Chapter 1

Introduction

This work is motivated by Balmer’s construction of the triangular spectrum. We have tried to understand geometric objects coming from a derived category of coherent sheaves and some triangulated categories. The study of geometric objects coming from triangulated category is interesting for many reasons. One reason is to extend the study of schemes from commutative algebras to non-commutative algebras. Another reason is to capture some geometry which is not visible via scheme structure; the formulation of homological mirror symmetry by Kontsevich is one such example. There are other examples outside algebraic geometry which make the study of such geometric association interesting.

The notion of triangulated categories was first axiomatized by Grothendieck and Verdier in order to develop Serre duality in a relative setting. It was an idea of Grothendieck to extract more homological information from the totality of complexes. Verdier and Illusie had developed triangulated categories, like the derived category of coherent sheaves and perfect complexes respectively, for better understanding of dualities.

Now triangulated categories have found many applications in other branches of mathematics outside algebraic geometry.

In algebraic geometry triangulated categories arose as the derived category of coherent sheaves and also as the category of perfect complexes on schemes; which was later realized as compact objects in the derived category of quasi-coherent sheaves over quasi compact and separated schemes by Neeman[33]. In modular representation theory triangulated categories appear in the form of the stable module category. The introduction to Balmer[3] contains more examples and motivation to study such abstract objects.

Gabriel[15] and later Rosenberg[37] proved that abelian category of quasi-coherent sheaves completely determines the underlying variety. Mukai[29] had given examples of two non-isomorphic varieties, which have equivalent
derived categories of coherent sheaves. So the derived category of a coherent sheaves cannot reconstruct the underlying variety for these cases. However Bondal and Orlov[9] proved it does determine the scheme whenever the canonical sheaf or the anti-canonical sheaf is ample. Still it is an interesting question to find an optimal class of varieties with such reconstruction results.

Since the triangulated structure alone is not enough for reconstruction Balmer[3] used tensor structure to reconstruct a smooth scheme from its bounded derived category of coherent sheaves. In fact, Balmer used the category of perfect complexes for quasi compact and quasi separated schemes to reconstruct the underlying schemes. The category of perfect complexes with tensor structure contains enough information about the scheme even in singular case. Balmer first constructs a topological space, which he called the triangular spectrum, associated with any essentially small tensor triangulated category. Balmer proved this topological space gives the universal support data on a triangulated category. Construction of the triangular spectrum depends on a classification of certain thick subcategories. Balmer used the classification of thick subcategories given by Thomason[39], which was motivated by results of Neeman[31] and Hopkins[21] for the affine case, to relate the triangular spectrum with the underlying scheme. Further, using the localisation result of Thomason[40], Balmer defined a sheaf of local triangulated categories. By considering endomorphisms of this sheaf of triangulated categories, Balmer constructs a sheaf of rings which reconstructs the scheme structure. Balmer firstly gave the reconstruction of quasi compact and quasi separated schemes using atomic subcategories, (see [2]). Later Balmer generalized the notion of prime ideal from commutative algebra to this abstract setting and demonstrated the usefulness of this concept in this generality, (see [3]). Balmer proved his reconstruction theorem under the assumption that the space is topologically Noetherian which was later removed by Krause et. al.[11]. Balmer proved later that this construction always gives a locally ringed space. He also gave an example coming from topology where this locally ringed space fails to be a scheme.

Using his definition of triangular spectrum Balmer applied many techniques from algebraic geometry to modular representation theory like gluing and the Picard group[5]. One question that naturally arises is how good is Spec as an invariant of the tensor triangulated category? It turns out that there do exist pairs of tensor triangulated categories which have isomorphic Specs (isomorphic as ringed spaces). We give two such examples. This motivates the need for some other finer geometric object attached to tensor triangulated categories. We shall compute the triangular spectrum in an equivariant setting, and for some superschemes by relating it with already known triangular spectrums. This computation is the starting point of this
work and occupies a large part of it. More precisely, the first example consists of smooth quasi projective scheme, say $X$, with action of a finite group $G$ as an automorphism. Hence we get the finite map $\pi : X \to Y := X/G$ which will give an exact functor

$$\pi^* : D^\text{per}(X/G) \to D^G(X).$$

We prove the following theorem.

**Theorem 1.0.1.** Assume that the scheme $X$ is smooth quasi projective and $G$ is a finite group acting on $X$. The induced map

$$\text{Spec}(\pi^*) : \text{Spec}(D^G(X)) \to \text{Spec}(D^b(X/G))$$

is an isomorphism of locally ringed spaces.

Here Spec denotes the construction due to Balmer[3]. The proof involves a computation using some results from representation theory. The second example is a computation of the Balmer spectrum for a split superscheme $X$. Superschemes, defined by Manin and Deligne (see for example [27]), are also an important object of study in modern algebraic geometry, specially due to applications in physics. We consider the triangulated category $D^\text{per}(X)$ of “perfect complexes” (the definition being modified appropriately in the super setting) on this superscheme.

**Theorem 1.0.2.** Let $X$ be a split superscheme. Let $X_0 = (X, O_{X,0})$ be the 0-th part of this superscheme. $X_0$ is by definition a scheme. Then we have an isomorphism of locally ringed spaces

$$f : X_0 \to \text{Spec}(D^\text{per}(X)).$$

The proof of homeomorphism adapts the classification of thick tensor ideals due to Thomason[39] as demonstrated by Balmer[3]. Again, following Balmer[3] we use the generalized localization theorem of [Theorem 2.1, Neeman[33]] to finish the proof.

We also tried to explore some ways to strengthen the geometric association of Balmer so that we can recover the tensor triangulated category. This is the problem of categorical reconstruction, of realizing the tensor triangulated category which we started with, as the tensor triangulated category canonically associated with a geometric object.
Overview of thesis

Now we shall give the content of each chapter for the convenience of the reader.

The first chapter recalls various preliminaries which are well known. We shall start with the definition of triangulated categories and exact functors between them. We shall also recall the definition of a derived category which is an important example of a triangulated category. Then we state the existence of derived functors and various relations between them. We recall the definitions of various functors in the theory of schemes. Next, by relating the sheaves of modules over superschemes with usual schemes, we give generalizations of many definitions and properties of perfect complexes for the super scheme case.

The second chapter recalls the definition of spectrum of a tensor triangulated category defined by Balmer. We recall various properties and general results from the papers of Balmer. We state the reconstruction result of Balmer as extended to non-Noetherian case by Krause[11]. We recall the definition of support data and the universal property of Balmer spectrum which is used crucially for the reconstruction. We state the functoriality result for the Balmer spectrum under non-unital functors (but the proof is not very different).

The third chapter contains new results which relate the spectrum of the bounded derived category of equivariant sheaves over a smooth quasi projective scheme with the spectrum of perfect complexes over the orbit space. First, we recall some basics on equivariant sheaves and we then prove the main theorem by dividing it into three cases - trivial action, free action and the general case. We also give the proof for curves as an interesting and important example.

The fourth chapter computes the spectrum for the tensor triangulated category of a split superscheme. In fact, this computation follows the steps laid down by Balmer for usual schemes. We get a relation between the spectrum of the underlying even scheme with the spectrum of a superscheme. We also recall the result of Neeman which is a generalization of the localization result of Thomason. The classification of radical thick tensor ideals is given by relating them with usual schemes. Here we use the generalization of the category of perfect complexes given earlier.

The fifth chapter contains some of our suggestions for the enrichment of Balmer spectrum. In the first section we give two ways of defining generalized spaces using the underlying topological space of the Balmer spectrum. In second section we give a functor of points approach to the Balmer spectrum.