CHAPTER VI

LOVE WAVE PROPAGATION DUE TO AN IMPEDANCE IN 
THE INTERFACE BETWEEN A SOLID LAYER AND A 
SOLID QUARTER-SPACE

6.1. Introduction

We propose to discuss, in this chapter, Love wave propagation due to an impedance \((z = 0, x \geq 0)\) in the interface between a solid layer \(-H \leq z \leq 0, x \geq 0\) and a solid quarter-space \(z \geq 0, x \geq 0\). The rigid surface occupies a quarter space and its boundary in contact with the elastic quarter space is a plane boundary. The evaluation of the integrals along appropriate contour in the complex plane gives the reflected and transmitted waves.

6.2. Formulation of the problem

The problem is two dimensional, the \(zx\)-plane is the vertical plane and Love waves are propagated parallel to the \(x\)-axis. A solid layer of thickness \(H (-H \leq z \leq 0, x \geq 0)\) lies over a solid quarter space \((z \geq 0, x \geq 0)\). The media are homogeneous, isotropic and slightly dissipative. The velocities and rigidites of shear waves in the quarter space and solid layer are \(V_1, \mu_1\) and \(V_2, \mu_2\) respectively as shown in Fig. 6.1. Let the incident wave be (Sato (1961))

\[ v_{0,1} = A \cos \beta_{2N} \exp(-\beta_{1N}z - ik_{1N}x), \ z \geq 0 \quad (6.1) \]

\[ v_{0,2} = A \cos \beta_{2N}(z+H) \exp(-ik_{1N}x), \ -H \leq z \leq 0 . \quad (6.2) \]
FIG. 6.1 Geometry of the model.
The wave equation in two dimension is
\[(\nabla^2 + k^2 - k^2)\psi = 0\]

6.3. Boundary conditions

Let the displacement in the media be
\[v(x, z) = v_{0,1} + v_1(x, z), \quad z \geq 0, \quad x \geq 0\]
\[= v_{0,2} + v_2(x, z), \quad -H \leq z \leq 0, \quad x \geq 0\]

The conditions on the boundaries are

(i) \[\frac{\partial}{\partial z} (v_{0,1} + v_1) = a(v_{0,1} + v_1(x, z)), \quad z = 0, \quad x \geq 0\] (6.4)

(ii) \[\frac{\partial}{\partial z} (v_{0,2} + v_2) = a(v_{0,2} + v_2(x, z)), \quad z = 0, \quad x \geq 0\] (6.5)

(iii) \[\frac{\partial}{\partial z} v_2 = 0, \quad z = -H, \quad x \geq 0\] (6.6)

(iv) \[v_{0,1} + v_1 = 0, \quad z \geq 0, \quad x = 0\] (6.7)

(v) \[v_{0,2} + v_2 = 0, \quad -H \leq z \leq 0, \quad x = 0\] (6.8)

where 'a' is a constant , the interpretation of 'a' is given in section (4.3) of Chapter 4. The condition (6.7) and (6.8) imply that there is no displacement across the rigid plane boundary. These conditions can be simplified to

\[v_1(x, z) = -A \cos \beta_{2N}He^{-\beta_{1N}z - ik_{1N}x}, \quad z \geq 0, \quad x = 0\] (6.9)

\[v_2(x, z) = -A \cos \beta_{2N}(z+H)e^{-ik_{1N}x}, \quad -H \leq z \leq 0, \quad x = 0\] (6.10)
6.4. Solution of the problem

We multiply (6.3) by $e^{ipx}$ and integrate it between $x = 0$ and $x = \infty$ to find ($j = 1$)

$$\frac{d^2}{d^2z^2} \bar{v}_{1+}(p,z) - \beta_1^2 \bar{v}_{1+}(p,z) = (\frac{\partial}{\partial x})_{x = 0} - ip(v_1)_x = 0 \quad (6.11)$$

where $\beta_1 = \pm(p^2 - k_1^2)^{1/2} \quad (6.12)$

Changing $p$ to $-p$ in (6.11) and subtracting the resulting equation from it, we obtain

$$\frac{d^2}{d^2z^2} \left[ \bar{v}_{1+}(p,z) - \bar{v}_{1+}(-p,z) \right] - \beta_1^2 \left[ \bar{v}_{1+}(p,z) - \bar{v}_{1+}(-p,z) \right] = -2ip(v_1)_x = 0 \quad (6.13)$$

Using (6.9), we find

$$\frac{d^2}{d^2z^2} \left[ \bar{v}_{1+}(p,z) - \bar{v}_{1+}(-p,z) \right] - \beta_1^2 \left[ \bar{v}_{1+}(p,z) - \bar{v}_{1+}(-p,z) \right] = 2ipA\cos\beta_{2N}H e^{-\beta_{1N}z} \quad (6.14)$$

The solution to (6.14) is

$$\bar{v}_{1+}(p,z) - \bar{v}_{1+}(-p,z) = E_1(p) e^{-\beta_1z} + E_2(p) e^{\beta_1z}$$

$$= \frac{2ipA\cos\beta_{2N}H e^{-\beta_{1N}z}}{p^2 - k_{1N}^2} \quad (6.15)$$

In order that $\bar{v}_{1+}(p,z) - \bar{v}_{1+}(-p,z)$ is bounded as $z \to \infty$, so $E_2(p) = 0$. The solution (6.15) is now

$$\bar{v}_{1+}(p,z) - \bar{v}_{1+}(-p,z) = E_1(p) e^{-\beta_1z} - \frac{2ipA\cos\beta_{2N}H e^{-\beta_{1N}z}}{p^2 - k_{1N}^2} \quad (6.16)$$
Putting $z = 0$ in (6.16) and eliminating $E_j(p)$ from (6.16) and resulting equation, we get

\[
\bar{v}_{1+}(p,z) - \bar{v}_{1+}(-p,z) = \left( \bar{v}_{1+}(p) - \bar{v}_{1+}(-p) + \frac{2ipA \cos \beta_{2N}H}{p^2 - k_{1N}^2} e^{-\beta_1 z} \right) e^{-\beta_1 z} \tag{6.17}
\]

where $\bar{v}_j(p,0)$ and $\bar{v}'_j(p,0)$ are represented by $\bar{v}_j(p)$ and $\bar{v}'_j(p)$ respectively. Differentiating (6.17) w.r.t $z$ and putting $z = 0$ in the resulting equation, we obtain

\[
\bar{v}'_{1+}(p) - \bar{v}'_{1+}(-p) = -\beta_1 \left( \bar{v}_{1+}(p) - \bar{v}_{1+}(-p) + \frac{2ipA \cos \beta_{2N}H}{p^2 - k_{1N}^2} \right) + \frac{2ipA \beta_{1N} \cos \beta_{2N}H}{p^2 - k_{1N}^2} \tag{6.18}
\]

We multiply (6.4) by $e^{ipx}$ and integrate it between $x = 0$ and $x = \infty$ to find $(z = 0)$

\[
\mu_1 \int_v \int_{v_1(x,z)} e^{ipx} \, dx - A \mu_1 \beta_{1N} \cos \beta_{2N}H e^{-\beta_{1N} x} \int_0^\infty e^{i(p-k_{1N})x} \, dx \, dz = 0
\]

or

\[
\mu_1 \bar{v}'_{1+}(p) - \frac{iA \mu_1 \beta_{1N} \cos \beta_{2N}H}{p - k_{1N}} = a \bar{v}_{1+}(p) + \frac{iA \cos \beta_{2N}H}{p - k_{1N}} \tag{6.20}
\]
or
\[
\mu_1 \vec{v}'_1(p) = \frac{iA}{p-k_1N} \vec{v}_1(p) + \frac{\mu_1 \beta_{1N} + a}{p-k_1N} \cos \beta_{2N}H \quad (6.21)
\]

Changing \( p \) to \(-p\) in (6.21) and subtracting the resulting equation from it, we find

\[
\mu_1(\vec{v}'_1(p) - \vec{v}'_1(-p)) = a(\vec{v}_1(p) - \vec{v}_1(-p)) + \frac{2ipA(\mu_1 \beta_{1N} + a) \cos \beta_{2N}H}{p^2-k_1N^2} \quad (6.22)
\]

From (6.18) and (6.22), we get

\[
-\beta_1(\vec{v}_1(p) - \vec{v}_1(-p)) + \frac{2ipA \cos \beta_{2N}H}{p^2-k_1N^2} + \frac{2ipA \beta_{1N} \cos \beta_{2N}H}{p^2-k_1N^2}
= (a/\mu_1)(\vec{v}_1(p) - \vec{v}_1(-p)) + \frac{2ipA(\mu_1 \beta_{1N} + a) \cos \beta_{2N}H}{\mu_1(p^2-k_1N^2)} \quad (6.23)
\]

or

\[
\left[\vec{v}_1(p) - \vec{v}_1(-p)\right] \left(\frac{a+\mu_1 \beta_1}{\mu_1}\right) = -\frac{2ipA \beta_{1N} \cos \beta_{2N}H}{\mu_1(p^2-k_1N^2)} + \frac{2ipA \beta_{1N} \cos \beta_{2N}H}{p^2-k_1N^2} - \frac{2ipA(\mu_1 \beta_{1N} + a) \cos \beta_{2N}H}{\mu_1(p^2-k_1N^2)} \quad (6.24)
\]

or

\[
\vec{v}_1(p) - \vec{v}_1(-p) = -\frac{2ipA \cos \beta_{2N}H}{p^2-k_1N^2} \quad (6.25)
\]
Using (6.25) in (6.17), we obtain
\[
\overline{v}_1^+(p,z) - \overline{v}_1^+(-p,z) = - \frac{2ip\cos\beta_2 N H e^{-\beta_1 N z}}{p^2 - k_{1N}^2} \tag{6.26}
\]

We again multiply (6.3) by $e^{ipx}$ and integrate it between $x = 0$ and $x = \infty$ to find ($j = 2$)
\[
\frac{d^2}{dz^2} \overline{v}_2^+(p,z) - \beta_2^2 \overline{v}_2^+(p,z) = \left( \frac{\partial v_2}{\partial x} \right)_{x=0} - ip(v_2)_{x=0} \tag{6.27}
\]

where $\beta_2 = \pm(p^2-k_2^2)^{1/2}$

Changing $p$ to $-p$ in (6.27) and subtracting the resulting equation from it, we get
\[
\frac{d^2}{dz^2} \left( \overline{v}_2^+(p,z) - \overline{v}_2^+(-p,z) \right) - \beta_2^2 \left( \overline{v}_2^+(p,z) - \overline{v}_2^+(-p,z) \right) = -2ip(v_2)_{x=0} \tag{6.28}
\]

Using (6.8), in (6.28), we obtain
\[
\frac{d^2}{dz^2} \left( \overline{v}_2^+(p,z) - \overline{v}_2^+(-p,z) \right) - \beta_2^2 \left( \overline{v}_2^+(p,z) - \overline{v}_2^+(-p,z) \right) = 2ip\cos\beta_2 N(z+H) \tag{6.29}
\]

The solution of (6.29) is
\[
\overline{v}_2^+(p,z) - \overline{v}_2^+(-p,z) = F_1(p) e^{-\beta_2 z} + F_2(p) e^{\beta_2 z}
\]

\[
\frac{2ip\cos\beta_2 N(z+H)}{p^2 - k_{1N}^2} \tag{6.30}
\]

Now taking the Fourier transform of (6.6), we find
\[ v'_{2^+}(p,-H) = 0 \], also \[ v'_{2^+}(-p,-H) = 0 \]

so that
\[ v'_{2^+}(p,-H) - v'_{2^+}(-p,-H) = 0 \] (6.31)

Differentiating (6.30) w.r.t \( z \), putting \( z = -H \) and using (6.31), we obtain

\[ v_{2^+}(p,z) - v_{2^+}(-p,z) = \frac{2ipA\cos \beta_{2N}(z+H)}{p^2 - k_{1N}^2} \] (6.32)

Putting \( z = 0 \) in (6.32) and eliminating \( F(p) \) between (6.32) and the resulting equation, we obtain

\[ v_{2^+}(p,z) - v_{2^+}(-p,z) = \frac{\cosh \beta_{2}(z+H)}{\cosh \beta_{2}H} \] [ \[ v_{2^+}(p) - v_{2^+}(-p) \] (6.33)

Differentiating (6.33) w.r.t \( z \) and putting \( z = 0 \) in the resulting equation, we obtain

\[ v'_{2^+}(p) - v'_{2^+}(-p) = \beta_2 \tanh \beta_2 H \] [ \[ v_{2^+}(p) - v_{2^+}(-p) \] (6.34)

Taking the Fourier transform of (6.5), we get \( (z = 0) \)

\[ \mu_2 \, \tilde{v}'_{2^+}(p) = a \, \tilde{v}_{2^+}(p) + \frac{Ai}{p - k_{1N}} (\mu_2 \beta_{2N} \sin \beta_{2N}H + a \cos \beta_{2N}H) \] (6.35)
Changing \( p \) to \( -p \) in (6.35) and subtracting the resulting equation from it, we find

\[
\mu_2 (\bar{v}_2^+(p) - \bar{v}_2^+(-p)) = a (\bar{v}_2^+(p) - \bar{v}_2^+(-p))
\]

\[
2 \rho A + \frac{(2 \mu \beta_2 \sin \beta_2 N H + a \cos \beta_2 N H)}{p^2 - k_1 N^2} \tag{6.36}
\]

From equation (6.34) and (6.36), we obtain

\[
\bar{v}_2^+(p) - \bar{v}_2^+(-p) = \frac{\cosh \beta_2 N H}{(\cosh \beta_2 H - \mu \beta_2 \sinh \beta_2 H)} \left[ - \frac{2 \rho A \cos \beta_2 N H}{p^2 - k_1 N^2} \right] \tag{6.37}
\]

Using (6.37) in (6.33), we get

\[
\bar{v}_2^+(p, z) - \bar{v}_2^+(-p, z) = - \frac{2 \rho A \cos \beta_2 N (z + H)}{p^2 - k_1 N^2} \tag{6.38}
\]

By inverse Fourier transform, we have

\[
v_2(x, z) = \frac{1}{2 \pi} \int_{-\infty + i\beta}^{\infty + i\beta} \bar{v}_2^+(p, z) e^{-ipx} dp, \quad x \geq 0 \tag{6.39}
\]

Since \( \bar{v}_2^+(-p, z) \) is analytic in the lower half of the complex plane, we find

\[
(1/2\pi) \int_{-\infty + i\beta}^{\infty + i\beta} \bar{v}_2^+(-p, z) e^{-ipx} dp = 0 \tag{6.40}
\]
Subtracting (6.40) from (6.39), we obtain

\[
\begin{align*}
v_2(x,z) & = 1/2\pi \int_{-\infty+\imath\beta}^{\infty+\imath\beta} \left( \bar{v}_2+(p,z) - \bar{v}_2+(-p,z) \right) e^{-\imath px} \, dp \\
& = 1/2\pi \int_{-\infty+\imath\beta}^{\infty+\imath\beta} \left( v_2+(p,z) - v_2+(-p,z) \right) e^{-\imath px} \, dp
\end{align*}
\]

(6.41)

Similarly, we find

\[
\begin{align*}
v_1(x,z) & = (1/2\pi) \int_{-\infty+\imath\beta}^{\infty+\imath\beta} \left( \bar{v}_1+(p,z) - \bar{v}_1+(-p,z) \right) e^{-\imath px} \, dp \\
& = (1/2\pi) \int_{-\infty+\imath\beta}^{\infty+\imath\beta} \left( v_1+(p,z) - v_1+(-p,z) \right) e^{-\imath px} \, dp
\end{align*}
\]

(6.42)

where \( \bar{v}_2+(p,z) - \bar{v}_2+(-p,z) \) and \( \bar{v}_1+(p,z) - \bar{v}_1+(-p,z) \) are given by (6.38) and (6.26) respectively.

6.5. Reflected and transmitted waves

We now evaluate the line integrals (6.41) and (6.42) along a suitable closed contour in the complex plane. The residues of the poles of (6.41) at \( p = \pm k_{1N} \) contribute

\[
v_{2,1} = A \cos \beta_{2N}(z+H) e^{-\imath k_{1N}x}, \quad x < 0
\]

(6.43)

and

\[
v_{2,2} = -A \cos \beta_{2N}(z+H) e^{-\imath k_{1N}x}, \quad x > 0
\]

(6.44)

The residues of the poles of (6.42) at \( p = + k_{1N} \) are

\[
v_{1,1} = A \cos \beta_{2N} e^{-\beta_{1N}z - \imath k_{1N}x}, \quad x < 0
\]

(6.45)

and

\[
v_{1,2} = -A \cos \beta_{2N} e^{-\beta_{1N}z + \imath k_{1N}x}, \quad x > 0
\]

(6.46)

The transmitted wave in (6.43) and (6.45) due to impeding surface are same as the incident waves in (6.1) and (6.2) when the contour of integration is in the upper half of the complex
FIG: 6.2. The contour of integration in the complex $p$-plane with branch cuts.
branch cut $L_2$. This shows that no wave is scattered due to the impeding surface in the interface between the layer and the quarter-space.

6.7. Conclusions

The reflected wave in both the layer and the quarter space are of the form $-\text{Acos} \beta_2 N (z+H) e^{ik_1 x \beta_1 N} \text{ and } -\text{Acos} \beta_2 N H e^{ik_1 x \beta_1 N}$ respectively whose amplitude is same as the amplitude of incident waves in (6.1) and (6.2). The transmitted wave in (6.43) and (6.45) in both the layer and the quarter space is same as the incident waves in (6.1) and (6.2). No wave is scattered in the layer and the quarter space.
plane. The equation (6.44) and (6.46) represent the reflected Love waves appropriate to the impeding surface whose amplitude is same as the amplitude of incident waves in both the layer and the quarter-space.

6.6. Scattered waves

We now derive the scattered waves in the surface layer and the solid quarter-space. Let us first consider the case of solid quarter space. There is a branch point \( p = -k_1 \) in the lower half plane. For contribution around this point, we put \( p = -k_1 - iu \), \( u \) is small. The branch cut is obtained by taking \( \Re(\beta_1) = 0 \). \( \beta_1 \) is imaginary and \( \beta_1^2 \) is negative. Then

\[
\beta_1^2 = p^2 - k_1^2 = (-k_1 - iu)^2 - k_1^2 = 2iu(k_1' + ik''_1),
\]

\[ k_1 = k_1' + ik''_1 \]

Taking \( k_1' = 0 \) i.e. \( \beta_1 = \pm i(2k''_1u)^{1/2} = \pm i \beta \)  

\( \Im(\beta_1) \) has different sign on two sides of the branch cut \( L_1 \). Integrating (6.42) along two sides of the branch cut \( L_1 \) (Fig. (6.2)). We obtain

\[
v_{1,3} = (i/2\pi) \int_0^\infty \left[ ( \tilde{v}_{1+}(p,z) - \tilde{v}_{1+}(-p,z)) \beta_1 = i\beta_1 
- ( \tilde{v}_{1+}(p,z) - \tilde{v}_{1+}(-p,z)) \beta_1 = -i\beta_1 \right] e^{-ik''_1 x} e^{-ux} du
\]

\[ = 0 \]

The integral (6.42) vanishes when evaluated along two sides of the branch cut \( L_2 \) in Fig. (6.2) in the complex plane. Similarly, the integral (6.41) vanishes when evaluated along two sides of