CHAPTER 6

Domination in Eccentric Graphs and Digraphs

6.1 Introduction

The concept of domination in graphs is one among the several interesting concepts in Graph theory which has been well investigated by many researchers. Though it has its origin in 1850’s when chess players wanted to find the minimum number of queens such that every square on the chess board has a queen or is attacked by a queen, on the fact that queen can move any number of squares in any direction, only in the past few decades, several papers on this topic have been published. There are different varieties of domination numbers, to list a few, broadcast domination number, global domination number, independent domination number, roman domination number etc. The concept of domination is having a great practical applications. It has played a vital role in facility location problems, Monitoring Communication and Electrical networks and Land surveying etc. In this chapter, we have studied the domination number for eccentric graphs of Trees, Broom graphs, Dendrimers and Grid graphs. Also we have studied the domination number for eccentric graphs and digraphs of
diameter maximal graphs and the broadcast domination number for Grid graphs.

6.2 Grid Graphs

In this section, we obtain the eccentric graph of the class of grid graphs $P_m \times P_n$, where $m, n \geq 2$.

![Grid graph $P_3 \times P_3$](image)

Figure 6.1. Grid graph $P_3 \times P_3$

Theorem 6.2.1

Let $G = P_m \times P_n$, where $m$ and $n$ are odd. Then $G_e$ is a tri-partite graph with partitions

\[ \{(u_i, v_j)/1 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq n\}; \]
\[ \{(u_i, v_j)/\frac{m+3}{2} \leq i \leq m, 1 \leq j \leq n\} \text{ and } \{(u_{m+1}, v_j)/1 \leq j \leq n\}. \]

Proof

Let $G = P_m \times P_n$, where $m$ and $n$ are odd. Let $V(P_m) = \{u_1, u_2, u_3, \ldots, u_m\}$ where $u_1$ and $u_m$ are end vertices of $P_m$ and $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ where $v_1$ and $v_n$ are end vertices of $P_n$. Then $EP(G) = \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\}$, since $(u', v')$ is an eccentric vertex of $(u, v)$ in $G$ if and only if $u'$ is an
eccentric vertex of $u$ in $P_m$ and $v'$ is an eccentric vertex of $v$ in $P_n$. This implies that $(u_m, v_n)$ is the only eccentric vertex of the vertices $(u_i, v_j)$, $1 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq \frac{n-1}{2}$; $(u_m, v_1)$ is the only eccentric vertex of the vertices $(u_i, v_j)$, $1 \leq i \leq \frac{m-1}{2}, \frac{n+3}{2} \leq j \leq n$; $(u_1, v_1)$ is the only eccentric vertex of the vertices $(u_i, v_j)$, $\frac{m+3}{2} \leq i \leq m, \frac{n+3}{2} \leq j \leq n$ and $(u_1, v_n)$ is the only eccentric vertex of the vertices $(u_i, v_j)$, $\frac{m+3}{2} \leq i \leq m, \frac{n+3}{2} \leq j \leq n$.

So, we have

\[ E((u_i, v_{\frac{n+1}{2}})) = \{(u_m, v_1), (u_m, v_n)\} , 1 \leq i \leq \frac{m-1}{2} \]

\[ E((u_i, v_{\frac{n+1}{2}})) = \{(u_1, v_1), (u_1, v_n)\} , \frac{m+3}{2} \leq i \leq m \]

\[ E((u_{\frac{m+1}{2}}, v_j)) = \{(u_1, v_n), (u_m, v_n)\} , 1 \leq j \leq \frac{n-1}{2} \]

\[ E((u_{\frac{m+1}{2}}, v_j)) = \{(u_1, v_n), (u_m, v_n)\} , \frac{n+3}{2} \leq j \leq n \]
This implies that in $G_e$,

$$E((u_{m+1}, v_{n+1})) = \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\}$$

It follows that $G_e$ is a tri-partite graph with partitions

$$\{ (u_i, v_j) / 1 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq n \};$$

$$\{ (u_i, v_j) / \frac{m+3}{2} \leq i \leq m, 1 \leq j \leq n \}$$

and

$$\{ (u_{m+1}, v_j) / 1 \leq j \leq n \}.$$
Figure 6.4. Eccentric graph of grid graph $(P_3 \times P_4)_e$

**Remark 6.2.1**

We note from Figures 6.1 and 6.2 that

$EP(P_3 \times P_5) = \{(u_1, v_1), (u_1, v_5), (u_3, v_1), (u_3, v_5)\}$ and $(P_3 \times P_5)_e$ is a tri-partite graph with partitions $\{(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_5)\}$,

$\{(u_2, v_1), (u_2, v_2), \ldots, (u_2, v_5)\}$ and $\{(u_3, v_1), (u_3, v_2), \ldots, (u_3, v_5)\}$.

**Theorem 6.2.2**

Let $G = P_m \times P_n$, where $m$ is odd and $n$ is even. Then $G_e$ is a tri-partite graph with partitions $\{(u_i, v_j)/1 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq n\}$; $\{(u_i, v_j)/\frac{m+3}{4} \leq i \leq m, 1 \leq j \leq n\}$ and $\{(u_{\frac{m+1}{2}}, v_j)/1 \leq j \leq n\}$. 

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Figure 6.5. Grid graph $P_4 \times P_3$

Proof

Let $G = P_m \times P_n$, where $m$ is odd and $n$ is even. Let $V(P_m) = \{u_1, u_2, u_3, \ldots, u_m\}$ where $u_1$ and $u_m$ are end vertices of $P_m$ and $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ where $v_1$ and $v_n$ are end vertices of $P_n$. Then $EP(G) = \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\}$, since $(u', v')$ is an eccentric vertex of $(u, v)$ in $G$ if and only if $u'$ is an eccentric vertex of $u$ in $P_m$ and $v'$ is an eccentric vertex of $v$ in $P_n$. This implies that $(u_m, v_n)$ is the only eccentric vertex of the vertices $(u_i, v_j)$, $1 \leq i \leq m - 1$, $1 \leq j \leq \frac{n}{2}$; $(u_m, v_1)$ is the only eccentric vertex of the vertices $(u_i, v_j)$, $1 \leq i \leq m - 1$, $\frac{n}{2} + 1 \leq j \leq n$; $(u_1, v_1)$ is the only eccentric vertex of the vertices $(u_i, v_j)$, $\frac{m+3}{2} \leq i \leq m$, $\frac{n}{2} + 1 \leq j \leq n$; $(u_1, v_n)$ is the only eccentric vertex of the vertices $(u_i, v_j)$, $\frac{m+3}{2} \leq i \leq m$, $\frac{n}{2} + 1 \leq j \leq n$.

So, we have

$E((u_1, v_{n/2})) = \{(u_m, v_1), (u_m, v_n)\}, 1 \leq i \leq \frac{m-1}{2}$

$E((u_i, v_{n/2})) = \{(u_1, v_1), (u_1, v_n)\}, \frac{m+3}{2} \leq i \leq m$
Figure 6.6. Eccentric graph of grid graph \((P_4 \times P_3)_e\)

\[
E((u_{\frac{m+1}{2}}, v_j)) = \{(u_1, v_n), (u_m, v_n)\}, 1 \leq j \leq \frac{n}{2}
\]

\[
E((u_{\frac{m+1}{2}}, v_j)) = \{(u_1, v_n), (u_m, v_n)\}, \frac{n}{2} + 1 \leq j \leq n
\]

\[
E((u_{\frac{m+1}{2}}, v_{\frac{n+1}{2}})) = \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\}
\]

This implies that in \(G_e\),

\[
N((u_i, v_j)) = \{(u_1, v_1)\}, \frac{m+3}{2} \leq i \leq m, \frac{n+3}{2} \leq j \leq n
\]

\[
N((u_i, v_j)) = \{(u_1, v_n)\}, \frac{m+3}{2} \leq i \leq m, \frac{n+3}{2} \leq j \leq n
\]

\[
N((u_i, v_j)) = \{(u_m, v_1)\}, 1 \leq i \leq \frac{m-1}{2}, \frac{n+3}{2} \leq j \leq n
\]

\[
N((u_i, v_j)) = \{(u_m, v_n)\}, 1 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq \frac{n-1}{2}
\]

\[
N((u_i, v_{\frac{n+1}{2}})) = \{(u_m, v_1), (u_m, v_n)\}, 1 \leq i \leq \frac{m-1}{2}
\]

\[
N((u_i, v_{\frac{n+1}{2}})) = \{(u_1, v_1), (u_1, v_n)\}, \frac{m+3}{2} \leq i \leq m
\]

\[
N((u_{\frac{m+1}{2}}, v_j)) = \{(u_1, v_n), (u_m, v_n)\}, 1 \leq j \leq \frac{n-1}{2}
\]

\[
N((u_{\frac{m+1}{2}}, v_j)) = \{(u_1, v_n), (u_m, v_n)\}, \frac{n+3}{2} \leq j \leq n
\]

\[
N((u_{\frac{m+1}{2}}, v_{\frac{n+1}{2}})) = \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\}
\]

It follows that \(G_e\) is a tri-partite graph with partitions

\[
\{(u_i, v_j) : 1 \leq i \leq \frac{m-1}{2}, 1 \leq j \leq n\};
\]

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\( \{(u_i, v_j)/\frac{m+3}{2} \leq i \leq m, 1 \leq j \leq n\} \) and \( \{(u_{\frac{m+1}{2}}, v_j)/1 \leq j \leq n\} \).

Remark 6.2.2

We note from Figures 6.3 and 6.4 that

\[ EP(P_3 \times P_4) = \{(u_1, v_1), (u_1, v_4), (u_3, v_1), (u_3, v_4)\} \text{ and } (P_3 \times P_4)_e \text{ is a tri-partite}\]

graph with partitions \( \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_1, v_4)\}, \)

\( \{(u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4)\} \text{ and } \{(u_3, v_1), (u_3, v_2), (u_3, v_3), (u_3, v_4)\} \).

Theorem 6.2.3

Let \( G = P_m \times P_n \), where \( m \) is even and \( n \) is odd. Then \( G_e \) is a bipartite graph with

partitions \( \{(u_i, v_j), 1 \leq i \leq \frac{m}{2}, 1 \leq j \leq n\} \) and \( \{(u_i, v_j), \frac{m}{2} < i \leq m, 1 \leq j \leq n\} \).

Proof

Let \( G = P_m \times P_n \), where \( m \) is even and \( n \) is odd. Let \( V(P_m) = \{u_1, u_2, u_3, \ldots, u_m\} \)

where \( u_1 \) and \( u_m \) are end vertices of \( P_m \) and \( V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\} \) where \( v_1 \)
Figure 6.8. Eccentric graph of grid graph \((P_4 \times P_6)_e\)

and \(v_n\) are end vertices of \(P_n\). Then \(EP(G) = \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\}\), since \((u', v')\) is an eccentric vertex of \((u, v)\) in \(G\) if and only if \(u'\) is an eccentric vertex of \(u\) in \(P_m\) and \(v'\) is an eccentric vertex of \(v\) in \(P_n\). This implies that

\[
E((u_i, v_j)) = \{(u_1, v_1)\}, \frac{m}{2} < i \leq m, \frac{n+3}{2} \leq j \leq n
\]

\[
E((u_i, v_j)) = \{(u_1, v_n)\}, \frac{m}{2} < i \leq m, 1 \leq j \leq \frac{n-1}{2}
\]

\[
E((u_i, v_j)) = \{(u_m, v_1)\}, 1 \leq i \leq \frac{m}{2}, 1 \leq j \leq \frac{n-1}{2}
\]

\[
E((u_i, v_j)) = \{(u_m, v_n)\}, 1 \leq i \leq \frac{m}{2}, \frac{n+3}{2} \leq j \leq n
\]

\[
E((u_i, v_{n+1})) = \{(u_m, v_1), (u_m, v_n)\} 1 \leq i \leq \frac{m}{2}
\]
\[
E((u_i, v_{n+1})) = \{(u_1, v_1), (u_1, v_n)\} \quad \frac{m}{2} < i \leq m
\]

This implies that in \(G_e\)

\[
N((u_i, v_j)) = \{(u_1, v_1)\}, \quad \frac{m}{2} < i \leq m, \quad \frac{n+3}{2} \leq j \leq n
\]

\[
N((u_i, v_j)) = \{(u_1, v_n)\}, \quad \frac{m}{2} < i \leq m, \quad \frac{n+1}{2} \leq j \leq n
\]

\[
N((u_i, v_j)) = \{(u_m, v_1)\}, \quad 1 \leq i \leq \frac{m}{2}, \quad \frac{n+1}{2} \leq j \leq n
\]

\[
N((u_i, v_j)) = \{(u_m, v_n)\}, \quad 1 \leq i \leq \frac{m}{2}, \quad \frac{n+3}{2} \leq j \leq n
\]

\[
N((u_i, v_{n+1})) = \{(u_m, v_1), (u_m, v_n)\} \quad 1 \leq i \leq \frac{m}{2}
\]

\[
N((u_i, v_{n+1})) = \{(u_1, v_1), (u_1, v_n)\} \quad \frac{m}{2} < i \leq m
\]

It follows that \(G_e\) is a bi-partite graph with partitions

\[
\{(u_i, v_j), 1 \leq i \leq \frac{m}{2}, 1 \leq j \leq n\} \quad \text{and} \quad \{(u_i, v_j), \frac{m}{2} < i \leq m, 1 \leq j \leq n\}.
\]

\[\square\]

**Remark 6.2.3**

We note from Figures 6.5 and 6.6 that

\[EP(P_4 \times P_3) = \{(u_1, v_1), (u_1, v_3), (u_4, v_1), (u_4, v_3)\}\] and \((P_4 \times P_3)_e\) is a bipartite graph with partitions \(\{(u_1, v_1), (u_1, v_2), \ldots, (u_2, v_3)\}\) and

\[\{(u_3, v_1), (u_3, v_2), \ldots, (u_4, v_3)\}\]

**Theorem 6.2.4**

Let \(G = P_m \times P_n\), where \(m\) and \(n\) are even. Then \(G_e\) is a union of two double stars.

**Proof**

Let \(G = P_m \times P_n\), where \(m\) and \(n\) are even. Since \(P_m\) and \(P_n\) are u.e.p graphs, so \(G = P_m \times P_n\) is a u.e.p graph. Further, \(G\) is non-self centered graph and

\[EP(G) = \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\}\]. Now, by Theorem 2.2.2, \(G_e\) is a
union of two double stars.

Remark 6.2.4

We note from Figures 6.7 and 6.8 that

\[ EP(P_4 \times P_6) = \{(u_1, v_1), (u_1, v_6), (u_4, v_1), (u_4, v_6)\} \]

and \((P_4 \times P_6)_e\) is a double star.

6.3 Domination Number in Eccentric Graphs and Digraphs

A set \(S \subseteq V(G)\) of vertices in a graph \(G = (V, E)\) is said to be a dominating set if every vertex \(v \in V\) is an element of \(S\) or adjacent to an element of \(S\). In other words, A set \(S \subseteq V(G)\) is said to be a dominating set if \(N[S] = V(G)\). A dominating set is said to be a minimal dominating set if there is no proper subset \(S' \subset S\) is a dominating set. The domination number \(\gamma(G)\) is the cardinality of a minimal dominating set.

Theorem 6.3.1

\[ |\gamma(G_e)| \leq |EP(G)|. \]

Proof

Let \(G\) be any graph with \(EP(G) = \{v_1, v_2, v_3, \ldots , v_m\}\). Let us assume that \(G_e\) be connected. Then by the definition of eccentric graph, every vertex of \(G_e\) is adjacent to at least one vertex of \(EP(G)\). This implies that the set \(\{v_1, v_2, v_3, \ldots , v_m\}\) is a dominating set of \(G_e\).

Therefore \( |\gamma(G_e)| \leq m = |EP(G)|. \)
Theorem 6.3.2

Let $G = T_{k,d}$ ($d \geq 2$) be a regular dendrimer. Then the domination number of $G_e$ is $\gamma(G_e) = 2$.

Proof

Let $T_{k,d} = (V, E)$ be a regular dendrimer with unique central vertex $v_0$. Let $V = V_1 \cup V_2 \cup V_3$ where $V_1 = \{v_0\}$, $V_2$ is the set of all pendant vertices and $V_3$ is the set of all vertices that are neither pendant nor central vertices. Then by Theorem 5.5.2, the eccentric graph of $T_{k,d}$ is a $d+1$ partite graph, where one partition has only one vertex $\{v_0\}$ and the remaining $d$ partitions, $P_1, P_2, P_3, \ldots, P_d$, each has $\frac{(d-1)^k-1}{d-2}$ vertices. Further each $P_i$ has $(d-1)^k-1$ vertices of $V_2$. This implies that any vertex in $P_i \cap V_2$ dominates $v_0$ and all other vertices $v \in P_j, 1 \leq j \leq d, j \neq i$ and the set $S = \{u, v/u, v \in V_2$ and they are not in the same $d$ partitions $P_i\}$. This implies that $S$ is the minimal dominating set in $(T_{k,d})_e$. Therefore $\gamma((T_{k,d})_e) = 2$. $\blacksquare$

Theorem 6.3.3

$$\gamma(B_{n,d}) = \begin{cases} 
\frac{d+1}{2}, & \text{if } d \text{ is odd} \\
\frac{d}{2}, & \text{if } d \text{ is even}
\end{cases}$$

Proof

Let $G = B_{n,d} = (V, E)$ be a broom graph with $V(G)$ is as in proof of Theorem 5.5.1. It is clear that when $d$ is respectively odd and even, the sets $\{u_1, u_3, u_5, \ldots, u_d\}$ and $\{u_2, u_4, \ldots, u_d\}$ are minimal dominating sets of $G$. Thus
we have the domination number as given in the statement of the theorem. □

**Theorem 6.3.4**

If $G = B_{n,d}$ is a broom graph, then $\gamma(G_e) = 2$;

**Proof**

Let $G = B_{n,d}$ be a broom graph with $V(G)$ is as in proof of Theorem 5.5.1. Now the eccentric graph $G_e$ has the structure as given in Theorem 5.5.1. So whether $d$ is odd or even, the set consisting of $u_1$ and any one of $v'_j$s is a minimal dominating set. Therefore $\gamma(G_e) = 2$. □

**Remark 6.3.1**

We note from Theorems 6.3.3 and 6.3.4 that the domination number of a given broom graph $B_{n,d}$ depends on $d$ whereas this number is a constant namely two for the corresponding eccentric graph.

**Theorem 6.3.5**

If $T$ is a tree on $n$ vertices with $|P(T)| = l$, then the domination number $\gamma(ED(T)) = n - l$ and $\gamma(ED^2(T)) = l$.

**Proof**

Let $T$ be a tree on $n$ vertices with $|P(T)| = l$. Then by Lemma 3.2.1, $|EP(T)| = l$. Also it is clear that in $ED(T)$, $n - l$ vertices are of in-degree zero. Hence $\gamma(ED(T)) = n - l$. Furthermore, by Theorem 3.2.2, $ED^2(T) = K_l \oplus K^*_{n-l}$. This implies that $\gamma(ED^2(T)) = l$. □

We now obtain the domination number of a diameter maximal graph and its
eccentric graphs and digraphs.

**Theorem 6.3.6**

The domination number of a diameter maximal graph with odd diameter $d$ is $\lceil (d + 1)/3 \rceil$.

**Proof**

Let $G$ be a diameter maximal graph with odd diameter $d$. Then by Lemma 2.1.2, $G$ has the form $G = G_o + G_1 + G_2 + \cdots + G_{d-1} + G_d$ where each $G_i = K_{a_i}$, $i = 1, 2, 3, 4, \cdots, d - 1$ and $G_o = K_1 = G_d$. Now, let us consider some point $u$ in $G_1$. It must be adjacent to all the points of $G_o, G_1$ and $G_2$. So any point of $G_1$ dominates every point of $G_o, G_1$ and $G_2$. Similarly, any point of $G_4$ dominates every point of $G_3, G_4$ and $G_5$. It follows that for every three consecutive cliques, any point of the middle clique is a dominating point. Therefore $\gamma(G) = \lceil (d + 1)/3 \rceil$ since there are $d + 1$ cliques. \(\square\)

**Theorem 6.3.7**

If $G$ is a diameter maximal graph with odd diameter then

(i) $\gamma(ED(G)) = m + n$ and

(ii) $\gamma(ED^2(G)) = 2$.

**Proof**

Let $G$ be a diameter maximal graph with odd diameter. In order to prove statement (i), we note that by Theorem 3.3.1, the eccentric digraph $ED(G)$ is a $K^*_2$ with one end having $m$ independent in-neighbors and other end having $n$
independent in-neighbors. This implies that \( \gamma(ED(G)) = m + n \).

Now to prove statement (ii), we have, as in the proof of Theorem 3.3.2,
\[
ED^2(G) = K_2^* \oplus K_{m+n}^*.
\]
Hence \( \gamma(ED^2(G)) = 2 \).
\[\square\]

**Theorem 6.3.8**

Let \( G \) be a diameter maximal graph with odd diameter. Then the domination number of the eccentric graph \( G_e \) is \( \gamma(G_e) = 2 \).

**Proof**

Let \( G \) be a diameter maximal graph with odd diameter. Then by Theorem 2.3.1, \( G_e \) is a double star with two central points \( u \) and \( v \). Now the set \( S = \{u, v\} \) is a minimal dominating set of \( G_e \). Thus \( \gamma(G_e) = 2 \).
\[\square\]

**Corollary 6.3.1**

The domination number of the eccentric graph of \( P_{2n}, n > 1 \) is two.

**Proof**

The path \( P_{2n}, n > 1 \), is clearly a diameter maximal graph with odd diameter. Hence by Theorem 6.3.8, \( \gamma((P_{2n})_e) = 2 \).
\[\square\]

**Remark 6.3.2**

We note from Theorems 6.3.6, 6.3.7 and 6.3.8 that the domination number of given diameter maximal graph depends on its odd diameter whereas this number is a constant namely two for the corresponding eccentric graph and second iterated eccentric digraph.
Theorem 6.3.9

Let \( G = P_m \times P_n \) where at least one of \( m \) and \( n \) is odd. Then the domination number, \( \gamma(G_e) = 4 \).

Proof

Let \( G = P_m \times P_n \) where at least one of \( m \) and \( n \) is odd.

Let \( V(P_m) = \{u_1, u_2, u_3, \ldots, u_m\} \), where \( u_1 \) and \( u_m \) are end vertices and

\( V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\} \) where \( v_1 \) and \( v_n \) are end vertices.

Then \( EP(G) = \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\} \). Now, by Theorem 6.2.1, 6.2.2 and 6.2.3, every vertex in \( G_e \) has at least one neighbor in

\( \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\} \).

Therefore, the set \( \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\} \) is a minimal dominating set in \( G_e \). Hence \( \gamma(G_e) = 4 \).

\[ \Box \]

6.4 Broadcast Domination Number of Eccentric Graph of Grid Graphs

We now recall the following definitions [28].

Let \( G \) be a graph. Then a function \( f : V \rightarrow \{0, 1, 2, \ldots, diam(G)\} \) is a broadcast if for every vertex \( v \in V(G) \), \( f(v) \leq e(v) \). The broadcast neighborhood of a vertex \( u \) is defined as \( N_f[u] = \{v \mid d(u, v) \leq f(u)\} \). The cost of a broadcast is defined as \( f(V) = \sum_{v \in V} f(v) = \sum_{v \in V^+} f(v) \), where \( V^+ = \{u \in V \mid f(u) > 0\} \).

A broadcast \( f \) is dominating if \( N_f[V^+] = V(G) \). The minimum cost \( f(V) \) of a dominating broadcast \( f \) of a graph \( G \) is the broadcast domination number \( \gamma_b(G) \).
In this section we obtain the broadcast domination number for the eccentric graph of grid graphs.

**Lemma 6.4.1** [28]

The broadcast domination number of grid graphs $P_m \times P_n$ is $\gamma(P_m \times P_n) = r$, $2 < m < n$.

**Theorem 6.4.1**

Let $G = P_m \times P_n$ be a grid graph. Then the broadcast domination number of $G_e$ is as follows

(i) $\gamma_b(G_e) = 2$, when both $m$ and $n$ are odd.

(ii) $\gamma_b(G_e) = 3$, when at least one of $m$ and $n$ is odd.

**Proof**

Let $G = P_m \times P_n$ be a grid graph. Now, let us prove (i), if both $m$ and $n$ are odd, then by Theorem 6.2.1 $G_e$ is a tripartite graph. Moreover, in $G_e$ the vertex $(u_{m+1}/2, v_{n+1}/2)$ has eccentricity two. Now, let $f$ be a function from $V(G_e)$ to $\{0, 1, 2, 3, \ldots, m \times n\}$ such that

\[
 f(u_i, v_j) = \begin{cases} 
 2, & \text{when } i = \frac{m+1}{2} \text{ and } j = \frac{n+1}{2} \\
 0, & \text{otherwise}
\end{cases}
\]

Therefore, $\gamma_b(G_e) = 2$.

Now, let us prove (ii). If $m$ is odd and $n$ is even, then by Theorem 6.2.2 $G_e$ is a tripartite graph. Moreover, in $G_e$, the vertex $(u_{m+1}/2, v_1)$ has eccentricity three.
Now, let \( f \) be a function from \( V(G_e) \) to \( \{0, 1, 2, 3, \ldots, m \times n\} \) such that
\[
f(u_i, v_j) = \begin{cases} 
3, & \text{when } i = \frac{m+1}{2} \text{ and } j = \frac{n+1}{2} \\
0, & \text{otherwise}
\end{cases}
\]

Therefore, \( \gamma_b(G_e) = 3. \)

If \( m \) is even and \( n \) is odd, then by Theorem 6.2.3 \( G_e \) is a bipartite graph. Moreover, in \( G_e \), the vertex \( (u_{\frac{m+1}{2}}, v_{\frac{n+1}{2}}) \) has eccentricity three. Now, let \( f \) be a function from \( V(G_e) \) to \( \{0, 1, 2, 3, \ldots, m \times n\} \) such that
\[
f(u_i, v_j) = \begin{cases} 
3, & \text{when } i = \frac{m+1}{2} \text{ and } j = \frac{n+1}{2} \\
0, & \text{otherwise}
\end{cases}
\]

Therefore, \( \gamma_b(G_e) = 3. \) \( \square \)

**Remark 6.4.1**

We note from Lemma 6.4.1 and Theorem 6.4.1 that the broadcast domination number of grid graphs \( P_m \times P_n \) where \( 2 < m < n \), depends on its radius \( r \) whereas for the eccentric graph of \( P_m \times P_n \), this number is a constant namely 2 when both \( m \) and \( n \) are odd and 3 when only one of \( m \) and \( n \) is odd.