3.1. Introduction

It is a general observation that both reflected and refracted P and SV waves are generated when a wave of either P or SV type impinges upon an interface between two different media. This phenomenon for isotropic/anisotropic half-spaces was considered by many investigators, i.e. Knott (1899), Stoneley (1924), Sezawa and Kanai (1932, 1934), Dix (1939), Muskat and Mercs (1940), McNiven and Mengi (1968), Henneke (1972), Červeny (1974) and others. In all these investigations it is assumed that the medium under consideration is not prestressed.

For studying the reflection and transmission of plane waves at a plane interface between two initially stressed elastic half-spaces, Bhattacharya and Sengupta (1973), Dey and Addy (1979) and Chakraborty and Dey (1980) used the method of potentials to decouple the P and SV motion. In general, potential method is not applicable for prestressed media. Therefore, the results of these authors are not acceptable.

In this chapter, the problem of the reflection and transmission of plane waves at a plane interface between two
prestressed elastic semi-infinite media has been solved by a direct method, without the introduction of potentials. The expressions for the reflection and transmission coefficients are derived. It is shown analytically how the amplitudes of the reflected and transmitted waves are affected by the initial stress.

3.2. Basic Equations

Consider two semi-infinite prestressed media, H and H', in welded contact (Fig. 3.1). The interface between H and H' is taken as the xz-plane, i.e., \( z = 0 \) with reference to a rectangular cartesian co-ordinate system \((x,y,z)\) in which the positive z-axis is directed towards the prestressed half-space H.

Let the principal directions of initial stress be chosen to coincide with the directions of elastic symmetry and the co-ordinate axes. The state of initial stress is, therefore, defined by the principal components \( S_{11}, S_{22}, \) and \( S_{33} \) of the initial stress. Here we assume \( S_{22} = S_{33}, S_{22} \) and \( S_{33} \) are constant. We restrict our analysis to plane strain parallel to the xz-plane with displacements \( u \) and \( w \) in the x and z directions, respectively. Let \( S_{11} \) and \( S_{33} \) be the initial stresses along the x and z directions respectively in the lower medium and \( S_{11}' \) and \( S_{33}' \) be the corresponding quantities in the upper medium. The equations of motion for incremental deformations in the lower medium are given by equation (1.9):
\[ B_{11} \frac{\partial^2 u}{\partial x^2} + A_3 \frac{\partial^2 w}{\partial x \partial z} + A_1 \frac{\partial^2 u}{\partial z^2} = \rho \frac{\partial^2 u}{\partial t^2}, \]

\[ B_{22} \frac{\partial^2 w}{\partial z^2} + A_3 \frac{\partial^2 u}{\partial z \partial x} + A_2 \frac{\partial^2 w}{\partial x^2} = \rho \frac{\partial^2 w}{\partial t^2}. \] (3.1)

\( B_{11}, B_{22}, B_{12} \) and \( q_2 \) are incremental elastic coefficients which are related with \( A_1, A_2, A_3 \) through equation (1.8). \( \rho \) is the density and \( P = S_{33} - S_{11} \) for the lower medium.

For the upper half-space \( H' \), we denote the displacements by \((u', 0, w')\), the density by \( \rho' \) and the incremental elastic coefficients by \((B'_{11}, B'_{22}, B'_{12}, q'_2)\) and \( P' = S'_{33} - S'_{11} \). Then, we can write the equations of motion for the upper medium similar to equation (3.1) in primes:

\[ B'_{11} \frac{\partial^2 u'}{\partial x^2} + A'_3 \frac{\partial^2 w'}{\partial x \partial z} + A'_1 \frac{\partial^2 u'}{\partial z^2} = \rho' \frac{\partial^2 u'}{\partial t^2}, \]

\[ B'_{22} \frac{\partial^2 w'}{\partial z^2} + A'_3 \frac{\partial^2 u'}{\partial z \partial x} + A'_2 \frac{\partial^2 w'}{\partial x^2} = \rho' \frac{\partial^2 w'}{\partial t^2}. \] (3.2)

3.3. Propagation of Plane Waves

For plane waves of circular frequency \( \omega \), wavenumber \( k \) and phase velocity \( C \), incident at an angle \( \theta \) with the \( z \)-axis from lower half-space \( H \), we may assume

\[ u = U_1 \exp(iP_1), \quad w = W_1 \exp(iP_1), \] (3.3)

where \( U_1 \) and \( W_1 \) are the amplitude factors and \( P_1 \) is the phase factor associated with the incident waves defined in
Equation (2.2). Equation (3.3) must satisfy the equations of motion (3.1) and after simplification, we get

$$-(D_1 - \rho C^2) U_1 + A_3 \sin \theta \cos \theta W_1 = 0,$$

$$A_3 \sin \theta \cos \theta U_1 - (D_2 - \rho C^2) W_1 = 0. \quad (3.4)$$

Similarly, for waves reflected at the interface $z = 0$, we have

$$u = U_r \exp(iP_r), \quad w = W_r \exp(iP_r), \quad (3.5)$$

where $U_r$ and $W_r$ are the amplitude factors and $P_r$ is the phase factor associated with the reflected waves, defined in equation (2.6). Equation (3.5) also satisfies the equations of motion (3.1) and on simplification gives

$$-(D_1 - \rho C^2) U_r + A_3 \sin \theta \cos \theta W_r = 0,$$

$$A_3 \sin \theta \cos \theta U_r - (D_2 - \rho C^2) W_r = 0. \quad (3.6)$$

where

$$D_1(\theta) = B_{11} \sin^2 \theta + A_1 \cos^2 \theta,$$

$$D_2(\theta) = B_{22} \cos^2 \theta + A_2 \sin^2 \theta. \quad (3.7)$$

For transmitted waves propagating in the upper half-space $H'$ making an angle $\theta$ with $z$-axis in the negative direction, we have

$$u' = U_t \exp(ip'), \quad w' = W_t \exp(ip'), \quad (3.8)$$
where $U_t$ and $W_t$ are the amplitude factors and

$$P' = k^i t C't - (x \sin \theta - z \cos \theta)^2,$$  \hspace{1cm} (3.9)

is the phase factor associated with the transmitted waves.

Equation (3.8) must satisfy the equations of motion (3.2) and after simplification, we get

$$-(D' - \rho C'^2) U_t + A_3 \sin \theta \cos \theta W_t = 0,$$

$$A_3 \sin \theta \cos \theta U_t - (D' - \rho C'^2) W_t = 0,$$  \hspace{1cm} (3.10)

where

$$D'(\theta) = B_{11} \sin^2 \theta + A_1^2 \cos^2 \theta,$$

$$D'(\theta) = B_{22} \cos^2 \theta + A_2^2 \sin^2 \theta.$$  \hspace{1cm} (3.11)

The set of equations (3.4) and (3.6) have a non-trivial solution only if

$$2 \rho C^2(\theta) = \left(D_1 + D_2 \right) \pm \left[ \left(D_1 - D_2 \right)^2 + 4A_3^2 \sin^2 \theta \cos^2 \theta \right]^{1/2}.$$  \hspace{1cm} (3.12)

Similarly the set of equations (3.10) has a non-trivial solution only if

$$2 \rho C'^2(\theta) = \left(D'_1 + D'_2 \right) \pm \left[ \left(D'_1 - D'_2 \right)^2 + 4A_3^2 \sin^2 \theta \cos^2 \theta \right]^{1/2}.$$  \hspace{1cm} (3.13)

Thus, in general, whether we consider incident or reflected waves, in the two-dimensional model of the prestress medium,
there are two types of plane waves whose phase velocities depend on the angle $\theta$. Let $C_p(\theta)$ and $C_{3V}(\theta)$ be the two values of $C(\theta)$ associated with the upper and lower signs, respectively, in equation (3.12). Similarly for transmitted waves, there are two types of plane waves whose phase velocities depend on the angle $\theta$. Let $C'_p(\theta)$ and $C'_{3V}(\theta)$ be the two values of $C'(\theta)$ associated with the upper and lower signs, respectively, in equation (3.13). $C_p(\theta)$ and $C'_{3V}(\theta)$ are associated with what are known as quasi-P waves and $C_{3V}(\theta)$ and $C'_p(\theta)$ are associated with quasi-SV waves.

3.4. Plane Waves Incident at the Interface

Consider the incident wave pair $P$, SV, shown in Fig. (3.1), incident at the interface between two prestressed elastic half-spaces of different material properties. We assume the total displacement field to be of the form

\[ u = U_{1L} \exp(iP_1) + U_{12} \exp(iP_2) + U_{r1} \exp(iQ_1) + U_{r2} \exp(iQ_2), \]

\[ w = W_{1L} \exp(iP_1) + W_{12} \exp(iP_2) + W_{r1} \exp(iQ_1) + W_{r2} \exp(iQ_2), \]

\[ u' = U_{t1} \exp(iP'_1) + U_{t2} \exp(iP'_2), \]

\[ w' = W_{t1} \exp(iP'_1) + W_{t2} \exp(iP'_2), \]

where $P_1$, $Q_1$, $P_2$ and $Q_2$ are defined in equation (2.10) and

\[ P'_1(x, z) = \frac{C_Q}{C_p} \left\{ C_p t - (x \sin e' - z \cos e') \right\}^2, \]
\[ P_2'(x, z) = \frac{\omega}{c_{SV}'} \left( x \sin f' - z \cos f' \right) \] (3.15)

\[ P_1' = P_1'(x, z) = \frac{\omega}{c_{P}'} \left( x \sin f - z \cos f \right) \] (3.16)

\[ P\text{' and } P^* \text{ are the phase factors associated with the transmitted quasi-P and quasi-SV waves, respectively and } e^* \text{ and } f^* \text{ are the corresponding angles with } z\text{-axis in the upper half-space. } (U^*_t, W^*_t) \text{ and } (U^*_t, W^*_t) \text{ are the amplitude factors associated with the transmitted quasi-P and quasi-SV waves, respectively.} \]

Since incident, reflected and transmitted waves must satisfy the equations of motion (3.1) and (3.2), separately, we have, from the first members in equations (3.4), (3.6) and (3.10),

\[ -\left( \frac{D}{c_{P}'} \right) U_{p1} + A_3 \sin e \cos e W_{t1} = 0 \]

\[ -\left( \frac{D}{c_{P}'} \right) U_{p2} + A_3 \sin f \cos f W_{t1} = 0 \]

\[ -\left( \frac{D}{c_{SV}'} \right) U_{r1} + A_3 \sin e \cos e W_{r1} = 0 \]

\[ -\left( \frac{D}{c_{SV}'} \right) U_{r2} + A_3 \sin f \cos f W_{r1} = 0 \]

\[ -\left( \frac{D}{c_{P}'(e')} \right) U_{t1} + A_3 \sin e' \cos e' W_{t1} = 0 \]

\[ -\left( \frac{D}{c_{SV}'(f')} \right) U_{t2} + A_3 \sin f' \cos f' W_{t1} = 0 \]

Equation (3.16) can be written as
\[ U_{11} = F_1 W_{11}, \quad U_{12} = F_2 W_{12}, \]
\[ U_{r1} = -F W_{r1}, \quad U_{r2} = -F W_{r2}, \]
\[ U_{t1} = F W_{t1}, \quad U_{t2} = F W_{t2}, \]

where \( F_1 \) and \( F_2 \) are defined in equation (2.13) and
\[
F_3 = \frac{A' \sin \theta' \cos \theta'}{D'(e') - \rho C_2^2(e')}, \quad F_4 = \frac{A' \sin \phi' \cos \phi'}{D'(f') - \rho C_2^2(f')}.
\]

3.5. Boundary Conditions

We assume that the half-spaces are in welded contact at \( z = 0 \). Hence the displacements and incremental boundary forces per unit initial area must be continuous there, that is,
\[
\begin{align*}
U_t &= U_{t1}, \\
W &= W_{t1}, \\
\Delta f_x &= \Delta f'_{x1}, \\
\Delta f_z &= \Delta f'_{z1},
\end{align*}
\]

at \( z = 0 \),

where \([\text{cf. Eq. (2.16)}]\)
\[
\begin{align*}
\Delta f_x &= s_{13} + s_{33} \omega - s_{11} e_{zx}, \\
\Delta f_z &= s_{33} + s_{33} e_{xx},
\end{align*}
\]
are the incremental boundary forces per unit initial area in
the lower medium and
\[ \Delta f'_x = s'_{13} + s'_{33} \omega_{y'} - s'_{11} e'_x , \]
\[ \Delta f'_z = s'_{33} + s'_{33} e'_{xx} , \]  
(3.21)
are the incremental boundary forces per unit initial area in
the upper medium. \( s'_{13}, s'_{33}, s'_{13}, s'_{33} \) are incremental
stresses, \( e'_{xx}, e'_{xx}, e'_{xx}, e'_{xx} \) are incremental strains and
\( \omega_x, \omega_y \) are the incremental rotation components parallel to
the \( xz \)-plane. Using \( v = 0 \) and \( \frac{\partial}{\partial y} = 0 \) in equation (1.2), we
get the explicit expressions for \( s'_{13}, s'_{33}, s'_{13}, s'_{33} \) in terms
of \( u \) and \( w \) as
\[ s'_{33} = (B_{12} - \rho) \frac{\partial u}{\partial x} + B_{22} \frac{\partial w}{\partial z} , \]
\[ s'_{13} = q' \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) , \]  
(3.22)
with similar expressions for \( s'_{33} \) and \( s'_{13} \)
\[ s'_{33} = (B'_{12} - \rho') \frac{\partial u'}{\partial x} + B'_{22} \frac{\partial w'}{\partial z} , \]
\[ s'_{13} = q' \left( \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} \right) , \]  
(3.23)
From equations (1.3) and (2.16) we have
\[ e_{xx} = a_{11} = \frac{\partial u}{\partial x} , \]
with similar expressions for $e'_{xx}$, $e'_{xz}$, and $\omega'_{y}$:

\[ e'_{xx} = \frac{\partial u'}{\partial x}, \]
\[ e'_{xz} = \frac{1}{2} \left( \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} \right), \]
\[ \omega'_{y} = \frac{1}{2} \left( \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} \right). \tag{3.25} \]

3.6. **Formulation of the Problem**

The total displacement field given in equation (3.14) must satisfy the boundary condition (3.19). Making use of equations (2.10), (3.14) - (3.15) and (3.20) - (3.25), we obtain:

\[
(U_{11} + U_{r1}) e_{1} + (U_{12} + U_{r2}) e_{2} = U_{t1} e_{1} + U_{t2} e_{2},
\]
\[
(W_{11} + W_{r1}) e_{1} + (W_{12} + W_{r2}) e_{2} = W_{t1} e_{1} + W_{t2} e_{2},
\]
\[
(iQ'_{1}(x,0)) e_{1} + (iQ'_{2}(x,0)) e_{2} = (iP'_{1}(x,0)) e_{1} + (iP'_{2}(x,0)) e_{2},
\]
\[
(iQ_{2}(x,0)) e_{1} + (iQ_{2}(x,0)) e_{2} = (iP_{2}(x,0)) e_{1} + (iP_{2}(x,0)) e_{2},
\]
\[
\text{and} \quad (iQ_{2}(x,0)) e_{1} + (iQ_{2}(x,0)) e_{2}.
\tag{3.26}
\[
\begin{align*}
&\frac{1}{C_p} \left\{(2q + p)(U_{11} - U_{12})\cos e + (2q - p)(W_{11} + W_{12})\sin e\right\} e^{ip_1(x,0)} \\
&\quad + \frac{1}{C_{SV}} \left\{(2q + p)(U_{21} - U_{22})\cos f + (2q - p)(W_{21} + W_{22})\sin f\right\} e^{ip_2(x,0)} \\
&= \frac{1}{C_p} \left\{- (2q + p')U_{11} \cos e' + (2q - p')W_{11} \sin e'\right\} e^{ip'_1(x,0)} \\
&\quad + \frac{1}{C_{SV}} \left\{- (2q + p')U_{21} \cos f' + (2q - p')W_{21} \sin f'\right\} e^{ip'_2(x,0)} \\
&\quad + \frac{1}{C_p} \left\{(B_{11} + S_{11})(U_{11} + U_{12})\sin e + B_{22}(W_{11} - W_{12})\cos e\right\} e^{ip_1(x,0)} \\
&\quad + \frac{1}{C_{SV}} \left\{(B_{12} + S_{12})(U_{21} + U_{22})\sin f + B_{22}(W_{12} - W_{12})\cos f\right\} e^{ip_2(x,0)} \\
&= \frac{1}{C_p} \left\{B_{11} + S_{11}\right\} U_{11} \sin e' - B_{22} W_{11} \cos e' e^{ip'_1(x,0)} \\
&\quad + \frac{1}{C_{SV}} \left\{B_{12} + S_{12}\right\} U_{21} \sin f' - B_{22} W_{21} \cos f' e^{ip'_2(x,0)}
\end{align*}
\]

where
\[
\begin{align*}
C_p &= C_p(e), \quad C_{SV} = C_{SV}(f), \\
C_p' &= C_p(e'), \quad C_{SV}' = C_{SV}(f'), \\
R &= S_{11} + S_{33}, \quad R' = S_{11}' + S_{33}'
\end{align*}
\]

and where we have made use of the result
\[
\begin{align*}
P_1(x,0) &= Q_1(x,0), \quad P_2(x,0) = Q_2(x,0).
\end{align*}
\]
since equation (3.26) must be satisfied for all values of \( x \), we have
\[
0(x,0) = Q(x,0) = P'(x,0) = P''(x,0),
\]
which, on using equations (2.10) and (3.15), imply
\[
\frac{\sin e}{C_p(e)} = \frac{\sin f}{C_{SV}(f)} = \frac{\sin e'}{C_p'(e')} = \frac{\sin f'}{C_{SV}'(f')}.
\]
This is the form of Snell's law for initially stressed media.

Equations (3.17), (3.26), (3.29) and (3.30) yield
\[
F_1 W_1 - F_2 W_2 + F_3 W_3 - F_4 W_4 = 0,
\]
\[
W_1 + W_1' + W_1' + W_1' + W_1' + W_1' + W_1' = 0,
\]
\[
a W_1 + a W_1 + a W_2 + a W_2 - a W_1 - a W_1 = 0,
\]
\[
b W_1 - b W_1 + b W_1 - b W_2 + b W_2 + b W_2' = 0,
\]
where
\[
C_p a_1 = -(P + 2q_2) F_1 \cos e + (2q_2 - R) \sin e,
\]
\[
C_{SV} a_2 = -(P + 2q_2) F_2 \cos f + (2q_2 - R), \sin f,
\]
\[
C_p b_1 = (B_{12} + S_{11}) F_1 \cos e - B_{22} \cos e,
\]
\[
C_{SV} b_2 = (B_{12} + S_{11}) F_2 \sin f - B_{22} \cos f,
\]
\[
C_p a_3 = -(2q_2' + p') F_3 \cos e' + (2q_2 - R) \sin e',
\]
\[
C_{SV} a_4 = -(2q_2' + p') F_4 \cos f' + (2q_2' - R') \sin f',
\]
3.7. Quasi-P Waves Incident at the Interface

We assume now that only a quasi-P wave is incident at the interface. Hence setting \( w_{21} = 0 \) in equation (3.31), it can be written as

\[
P_1 \frac{w_{r_1}}{w_{1i}} + P_2 \frac{w_{r_2}}{w_{1i}} + P_3 \frac{w_{t_1}}{w_{1i}} + P_4 \frac{w_{t_2}}{w_{1i}} = P_1
\]

\[
\frac{w_{r_1}}{w_{1i}} + \frac{w_{r_2}}{w_{1i}} - \frac{w_{t_1}}{w_{1i}} - \frac{w_{t_2}}{w_{1i}} = -1
\]

\[
a_1 \frac{w_{r_1}}{w_{1i}} + a_2 \frac{w_{r_2}}{w_{1i}} - a_3 \frac{w_{t_1}}{w_{1i}} - a_4 \frac{w_{t_2}}{w_{1i}} = -a_1
\]

\[
-b_1 \frac{w_{r_1}}{w_{1i}} - b_2 \frac{w_{r_2}}{w_{1i}} - b_3 \frac{w_{t_1}}{w_{1i}} - b_4 \frac{w_{t_2}}{w_{1i}} = -b_1
\]

Equations (3.33) are easily solved by Cramer's rule of determinants. The results for the reflection coefficients are

\[
\frac{w_{r_1}}{w_{1i}} = \frac{\Delta_1}{w_{1i}}, \quad \frac{w_{r_2}}{w_{1i}} = \frac{\Delta_2}{w_{1i}}, \quad \frac{w_{t_1}}{w_{1i}} = -\frac{\Delta_1}{w_{1i}}, \quad \frac{w_{t_2}}{w_{1i}} = -\frac{\Delta_1}{w_{1i}}
\]
\[ \begin{align*}
\frac{U_{r2}}{U_{i1}} &= -\frac{F_2}{F_1} \quad \frac{W_{t2}}{W_{i1}} = -\frac{F_2 \Delta_2}{F_1 \Delta} \quad \text{(3.34)}
\end{align*} \]

and for the transmission coefficients
\[ \begin{align*}
\frac{W_{t1}}{W_{i1}} &= \frac{\Delta_3}{\Delta} \quad \frac{W_{t2}}{W_{i1}} = \frac{\Delta_4}{\Delta} \quad \frac{U_{t1}}{U_{i1}} = \frac{F_3}{F_1} \quad \frac{W_{t1}}{W_{i1}} = \frac{F_3 \Delta_3}{F_1 \Delta} \\
\frac{U_{t2}}{U_{i1}} &= \frac{F_4}{F_1} \quad \frac{W_{t2}}{W_{i1}} = \frac{F_4 \Delta_4}{F_1 \Delta}
\end{align*} \quad \text{(3.35)} \]

where
\[ \Delta = \begin{vmatrix}
F_1 & F_2 & F_3 & F_4 \\
1 & 1 & -1 & -1 \\
a_1 & a_2 & -a_3 & -a_4 \\
-b_1 & -b_2 & -b_3 & -b_4
\end{vmatrix} \quad \text{(3.36)} \]

\[ = F_1 \left[ a_2(b_3-b_4) + a_3(b_4+b_2) - a_4(b_2+b_3) \right] + F_2 \left[ a_1(b_4-b_3) - a_3(b_1+b_4) + a_4(b_1+b_3) \right] + F_3 \left[ a_1(b_4+b_2) - a_2(b_4+b_1) + a_4(b_1-b_2) \right] + F_4 \left[ -a_1(b_2+b_3) + a_2(b_1+b_3) - a_3(b_1-b_2) \right], \]

and \( \Delta_j \) \( (j = 1,2,3,4) \) are obtained from \( \Delta \) on replacing the
elements of its jth column by $F_1$, $-1$, $-a_1$, $-b_1$, respectively.

On simplification, we get

$$\Delta_1 = F_1 \left[ a_2(b_3 - b_4) + a_3(b_4 + b_2) - a_4(b_2 + b_3) \right]$$

$$+ F_2 \left[ a_1(b_3 - b_4) + a_3(b_4 - b_1) + a_4(b_1 - b_3) \right]$$

$$+ F_3 \left[ a_2(b_4 - b_1) - a_1(b_2 + b_4) + a_4(b_1 + b_2) \right]$$

$$+ F_4 \left[ a_1(b_3 + b_2) + a_2(b_1 - b_3) - a_3(b_2 + b_1) \right], \quad (3.37)$$

$$\Delta_2 = 2F_1 \left[ a_1(b_4 - b_3) + (a_4b_3 - a_3b_4) \right] + 2b_1 \left[ F_3(a_1 - a_4) + F_4(a_3 - a_1) \right], \quad (3.38)$$

$$\Delta_3 = 2F_1 \left[ a_1(b_4 + b_2) - (a_4b_2 + a_2b_4) \right] + 2b_1 \left[ F_2(a_4 - a_1) + F_4(a_2 - a_1) \right], \quad (3.39)$$

$$\Delta_4 = 2F_1 \left[ -a_1(b_2 + b_3) + (a_2b_3 + a_3b_2) \right] + 2b_1 \left[ F_2(a_1 - a_3) + F_3(a_1 - a_2) \right]. \quad (3.40)$$

Equations (3.34) and (3.35) give the reflection and transmission coefficients when $P$ waves are incident at a plane interface between two prestressed semi-infinite elastic media in welded contact, respectively.

3.8. Quasi-$SV$ Waves Incident at the Interface

Assuming next that only a quasi-$SV$ wave is incident at the interface, we set $w_1 = 0$ in equation (3.31) which
produces a set of equations similar to equation (3.33) as

\[
\begin{align*}
& W_{r1} - W_{r2} = F_1 - F_2 + F_3 - F_4 = F_2, \\
& W_{t1} - W_{t2} = -1, \\
& a_1 + a_2 - \alpha_3 - a_4 = -a_2, \\
& b_1 + b_2 + b_3 + b_4 = b_2. \\
\end{align*}
\]

Solving equation (3.41), we get

\[
\begin{align*}
& W_{r1} = \frac{\Delta_5}{\Delta}, \quad W_{r2} = \frac{\Delta_6}{\Delta}, \quad U_{r1} = -\frac{F_1}{F_2} W_{r1} = -\frac{F_1}{F_2} \Delta_5, \\
& U_{r2} = -\frac{W_{r2}}{W_{t2}} = -\frac{\Delta_6}{\Delta}, \\
& U_{t1} = \frac{F_1}{F_2} W_{t1} = \frac{F_1}{F_2} \Delta_7, \\
& U_{t2} = \frac{F_1}{F_2} W_{t2} = \frac{F_1}{F_2} \Delta.
\end{align*}
\]
where $\Delta$ is given by equation (3.36) and $\Delta_j$ ($j = 5, 6, 7, 8$) are obtained from $\Delta$ on replacing the elements of its $j$th column by $F_2$, $-1$, $-a_2$, $-b_2$. On simplification, we get

\[
\Delta_5 = 2F_2 \left[ (a_2 - a_4) b_3 + (a_3 - a_2) b_4 \right] - 2b_2 \left[ F_3 (a_2 - a_3) - F_4 (a_2 - a_3) \right].
\]

(3.44)

\[
\Delta_6 = F_1 \left[ (a_3 b_3 - b_3) - a_3 (b_4 - b_2) + a_4 (b_3 - b_2) \right] \\
+ F_2 \left[ a_1 (b_4 - b_3) - a_3 (b_1 + b_4) + a_4 (b_1 + b_2) \right] \\
- F_3 \left[ a_1 (b_4 - b_2) - a_2 (b_1 + b_4) + a_4 (b_1 + b_2) \right] \\
+ F_4 \left[ a_1 (b_2 - b_3) - a_2 (b_1 + b_3) + a_3 (b_1 - b_2) \right],
\]

(3.45)

\[
\Delta_7 = 2F_2 \left[ (a_4 - a_2) b_1 + (a_1 - a_2) b_4 \right] - 2b_2 \left[ F_1 (a_4 - a_2) + F_4 (a_4 - a_2) \right],
\]

(3.46)

\[
\Delta_8 = 2F_2 \left[ (a_2 - a_3) b_1 - (a_1 - a_2) b_3 \right] - 2b_2 \left[ F_1 (a_2 - a_3) - F_3 (a_2 - a_3) \right].
\]

(3.47)

Equations (3.42) and (3.43) give the reflection and transmission coefficients when $SV$ waves are incident at a plane interface between two prestressed semi-infinite elastic media in welded contact.
3.9. Particular Cases

If the media are free from initial stresses, i.e. 
\[ S_{11} = S_{33} = S_{13} = S_{31} = 0 \], so that \( P = R = P' = R' = 0 \), the results in equation (1.43) are applicable. Setting these values in equations (3.18) and (3.32), we obtain

\[ C_p = \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad C_{SV'} = \beta = \sqrt{\frac{\mu}{\rho}}, \quad \alpha' = \frac{\sqrt{\lambda + 2\mu}}{\rho'}, \quad \beta' = \sqrt{\frac{\mu}{\rho'}} \]

\[ F_1 = -\tan e, \quad F_2 = \cot f, \quad F_3 = -\tan e', \quad F_4 = \cot f', \]

\[ a_1 = \frac{4\mu \sin e}{\alpha}, \quad a_2 = -\frac{2\mu \cos 2f}{\beta \sin f}, \quad a_3 = \frac{4\mu' \sin e'}{\alpha'}, \]

\[ a_4 = -\frac{2\mu' \cos 2f'}{\beta' \sin f'} \]

\[ b_1 = -\frac{\mu \alpha \cos 2f}{\beta \cos e}, \quad b_2 = -\frac{2\mu \cos f}{\beta}, \quad b_3 = -\frac{\mu' \alpha' \cos 2f'}{\beta' \cos e'}, \]

\[ b_4 = -\frac{2\mu' \cos f'}{\beta'} \]

where \( \alpha \) and \( \beta \) are the usual P and S wave velocities.

Using equation (3.43), we may express equation (3.33) in the form

\[-\tan e \frac{W_{r_1}}{W_{t_1}} + \cot f \frac{W_{r_2}}{W_{t_2}} - \tan e' \frac{W_{t_1}}{W_{t_2}} + \cot f' \frac{W_{t_2}}{W_{t_2}} = -\tan e \frac{W_{t_1}}{W_{t_1}}, \]

\[ W_{r_1} + W_{r_2} - W_{t_1} - W_{t_2} = -W_{t_1}, \]
2 \sin e \, W_1 - \frac{\alpha \cos 2f}{\beta \sin f} \, W_2 - \frac{2\rho'_\alpha}{\rho \alpha'} \left( \frac{\beta^1}{\beta} \right) \sin e' \, W_{t_1} \\
+ \frac{\rho'_\alpha}{\rho \beta'} \left( \frac{\beta^1}{\beta} \right)^2 \frac{\cos 2f'}{\sin f'} \, W_{t_2} = -2\sin e \, W_1 ,

\cos 2f \left( 1 - \frac{2\beta}{\cos f} \right) \frac{W_r}{\cos e} = \frac{\cos 2f}{\cos e} \, W_{l_1} . \quad (3.49)

Equation (3.49) is equivalent to the corresponding results of Ben-Menahem and Singh (1981, p. 102) for incident P waves at an interface.

In the case of incident SV waves, we get a set of equations similar to equation (3.49) from equations (3.41) and (3.43)

\begin{align*}
\tan e \, W_{r_1} + \cot f \, W_{r_2} - \tan e' \, W_{t_1} + \cot f' \, W_{t_2} &= \cot f \, W_{l_2} \\
- \frac{W_{r_1}}{W_{r_1}} - \frac{W_{r_2}}{W_{r_2}} + \frac{W_{t_1}}{W_{t_1}} + \frac{W_{t_2}}{W_{t_2}} &= W_{l_2} \\
\frac{2\beta}{\alpha} \frac{\sin e \, W_{r_1}}{\sin f} - \frac{\cos 2f}{\sin f} \frac{W_{r_2}}{\rho \alpha \beta} - \frac{2\rho' \beta^1}{\rho \beta} \sin e' \, W_{t_1} + \frac{\rho' \beta^1}{\rho \beta} \frac{\cos 2f'}{\sin f'} \, W_{t_2} &= \frac{\cos 2f}{\sin f} \, W_{l_2} 
\end{align*}
\[
\frac{a \cos 2f}{\beta \cos e} W_1 + 2 \cos f \frac{\cos 2f'}{r_2} + \frac{a' \rho'}{\beta' \cos e'} W_{t1}
\]

\[
+ \frac{2 \rho' \beta'}{\rho s} \cos f' W_{t2} = 2 \cos f W_{12} \quad (3.50)
\]

Equation (3.50) is equivalent to the corresponding results of Ben-Menahem and Singh (1981, p. 103) for incident SV waves at a solid-solid interface.

3.10. Conclusions

In this chapter we have derived the analytical expressions for the reflection and transmission coefficients when plane quasi-P and quasi-SV waves are incident at a plane interface between two homogeneous prestressed elastic half-spaces. It has been shown that in the absence of the prestress the expressions derived by us coincide with the known results.
Fig. 4.1