
\] (7.10)

The initial conditions are: At \(t=0\)

\[
\Pi^{(1)}(n, 0) = 0 \quad \text{for} \quad n \neq 1 \quad \text{i.e.} \quad n = 0, 2, 3, 4, \ldots
\]

\[
\Pi^{(1)}(1, 0) = 1
\]

and

\[
\Pi^{(2)}(n, 0) = 0 \quad \text{for} \quad n \neq 0 \quad \text{i.e.} \quad n = 1, 2, 3, 4, \ldots
\]

\[
\Pi^{(2)}(0, 0) = 1
\]

But for all \(t\), \(\Pi^{(1)}(0, t) = 0\). This means that for an electron initiated shower there is at least one electron at all thicknesses since we do not have an absorption probability for electrons. We apply these conditions to equations (7.7) and (7.8). Since

\[
\frac{\partial \Pi^{(2)}(1, t)}{\partial t} = \left\{\Pi^{(1)}(1, t) \Pi^{(1)}(0, t) + \Pi^{(1)}(0, t) \Pi^{(1)}(1, t) - \Pi^{(2)}(1, t)\right\}\alpha_2
\]

and \(\Pi(0, t) = 0\) for all \(t\), we have \(\Pi^{(2)}(1, t) = A e^{-\alpha_2 t}\).

But at \(t = 0\), \(\Pi^{(2)}(1, t) = 0\) whence \(A = 0\) and \(\Pi^{(2)}(1, t) = 0\) for all \(t\). Now applying this result to equation (7.7) we have

\[
\frac{d \Pi^{(1)}(2, t)}{dt} = -\alpha_1 (1 - e^{-\alpha_2 t}) \Pi^{(1)}(2, t)
\]

\[
= f(t) \Pi^{(1)}(2, t)
\]
where \( f(t) = -\alpha_1 (1 - e^{-\alpha_2 t}) \)

\[
\Pi^{(1)}(2, t) = A e^{\int_0^t f(t') dt'}
\]

But \( \Pi^{(1)}(2, t) = 0 \) at \( t = 0 \) whence \( A = 0 \) and \( \Pi^{(1)}(2, t) = 0 \) for all \( t \). By successive application of these conditions we get \( \Pi^{(2)}(3, t) = 0 \) for all \( t \), \( \Pi^{(1)}(4, t) = 0 \) for all \( t \) and in general

\[
\Pi^{(1)}(n, t) = 0 \text{ for all } t \text{ if } n \text{ is even}
\]

\[
\Pi^{(2)}(n, t) = 0 \text{ for all } t \text{ if } n \text{ is odd}.
\]

This corresponds to the physical situation; since electrons are produced in pairs an electron initiated shower should have an odd number of electrons and a photon initiated shower should have an even number.

It should be noted that the same equations represent the photon distribution but the equations will then satisfy different initial conditions. Thus if we are concerned with the number of photons at \( t \), interpreting \( \Pi^{(1)} \), \( \Pi^{(2)} \), \( G \) and \( F \) accordingly we write the initial conditions as

\[
\Pi^{(1)}(n, 0) = 1 \text{ if } n = 0 \quad \Pi^{(1)}(n, 0) = 0 \text{ if } n \neq 0
\]

\[
\Pi^{(2)}(n, 0) = 1 \text{ if } n = 1 \quad \Pi^{(2)}(n, 0) = 0 \text{ if } n \neq 1
\]

Under these conditions we find \( \Pi^{(1)}(n, t) \) and \( \Pi^{(2)}(n, t) \) exist for all values of \( t \), that is, the number of photons can be even or odd.
It is a special feature of Janossy's approach that the simultaneous equations connect a photon initiated shower of electrons with an electron initiated shower of electrons while the well-known equations of the cascade theory* connect the number of photons of a shower with the number of electrons.

Operating on $G$ and $F$ and writing

\[
\left( \frac{\partial G}{\partial u} \right)_{u=1} = \overline{n}, \quad \left( \frac{\partial F}{\partial u} \right)_{u=1} = \overline{m}
\]

\[
\left( \frac{\partial^2 G}{\partial u^2} \right)_{u=1} = \nu \overline{n}, \quad \left( \frac{\partial^2 F}{\partial u^2} \right)_{u=1} = \mu \overline{m}
\]

we find that $\overline{n}$, $\overline{m}$, $\nu$, $\mu$ satisfy the equations

\[
\frac{d\bar{n}}{dt} = \bar{m} \alpha, \quad \frac{d^2\bar{m}}{dt^2} + \beta \frac{d\bar{m}}{dt} - 2\alpha \beta \bar{m} = 0 \tag{7.11}
\]

\[
\frac{d^2\mu}{dt^2} + \beta \frac{d\mu}{dt} - 2\alpha \beta \mu = \varepsilon \overline{n} \overline{m} \alpha \beta \tag{7.12}
\]

\[
\nu - \overline{n}^2 = \int \mu \alpha \, dt
\]

According to the quantum-mechanical cross-sections now available $\alpha$ is infinite, contrary to physical facts. So, to build up our statistical model, we shall take $\alpha = \beta$ **. By choosing a suitable unit of thickness we can without loss of generality put $\alpha' = \gamma$.

* For literature on cascade theory see Janossy (1948)

** For $\alpha \neq \beta$ the author has not been able to find explicit solutions for the equations.
\[
\frac{\partial G}{\partial \tau} = GF - G, \quad \frac{\partial F}{\partial \tau} = G^2 - F, \quad (7.13)
\]

At \( \tau = 0 \), \( G = u, F = 1 \) for electron shower,

\[ G = 1, \quad F = u \quad \text{for photon shower.} \]

To solve the equation we put

\[ g(\tau) = G(\tau) e^\tau, \quad f(\tau) = F(\tau) e^\tau \quad \text{and} \quad y = e^{-\tau} \]

This yields

\[ \frac{\partial g}{\partial y} = -gf, \quad \frac{\partial f}{\partial y} = -g^2 \]

The solution of these equations is readily found to be

**Electron Distribution**

\[ G = \sqrt{1-u^2} e^{-\tau} \csc h \left\{ (e^{-\tau} - 1) \sqrt{1-u^2} + \coth^{-1} \frac{1}{\sqrt{1-u^2}} \right\} \]

\[ F = \sqrt{1-u^2} e^{-\tau} \coth \left\{ (e^{-\tau} - 1) \sqrt{1-u^2} + \coth^{-1} \frac{1}{\sqrt{1-u^2}} \right\} \]

**Photon Distribution**

\[ G = \sqrt{u^2-1} e^{-\tau} \csc h \left\{ (e^{-\tau} - 1) \sqrt{u^2-1} + \coth^{-1} \frac{u}{\sqrt{u^2-1}} \right\} \]

\[ F = \sqrt{u^2-1} e^{-\tau} \coth \left\{ (e^{-\tau} - 1) \sqrt{u^2-1} + \coth^{-1} \frac{u}{\sqrt{u^2-1}} \right\} \]

(7.14) (7.15)

In his symposium paper Bartlett (1949) obtained the differential equation for the generating function \( \Pi(Z, W, \tau) \)
for the joint distribution of electrons and photons, \( z \) and \( w \) being the corresponding variables. Ho obtained the equation (his equations (32) and (33) with \( \lambda_1 = \lambda_2 = 1, \mu_1 = \mu_2 = 0 \))

\[
\frac{\partial \Pi}{\partial t} = (ZW - Z) \frac{\partial \Pi}{\partial Z} + (Z^2 - W) \frac{\partial \Pi}{\partial W}
\]

It is quite clear that the equations (3) and (4) are the equations for the marginal distributions. In a private communication ho has pointed out that the above equation can be completely solved when the equations for the marginal distribution can be solved and these are mathematically the same as (3) and (4) with \( Z \) and \( W \) in place of \( \gamma_1 \) and \( \gamma_2 \). We give here the result,

Electron initiated shower

\[
\Pi(Z, W) = e^{-t \sqrt{W^2 - Z^2}} \cosh \left\{ \frac{e^{-t} \sqrt{W^2 - Z^2}}{2} \right\} \cosh \left\{ \frac{W}{\sqrt{W^2 - Z^2}} \right\}
\]

Photon initiated shower

\[
\Pi(Z, W) = e^{-t \sqrt{W^2 - Z^2}} \cosh \left\{ \frac{e^{-t} \sqrt{W^2 - Z^2}}{2} \right\} \coth \left\{ \frac{W}{\sqrt{W^2 - Z^2}} \right\}
\]

Putting \( Z=1 \) or \( W=1 \) we get the generating function of the marginal distributions.
J. Good (1949), in his contribution to the Symposium discussion has derived equations (3) and (4) as special case of a more general equation. This he obtained by a different approach by first considering the time (or thickness) parameter as discrete and passing to continuous as a limiting case.

The mean number of electrons and photons for electron and photon initiated showers are given below.

**Electron initiated shower**

- Mean number of electrons
  \[ \frac{2}{3} \left( e^{-\tau} + \frac{1}{2} e^{-2\tau} \right) \]
- Mean number of photons
  \[ e^{-\tau} - e^{-2\tau} \]

**Photon initiated shower**

- Mean number of electrons
  \[ e^{-\tau} - e^{-2\tau} \]
- Mean number of photons
  \[ \frac{2}{3} \left( e^{-\tau} + 2 e^{-2\tau} \right) \]

It is to be noted that if we do not distinguish between the photons and electrons the total number of particles is \( e^\tau \) which is as it should be. Bartlett considered also an approximate 'randomised' solution for the simplified problem of cosmic radiation but he has asked me to point out that the above solution for the mean number indicates that the right value for the constant occurring in his solution is \( \frac{1}{3} \), and not \( \frac{1}{4} \) as first suggested by him.

Calculating the mean and mean square deviation of the number of electrons and photons under the assumption \( e^t \gg e^{-t} \).
we find

\[ \text{Mean square deviation of electrons } \sim (\text{mean number of electrons})^2 \]

\[ \text{Mean square deviation of photons } \sim (\text{mean number of photons})^2 \]

...in the case of both photon and electron initiated showers.

If we have two stochastic variables \( X \) and \( Y \) and we assume that

\[ E(X^2) - \{E(X)\}^2 = \sigma^2(X) = \{E(X)\}^2 \]

\[ E(Y^2) - \{E(Y)\}^2 = \sigma^2(Y) = \{E(Y)\}^2 \]

and writing \( Z = X + Y \), if we have

\[ E(Z^2) - \{E(Z)\}^2 = \sigma^2(Z) = \{E(Z)\}^2 \quad (7.21) \]

it follows that \( E(XY) - E(X)E(Y) = E(X)E(Y) \). Since we know that the total number of particles obeys a Furry distribution* and that the number of electrons as well as the number of photons obey a Furry law approximately (and that \( E(X) \) and \( E(Y) \) are of the same order of magnitude) it follows that the correlation coefficient between electrons and photons is approximately 1.

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*By a Furry process we mean a simple multiplicative process of only one type of particle. The generating function of the process is given as the solution of \( \frac{\partial G}{\partial t} = \frac{G}{\mu} - \frac{G^2}{\mu} \) at \( t = 0 \) the 'birth' probability per unit \( t \) being arranged to be unity.

\[ E(n) = e^t, \quad \sigma^2(n) = E(n^2) - [E(n)]^2 = [E(n)]^2 - E(n) \]

If \( e^t \gg 1 \), \( \sigma^2(n) \sim [E(n)]^2 \)
In view of the peculiar nature of the generating functions obtained it is relevant to state a simple result of general validity regarding generating functions.

Assuming that a generating function $G(u)$ is regular at $u=c$, it can be expanded in the form of a Taylor series about $u=0$

$$G_i(u) = G(0) + G_1'(0)u + G_2''(0)\frac{u^2}{2!} + \ldots \ldots$$

But by definition

$$G_i(u) = P(0) + P(1)u + P(2)u^2 + \ldots$$

where $P(n)$ is the corresponding probability for a discrete distribution.

Hence

$$P(n) = \left\{ \frac{G_i^{(n)}(0)}{n!} \right\}$$ \hspace{1cm} (7.22)

It immediately follows that if $G_i(u) = G_i(-u)$. Then $P(n) = 0$ when $n$ is odd and that if $G_i(u) = -G_i(-u)$ then $P(n) = 0$ when $n$ is even.

In the case of electron initiated shower of electrons to recognise that $G$ is odd and $F$ is even we expand

$$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$$

Here

$$B = \cosh^{-1} \frac{1}{\sqrt{1-u^2}} = \cosh^{-1} \frac{1}{u} = \sinh^{-1} \frac{1}{\sqrt{1-u^2}}$$
Hence for the electron distribution of an electron initiated shower:

$$G(t) = \frac{u \sqrt{1 - u^2} e^{-\tau}}{\{ \sinh A + \sqrt{1 - u^2} \cos h A \}} \quad (7.23)$$

where

$$A = (e^{-\tau} - 1) \sqrt{1 - u^2}$$

By similar considerations we recognise $F$ to be an even function of $U$ when it represents the generating function for the distribution of photons.

3. The negative binomial distribution

We shall construct a statistical model which is slightly different from the Furry model. We shall assume that whenever the initial particle (ancestor) gives 'birth' to a new particle, these newly created or secondary particles and their descendents have a probability to break up (into two) equal to $\beta$ times the corresponding probability for the initial 'ancestor'. If $G$ is the generating function of the distribution of the total number of particles (omitting the ancestor) and $F$ that arising from each secondary particle (including that particle itself) then

$$\frac{dG}{dt} = GF - G, \quad (G = 1 \text{ at } t = 0) \quad (7.24)$$

* In a private communication D.G.Kendall has pointed out that these results were obtained by Consael using Method I.
where
\[ \frac{\partial F}{\partial t} = (F^2 - G) \beta \quad (F = U \text{ at } t = 0) \] (7.25)

If \( \beta = 1 \), \( F \) represents a Furry distribution, and \( G = F/U \)
If \( \beta = 0 \), then \( F = U \) and
\[ \frac{\partial G}{\partial t} = G(U - G), \quad G_1 = e^{(u-1)t} \] (7.26)
giving the generating function of a Poisson distribution. For general \( \beta \)
\[ G = \left[ u - (u-1)e^{\beta t} \right]^{-1} / \beta \] (7.27)

For \( \beta \to 0 \), \( G_1 = e^{(u-1)t} \) and for \( \beta \to 1 \), \( G_1 \to 1/[(u-(u-1)e^t] \)
and for general the mean number of particles omitting the ancestor is
\[ \Pi = \left[ e^{\beta t} - 1 \right] / \beta \]
The mean square deviation is \( \beta \bar{n}^2 + \bar{n} \), giving the deviation for this distribution which is variously known as the Polya or negative binomial or Pascal distribution.

We could have obtained the following difference-differential equation for \( \Pi(n, t) \), the probability that there are \( n \) secondaries produced by the primary,
\[ \frac{\partial \Pi(n, t)}{\partial t} = -[1 + \beta \bar{n}] \Pi(n, t) \]
\[ + [1 + \beta(n-1)] \Pi(n-1, t) \] (7.23)
\[ \Pi(n, 0) = 0 \quad \text{if } n \neq 0, \quad \Pi(n, 0) = 1 \quad \text{if } n = 0 \]
By successive operations we obtain

\[
\pi(n, t) = \left(1 - \frac{e^{-\beta t}}{\beta}\right)^n \frac{1}{n!} \left(1 + \beta \right) \left(1 + 2\beta \right) \cdots \left(1 + n\beta \right) e^{-t} \tag{7.29}
\]

\[
\pi(0, t) = e^{-t}
\]

Arley considered the distribution

\[
\pi(n, x) = \left(\frac{\lambda x}{1 + \beta \lambda x}\right)^n \frac{1}{n!} \left(1 + \beta \right) \left(1 + 2\beta \right) \cdots \left(1 + n\beta \right) \left(1 + \beta \lambda x\right)^{-n/\beta} \tag{7.30}
\]

Putting \(\beta = 1\)

\[
\pi(n, x) = \frac{1}{(1 + \lambda x)} \left\{1 - \frac{1}{1 + \lambda x}\right\}^n \tag{7.31}
\]

To interpret this as a Furry distribution he used the transformation

\[
e^{\lambda t} = (1 + \lambda x) \tag{7.32}
\]

This artificial transformation is not necessary if we take the model suggested above. The variables \(t\) and \(x\) are connected by the relation

\[
e^{\beta t} = (1 + \beta \lambda x) \tag{7.33}
\]

To obtain the Polya distribution Arley assumed the probability that a particle is born when there are \(n\) particles in
existence at \( X \) to be

\[
\frac{\lambda (1 + \beta \eta)}{(1 + \beta \lambda x)} \, dx
\]

This quantity is clearly equal to \((1 + \beta \eta) \, dt\) and hence we obtain the difference equation given above.

Recently Anscombe (1950) has recalled a number of ways in which the negative binomial distribution can arise in biology. One of them bears a close relation to the above derivation. 'If colonies or groups of individuals are randomly distributed over an area in a Poisson distribution we obtain a negative binomial distribution for the total count if the numbers of individuals in the colonies are distributed independently in a logarithmic series distribution'. The problem considered above corresponds to the case of a Poisson distribution of colonies arising in time; each colony, once formed, growing as a Furry distribution; this case is also briefly mentioned by Anscombe.

It should be pointed out that the process discussed above is essentially identical with the simple birth and immigration process of Kendall by regarding the initial ancestor as a source of 'immigrants'. The main object of the above discussion has been to compare it with Arluy's treatment.

4. The simple 'birth, death, and immigration' process of Kendall

If we assume the probability of the 'death' of a particle (per unit range of \( t \)) to be equal to \( \mu \), if we let the
corresponding rate of 'immigration' into the population be equal to \( \eta \), and if we take the probability per unit \( t \) of 'birth' to be unity, then by simple arguments we are led to the following equations for \( G \), the generating function for the number of individuals in a population generated by 'immigration' and the function \( F \) which denotes the generating function arising from each immigrant.

\[
\frac{\partial G}{\partial t} = (GF - G) \eta \quad \frac{\partial F}{\partial t} = (F-1)(F-\mu) \tag{7.34}
\]

when \( \mu \neq 1 \)

\[
F = 1 + \frac{m \alpha e^{\mu t}}{1 - \alpha e^{\mu t}} \quad G_i = \left( \frac{1 - \alpha e^{\mu t}}{1 - \alpha} \right)^{-\eta}
\]

when \( \mu = 1 \)

\[
F = \frac{u - (u-1)t}{1 - (u-1)t} \quad G_i = \left( 1 - (u-1)t \right)^{-\eta}
\]

If at \( t = 0 \), we have \( N_0 \) members of the population then the generating function for the distribution of the population at \( t \) is obviously \( GF^{N_0} \).

5. Multiple production

We shall now introduce a new feature into the type of stochastic process until now discussed — that of multiple
production. We assume that a particle is replaced by a statistical distribution of particles represented by $\phi(n)$ with a probability $\alpha$ per unit $t$. (Without loss of generality we can put $\alpha=1$ by suitably choosing a unit of $t$) $\phi(o)$ represent the probability per unit $t$ of absorption. Also $\phi(1)=0$ since 'replacement' and the existence of the particle are mutually exclusive. If $G$ is the generating function of the total number of particles at $t$ then $G$ obviously satisfies

$$\frac{\partial G}{\partial t} = \phi(G) - G \tag{7.35}$$

If we know that if a replacement occurs, exactly $N$ particles replace the existing particle (or in other words the existing particle produces $N-1$ particles then $\phi(u)=u^N$ and

$$\frac{\partial G}{\partial t} = G^{N-1} - G \tag{7.36}$$

If $n=2$ we get a Furry process. For a general $\phi$ it may be difficult to solve for $G$ completely but it is quite easy to get the first and second moments from $G$. Putting

$$\begin{align*}
\left(\frac{\partial G}{\partial u}\right)_{u=1} & = \bar{n} , \left(\frac{\partial^2 G}{\partial u^2}\right)_{u=1} = \bar{n}^2 - \bar{n}, \\
\left[\frac{\partial \phi(u)}{\partial u}\right]_{u=1} & = \bar{a} , \left[\frac{\partial^2 \phi(u)}{\partial u^2}\right]_{u=1} = \bar{a}^2 - \bar{a}
\end{align*}$$

we have

$$\bar{n} = e^{(\bar{a}-1)t}, \quad \bar{n}^2 - \bar{n} = \frac{\bar{a}^2 - 2\bar{a} + 1}{\bar{a} - 1}$$

If we assume that at $t=0$ there is a primary and the primary
produce n secondaries with a probability per unit t equal to \( \xi \), and that the secondaries do not have the capacity to multiply then if \( G \) be the generating function of the distribution of the total number of particles then

\[
\frac{\partial G}{\partial t} = (G^n - G) \xi, \quad G = e^{(u^n - 1) \xi t} \tag{7.37}
\]

Le Coutour's model for nuclear evaporation

There are many physical experiments, such as nuclear disintegration, in which a single primary particle gives rise to several secondary particles by a process which is itself unobservable. A possible, much simplified mathematical model is a population in which each member has probabilities \( \lambda(t)dt \) of giving birth, \( (\mu - \gamma)(t)dt \) of death and \( \tau(t)dt \) of emigration in the time interval \( t \) to \( t + dt \). Only the emigrants are supposed to be observable and one has to calculate the total number of emigrants out of a population generated by a single ancestor.

If \( G \) be the generating function of the distribution of emigrants then \( G \) satisfies the equation

\[
\frac{\partial G}{\partial t} = G^2 \lambda + (\mu - \gamma) + \mu \gamma - G(\mu + \lambda) \tag{7.38}
\]

If \( \lambda, \mu, \gamma \) are independent of \( t \), we can solve the equation completely.
\[
\frac{\partial G}{\partial t} = \lambda (G_i - \alpha)(G_i - \beta)
\]

where

\[
\alpha, \beta = \left(\frac{\mu + \lambda}{\lambda}\right) \pm \sqrt{\left(\frac{\mu + \lambda}{\lambda}\right)^2 - 4\left(\frac{\mu - \gamma + \nu \gamma}{\lambda}\right)}
\]

\[
G_i = \frac{\beta - \alpha}{(1-\alpha)(\alpha - \beta)\lambda t} + \beta
\]

at \( t \to \infty \), \( e^{(\alpha - \beta)\lambda t} \to 0 \) and \( G_i \to \beta \). Thus

\[
G \to \left(\frac{\mu + \lambda}{\lambda}\right) - \sqrt{\left(\frac{\mu + \lambda}{\lambda}\right)^2 - 4\left(\frac{\mu - \gamma + \nu \gamma}{\lambda}\right)}
\]

which is Moyal's solution of Le Couteur's problem.

6. A problem on counters

We shall now consider a simple stochastic problem relating to electron counters which are supposed to register random events such that the probability of an event in any small interval of time of length \( \Delta t \) (\( \Delta t \) is an infinitesimal quantity) independently of previous events is \( \alpha \Delta t + O\Delta t \) or simply \( \Delta t \) (provided we suitably choose the unit of \( t \)). The probability that \( n \) events occur during a time interval of length \( t \) is given by the Poisson law.
We shall assume that every event is followed by a dead time during which no event is registered. Our problem is to calculate \( \Pi(n, t) \) that \( n \) events are recorded in time \( t \). It is quite clear that \( \Pi(n, t) \) satisfies the integral equation

\[
\Pi(n, t) = \int_0^t e^{-\tau} \Pi(n, t-\tau) d\tau + \int_a^t e^{-\tau} \Pi(n-1, t-\tau) d\tau \tag{7.41}
\]

The equation is self-explanatory. If \( G(u, t) \) is the generating function of \( \Pi(n, t) \)

\[
G(u, t) = \int_0^t e^{-\tau} G(u, t-\tau) d\tau + \int_a^t e^{-\tau} u G(u, t-\tau) d\tau \tag{7.42}
\]

Differentiating with respect to \( t \) we have

\[
\frac{dG(u, t)}{dt} = (u-1) e^{-a} G(u, t-a) \tag{7.43}
\]

We know that \( G(u, \tau) = 1 \), for \( \tau < a \). So we choose an \( N \) such that \( t-Na < a \). By iteration we obtain

\[
G(u, t) = 1 + \frac{(u-1) e^{-a}}{1!} (t-a) + \ldots
\]

\[
\ldots + \frac{(u-1)^N e^{-Na}}{N!} (t-Na)^N \tag{7.44}
\]

\[
\Pi(n, t) = \sum_{m=n}^{\infty} (-1)^{m-n} \binom{m}{n} e^{-ma} (t-ma)^m \tag{7.45}
\]