References


Nilpotent graphs of genus one

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Let $R$ be a commutative ring with identity. The nilpotent graph of $R$, denoted by $\Gamma_N(R)$, is a graph with vertex set $Z_N(R)^*$, and two vertices $x$ and $y$ are adjacent if and only if $xy$ is nilpotent, where $Z_N(R) = \{x \in R : xy \text{ is nilpotent, for some } y \in R^*\}$. In this paper, we determine all isomorphism classes of finite commutative rings with identity whose $\Gamma_N(R)$ has genus one.

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1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past 20 years, leading to many fascinating results and questions. In the literature, there are many papers assigning graphs to groups, semigroups and rings. In 1988, Beck began to investigate the possibility of coloring a commutative ring $R$ by associating to the ring a zero-divisor graph, defined as a simple graph, the vertices of which are the elements of the ring $R$, with two distinct elements $x$ and $y$ being adjacent if and only if $xy = 0$ [4].

Retaining the original definition, the next decade brought little progress. However, in 1999, Anderson and Livingston [2] modified and studied the zero-divisor graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors $Z(R)^*$ of the commutative ring $R$. In [7], Chen defined a kind of graph structure of rings. He let all the elements of ring $R$ be the vertices of the graph and two vertices $x$ and $y$ are adjacent if and only if $xy$ is nilpotent. However, in 2010, Li and Li [8] modified and studied the nilpotent graph $\Gamma_N(R)$ of $R$ is a graph with vertex set $Z_N(R)^*$, and two vertices $x$ and $y$ are adjacent if and only if $xy$ is nilpotent, where
$Z_N(R) = \{x \in R : xy \text{ is nilpotent, for some } y \in R^*\}$. Note that the usual zero-divisor graph $\Gamma(R)$ is a subgraph of the graph $\Gamma_N(R)$. We denote the ring of integers modulo $n$ by $\mathbb{Z}_n$, the field with $q$ elements by $\mathbb{F}_q$. An element $x$ is called nilpotent if $x^m = 0$ for some positive integer $m$ and the set of all nilpotent elements in $R$ is denoted by $N(R)$. A ring $R$ is called local if it has a unique maximal ideal. Note that $R^\times$ be the set of all units in $R$ and $R^* = R - \{0\}$. For basic definitions on rings, one may refer [3].

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set $V$ and edge set $E$. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use $K_n$ to denote the complete graph with $n$ vertices. An $r$-partite graph is one whose vertex set can be partitioned into $r$ subsets so that no edge has both ends in any one subset. A complete $r$-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes $m$ and $n$ is denoted by $K_{m,n}$.

A split graph is a simple graph in which the vertices can be partitioned into a clique and an independent set. A graph $G$ is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski’s Theorem says that a graph $G$ is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$.

Let $S_k$ denote the sphere with $k$ handles, where $k$ is a nonnegative integer, that is, $S_k$ is an oriented surface of genus $k$. The genus of a graph $G$, denoted $\gamma(G)$, is the minimal integer $n$ such that the graph can be embedded in $S_n$. For details on the notion of embedding a graph in a surface, see [12]. Intuitively, $G$ is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A genus zero graph is called a planar graph and a genus one graph is called a toroidal graph. We note here that if $H$ is a subgraph of a graph $G$, then $\gamma(H) \leq \gamma(G)$. Two graphs $G$ and $H$ have disjoint vertex sets $V_1$, $V_2$ and edge sets $E_1$, $E_2$, respectively. Their join is denoted by $G + H$ and it consists of $G \cup H$ and all edges joining every vertex of $V_1$ with every vertex of $V_2$. For basic definitions on graphs, one may refer [6]. The following results are useful for further reference in this paper.

**Theorem 1.1 (Kuratowski).** A graph $G$ is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$.

**Theorem 1.2.** Let $G$ be a connected graph. Then $G$ is a split graph if and only if $G$ contains no induced subgraph isomorphic to $2K_2$, $C_4$, $C_5$.

**Theorem 1.3 ([5]).** If $R$ is a finite local ring, then $|R| = p^n$ for some prime $p$ and some positive integer $n \geq 2$.

**Theorem 1.4 ([3]).** If $(R, m)$ is an Artinian local ring, then $m$ is nil-ideal.
Example 1.5 ([1, 9]). Let \((R, m)\) be a finite local ring and \(\Gamma(R)\) be the zero-divisor graph of \(R\). Then

| \(|Z(R^*)|\) | \(R\) | \(|R|\) | \(\Gamma(R)\) |
|---|---|---|---|
| 1 | \(\mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}\) | 4 | \(K_1\) |
| 2 | \(\mathbb{Z}_9, \frac{\mathbb{Z}_3[x]}{(x^2)}\) | 9 | \(K_2\) |
| 3 | \(\mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{(x^4)}, \frac{\mathbb{Z}_4[x]}{(2x, x^2 - 2)}\) | 8 | \(K_{1,2}\) |
| 4 | \(\mathbb{Z}_25, \frac{\mathbb{Z}_5[x]}{(x^2)}\) | 25 | \(K_4\) |

Theorem 1.6 ([10, Theorem 3.7]). Let \(R = F_1 \times \cdots \times F_n\) be a finite ring, where each \(F_i\) is a field and \(n \geq 2\). Then \(\Gamma(R)\) is planar if and only if \(R\) is isomorphic to one of the following rings: \(\mathbb{Z}_2 \times F, \mathbb{Z}_3 \times F, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\) or \(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3\) where \(F\) is a finite field.

Theorem 1.7 ([11, Theorem 3.6.2]). Let \(R \cong F_1 \times \cdots \times F_n\) be a finite ring, where each \(F_j\) is a field and \(n \geq 2\). Then \(\gamma(\Gamma(R)) = 1\) if and only if \(R\) is isomorphic to one of the following rings: \(\mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5, \mathbb{F}_4 \times \mathbb{Z}_7, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7\) or \(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\).

Lemma 1.8. \(\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil\) if \(n \geq 3\). In particular, \(\gamma(K_n) = 1\) if \(n = 5, 6, 7\).

Lemma 1.9. \(\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil\) if \(m, n \geq 2\). In particular, \(\gamma(K_{4,4}) = \gamma(K_{3,n}) = 1\) if \(n = 3, 4, 5, 6\). Also \(\gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{m,4}) = 2\) if \(m = 7, 8, 9, 10\).

2. When is the \(\Gamma_N(R)\) Split Graph?

Lemma 2.1. Let \((R, m)\) be a finite local ring with \(|R| = p^n\) for some prime \(p\) and some positive integer \(n \geq 2\). Then \(\Gamma_N(R)\) is a split graph.

Proof. By Theorem 1.4, every element of \(m\) is nilpotent and so \(Z_N(R) = R\). By the definition of \(\Gamma_N(R)\), the subgraph induced by \(m^*\) is complete. Since \(R\) is finite, \(R \setminus m = R^\times\) and so the subgraph induced by \(R \setminus m\) is totally disconnected. Also every element of \(R \setminus m\) is only adjacent to every element of \(m^*\) and so \(\Gamma_N(R)\) is connected. Thus, \(Z_N(R)^*\) can be partitioned into two subsets \(m^*\) and \(R \setminus m\) such...
that \( \langle m^* \rangle \) is complete and \( R \setminus m \) is an independent set of \( \Gamma_N(R) \). Hence \( \Gamma_N(R) \) is a split graph.

The converse of the Lemma 2.1 is not true. For example, \( \Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \) is a split graph, but \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) is not a local ring.

**Corollary 2.2.** Let \( (R, m) \) be a finite local ring with \( |R| = p^n \) for some prime \( p \) and some positive integer \( n \geq 2 \). Then \( \Gamma_N(R) \cong K_{|m^*|} + \mathbb{K}_{|R^*|} \), where \( R^* = R \setminus m \) is the set of all units in \( R \).

**Theorem 2.3.** Let \( R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m \) be a finite commutative ring with identity but not a field, where each \( (R_i, m_i) \) is a local ring and \( F_j \) is a field. Then \( \Gamma_N(R) \) is a split graph if and only if \( R \) is isomorphic to one of the following rings: \( R \) is local or \( \mathbb{Z}_2 \times F \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \).

**Proof.** Suppose \( \Gamma_N(R) \) is a split graph. Suppose \( n + m \geq 4 \). Then \((1, 0, \ldots, 0) - (0, 0, \ldots, 0) - (0, v, 0, \ldots, 0) - (1, 0, \ldots, 0)\) is a subgraph of \( \Gamma_N(R) \) where \( u \) and \( v \) are units. By Theorem 1.2, \( \Gamma_N(R) \) is not a split graph, a contradiction. Hence \( n + m \leq 3 \).

**Case 1.** Consider \( R \cong F_1 \times F_2 \times F_3 \). Note that \( \Gamma_N(R) \cong \Gamma(R) \). Suppose \( |F_i| \geq 3 \) for some \( i = 1, 2, 3 \). Without loss of generality, we assume that \( |F_1| \geq 3 \). Then \((1, 0, 0) - (0, 1, 0) - (a, 0, 0) - (0, 1, 1) - (1, 0, 0)\) is a subgraph of \( \Gamma_N(R) \), where \( a \) is a unit in \( F_1 \). By Theorem 1.2, \( \Gamma_N(R) \) is not a split graph, a contradiction. Hence \( |F_1| = |F_2| = |F_3| = 2 \) and so \( R \cong \mathbb{Z}_2 \times F \), where \( F \) is a field.

**Case 2.** Consider \( R \cong R_1 \times \cdots \times R_n \). If \( n \geq 2 \), then \( |R_i| \geq 4 \) for all \( i \) and so \((1, 0, \ldots, 0) - (0, 1, 0, \ldots, 0) - (b, 0, \ldots, 0) - (0, c, 0, \ldots, 0) - (0, 1, \ldots, 0)\) is a subgraph of \( \Gamma_N(R) \), where \( b \in R_1^* \) and \( c \in R_2^* \). By Theorem 1.2, \( \Gamma_N(R) \) is not a split graph, a contradiction. Hence \( n = 1 \) and so \( R \) is local.

**Case 3.** Consider \( R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m \) and \( n, m \geq 1 \). If \( n + m = 3 \), then \( |R_i^*| \geq 2 \) and so \((0, 1, 0) - (0, d, 0, 0) - (0, 1, 0) - (1, 0, 0)\) is a subgraph of \( \Gamma_N(R) \), where \( d \in R_3^* \). By Theorem 1.2, \( \Gamma_N(R) \) is not a split graph, a contradiction. If \( n + m = 2 \), then \((0, 1) - (1, 0) - (z, 1) - (u_1, 0) - (0, 1)\) is a subgraph of \( \Gamma_N(R) \), where \( u_1 \in R_1^* \) and \( z \in m_1^* \). By Theorem 1.2, \( \Gamma_N(R) \) is not a split graph, a contradiction. Hence in all cases, we get \( R \) is either a local ring or \( \mathbb{Z}_2 \times F \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \). Converse is obvious.

3. Planarity of \( \Gamma_N(R) \)

**Theorem 3.1.** Let \( (R, m) \) be a finite local ring with \( |R| = p^n \) for some prime \( p \) and some positive integer \( n \geq 2 \). Then \( \Gamma_N(R) \) is planar if and only if \( R \) is isomorphic to one of the following rings: \( \mathbb{Z}_4 \), \( \mathbb{Z}_2 + \langle \frac{1}{\langle 2 \rangle} \rangle \), \( \mathbb{Z}_9 \) or \( \mathbb{Z}_{27} + \langle \frac{1}{\langle 3 \rangle} \rangle \).
Proof. Let $R$ be one of the rings $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{(x^2)}$. It is easy to see that $\Gamma_N(R)$ is planar (see Fig. 1).

Suppose $\Gamma_N(R)$ is planar. Note that $|m| = p^k$ for some $k < n$ and $|R^x| = |R \setminus m| = p^k(p^{n-k} - 1)$. By Corollary 2.2, $\Gamma_N(R) \cong K_{p^k-1} + \overline{K}_{p^k(p^{n-k}-1)}$. If $|m^*| \geq 3$, then $|R^x| \geq 4$ and so $K_{3,4}$ is a subgraph of $\Gamma_N(R)$, a contradiction. Thus, $|m^*| \leq 2$. If $|m^*| = 1$, then by Example 1.5, $R \cong \mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{(x^2)}$. If $|m^*| = 2$, then by Example 1.5, $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{(x^2)}$.

Theorem 3.2. Let $R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where each $(R_i, m_i)$ is a local ring and $F_j$ is a field, $m, n \geq 1$ and $m + n \geq 2$. Then $\Gamma_N(R)$ is planar if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_2$.

Proof. Let $R$ be one of the rings $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_2$. It is easy to see that $\Gamma_N(R)$ is planar (see Fig. 2).

Assume that $\Gamma_N(R)$ is planar. Suppose $n + m \geq 3$. Note that $|R_i| \geq 4$ and $|m_i^*| \geq 1$ for all $i$, $1 \leq i \leq n$. Let $a \in m_i^*$. Then $a$ is nilpotent. Consider $x_1 = (a, 0, \ldots, 0), x_2 = (0, 1, 0, \ldots, 0), x_3 = (a, 1, 0, \ldots, 0), y_1 = (0, 0, 0, \ldots, 0), y_2 = (a, 0, \ldots, 0, 1), y_3 = (1, 0, \ldots, 0) \in \mathbb{Z}_N(R)^*$. Let $\Omega = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Then the subgraph induced by $\Omega$ contains $K_{3,3}$ and so $K_{3,3}$ is a subgraph of $\Gamma_N(R)$, a contradiction. Hence $n + m = 2$ and so $R \cong R_1 \times F_1$.

Suppose $|m_1^*| \geq 2$. Note that $|F_1| \geq 2$. Let $u_1 = (a, 0), u_2 = (b, 0), u_3 = (u, 0), v_1 = (0, 1), v_2 = (a, 1)$ and $v_3 = (b, 1) \in \mathbb{Z}_N(R)^*$, where $a, b \in m_1^*, a \neq b$ and $u$ is unit in $R_1$. Consider $\Omega' = \{u_1, u_2, u_3, v_1, v_2, v_3\}$. Then the subgraph induced

Fig. 1. $\Gamma_N(\mathbb{Z}_4) \cong \Gamma_N(\frac{\mathbb{Z}_2[x]}{(x^2)})$

Fig. 2. $\Gamma_N(\mathbb{Z}_4 \times \mathbb{Z}_2) \cong \Gamma_N(\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_2)$. 

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by $\Omega'$ contains $K_{3,3}$ and so $K_{3,3}$ is a subgraph of $\Gamma_N(R)$, a contradiction. Hence $|m_1| = 1$ and so by Example 1.5, $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Suppose $|F_1| \geq 3$. Let $z \in m_1^3$ and $u \in R_1'$. Consider $a_1 = (z, 0), a_2 = (u, 0), a_3 = (1, 0), b_1 = (z, 1), b_2 = (0, f), b_3 = (0, 1) \in Z_N(R^*),$ where $f$ is unit in $F_1$. Let $\Omega' = \{ a_1, a_2, a_3, b_1, b_2, b_3 \}$. Then the subgraph induced by $\Omega'$ contains $K_{3,3}$ and so $K_{3,3}$ is a subgraph of $\Gamma_N(R)$, a contradiction. Hence $|F_1| = 2$, $F_1 \cong \mathbb{Z}_2$ and so $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2$.

**Corollary 3.3.** Let $R \cong F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where each $F_j$ is a field and $m \geq 2$. Then $\Gamma_N(R)$ is planar if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_2 \times F, \mathbb{Z}_3 \times F, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, where $F$ is a finite field.

**Proof.** Note that $R$ has no nonzero nilpotent elements. Therefore, $Z_N(R) = Z(R)$ and so $\Gamma_N(R) \cong \Gamma(R)$. Hence the proof follows from Theorem 1.6.

4. Genus of $\Gamma_N(R)$

**Theorem 4.1.** Let $(R, m)$ be a finite local ring. Then $\gamma(\Gamma_N(R)) = 1$ if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_8, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_4[x,y]/(x,y)^2$, or $\mathbb{Z}_4[x]/(x^2)$.

**Proof.** Let $R$ be one of the rings $\mathbb{Z}_8, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_4[x,y]/(x,y)^2$ or $\mathbb{Z}_4[x]/(x^2)$. Then by Example 1.5, $|R| = 8$, $|m^*| = 3$ and so $|R^x| = 4$. By Lemma 2.1, all these rings have same $\Gamma_N(R)$. Let $R = \{ 0, z_1, z_2, z_3, u_1, u_2, u_3, u_4 \}$, where each $u_i$ is unit in $R$ and each $z_i$ is a nonzero zero-divisors in $R$. By Lemma 2.1, $K_{3,3}$ is a subgraph of $\Gamma_N(R)$ and so by Lemma 1.8, $\gamma(\Gamma_N(R)) \geq 1$. However, we can draw $\Gamma_N(R)$ on the surface of a torus, see Fig. 3. Therefore, $\gamma(\Gamma_N(R)) = 1$.

Conversely, suppose $\gamma(\Gamma_N(R)) = 1$. Then $\Gamma_N(R)$ is non-planar and by Theorem 3.1, $|m^*| \geq 3$. If $|m^*| = 4$, then by Example 1.5, $R \cong \mathbb{Z}_{25}$ or $\mathbb{Z}_4[x]/(x^2)$ and so by Lemma 2.1, $K_{3,5}$ is a subgraph of $\Gamma_N(R)$. By Lemma 1.9, $\gamma(\Gamma_N(R)) > 1$, a contradiction.

If $|m^*| \geq 5$, then $|R^x| \geq 6$ and by Lemma 2.1, $K_{5,6}$ is a subgraph of $\Gamma_N(R)$ and so $\gamma(\Gamma_N(R)) > 1$, a contradiction. Thus, $|m^*| = 3$ and by Example 1.5, $R$ is isomorphic to one of the following rings: $\mathbb{Z}_8, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_4[x,y]/(x,y)^2, \mathbb{Z}_4[x]/(x^2)$ or $\mathbb{Z}_4[x]/(x^2 + x + 1)$. Suppose $R \cong \mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_4[x]/(x^2 + x + 1)$. Then by Example 1.5, $|R| = 16$ and $|R^x| = 12$. By Lemma 2.1, $K_{3,12}$ is a subgraph of $\Gamma_N(R)$ and by Lemma 1.9, $\gamma(\Gamma_N(R)) \geq 2$, a contradiction. Thus, $R$ is isomorphic to one of the following rings: $\mathbb{Z}_8, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_2[x,y]/(x^2 + x + 1)$ or $\mathbb{Z}_2[x]/(x^2 + x + 1)$ (see Fig. 3). □
Corollary 4.2. Let $R \cong F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where each $F_j$ is a field and $m \geq 2$. Then $\gamma(\Gamma_N(R)) = 1$ if and only if $R$ is isomorphic to one of the following rings: $F_4 \times F_4$, $F_4 \times Z_5$, $F_4 \times Z_7$, $Z_5 \times Z_5$, $Z_2 \times Z_3 \times Z_3$, $Z_3 \times Z_3 \times Z_3$, $Z_2 \times Z_3 \times F_4$, $Z_2 \times Z_2 \times Z_5$, $Z_2 \times Z_2 \times Z_7$ or $Z_2 \times Z_2 \times Z_2 \times Z_2$.

Proof. Note that $R$ has no nonzero nilpotent elements. Therefore, $Z_N(R) = Z(R)$ and so $\Gamma_N(R) \cong \Gamma(R)$. Hence the proof follows from Theorem 1.7.

Corollary 4.3. Let $(R, m)$ be a finite local ring. Then the following hold:

(i) If $|m^*| = 3$ and $\gamma(\Gamma_N(R)) \geq 2$, then $R \cong \frac{Z_3[x]}{(x^3)}$ or $\frac{Z_2[x]}{(x^2 + x + 1)}$.

(ii) If $|m^*| \geq 4$, then $\gamma(\Gamma_N(R)) \geq 2$.

Proof. (i) This follows from Example 1.5 and Theorem 4.1.

(ii) Since $|m^*| \geq 4$, $|R^\times| \geq 5$ and by Lemma 2.1, $K_{4,5}$ is a subgraph of $\Gamma_N(R)$. By Lemma 1.9, $\gamma(\Gamma_N(R)) \geq 2$.

Theorem 4.4. Let $R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where each $(R_i, m_i)$ is a local ring and $F_j$ is a field, $1 \leq i \leq n$, $1 \leq j \leq m$, $m, n \geq 1$ and $m + n \geq 2$. Then $\gamma(\Gamma_N(R)) = 1$ if and only if $R$ is isomorphic to one of the following rings: $Z_4 \times Z_3$, $\frac{Z_2[x]}{(x^2)} \times Z_3$, $\frac{Z_2[x]}{(x^2)} \times F_4$ or $Z_4 \times F_4$, where $F_4$ is a field with four elements.

Proof. Let $R$ be one of the rings $Z_4 \times Z_3$, $\frac{Z_2[x]}{(x^2)} \times Z_3$, $\frac{Z_2[x]}{(x^2)} \times F_4$ or $Z_4 \times F_4$. Note that $\Gamma_N(Z_4 \times Z_3) \cong \Gamma_N(\frac{Z_2[x]}{(x^2)}) \times Z_3)$ and $\Gamma_N(Z_4 \times F_4) \cong \Gamma_N(\frac{Z_2[x]}{(x^2)}) \times F_4)$. Clearly $K_{3,3}$ is a subgraph of $\Gamma_N(R)$ and so Lemma 1.9, $\gamma(\Gamma_N(R)) \geq 1$. However, we can draw $\Gamma_N(R)$ on the surface of a torus, see Figs. 5 and 6, $\gamma(\Gamma_N(R)) = 1$. 1450037-7
Assume that $\gamma(\Gamma_N(R)) = 1$. If $n + m \geq 3$, then $|R| \geq 16$, $|R_1| \geq 4$ and $|m_1*| \geq 1$. Let $a \in m_1^* \text{ and } b, c \in R_1^e \text{ with } b \neq c$. Consider $x_1 = (a, 0, \ldots, 0), x_2 = (b, 0, \ldots, 0), x_3 = (c, 0, 0, \ldots, 0), x_4 = (0, 1, 0, \ldots, 0), x_5 = (0, 0, 1, 0, \ldots, 0), x_6 = (0, 1, 0, \ldots, 0), x_7 = (a, 1, 0, \ldots, 0), x_8 = (a, 0, 1, 0, \ldots, 0), x_9 = (a, 1, 1, 0, \ldots, 0), x_{10} = (c, 0, 1, 0, \ldots, 0) \text{ and } x_{11} = (b, 0, 1, 0, \ldots, 0) \in Z_N(R)^e$. Let $\Omega = \{x_1, \ldots, x_{11}\}$. Then $G$ is a subgraph of $\langle \Omega \rangle$ in $\Gamma_N(R)$ (see Fig. 4) and so $\gamma(\Gamma_N(R)) \geq 2$, a contradiction. Hence $n + m = 2$ and so $R \cong R_1 \times F_1$. We first claim that $|m^*| = 1$.

Case 1. $|m_1^*| = 2$.
Then $R_1 \cong Z_9$ or $\frac{\phi_3(x)}{\phi_3(x)}$ and so $|R_1| = 9$. Let $a_1, a_2 \in m_1^* \text{ with } a_1 \neq a_2$. Let $w_1 = (a_1, 0), w_2 = (a_2, 0), w_3 = (u_1, 0), w_4 = (u_2, 0), w_5 = (u_3, 0), w_6 = (u_4, 0), w_7 = (u_5, 0), w_8 = (a_1, 1), w_9 = (a_2, 1) \text{ and } w_{10} = (0, 1), \text{ where } u_i \in R_1^e, 1 \leq i \leq 5 \text{ and } u_i \neq u_j \text{ for } i \neq j$. Consider $S = \{w_1, \ldots, w_{10}\}$. Then $K_{3, 7}$ is a subgraph of $\langle S \rangle$ in $\Gamma_N(R)$ and so $\gamma(\Gamma_N(R)) \geq 2$, a contradiction.

Case 2. $|m_1^*| \geq 3$.
Then $|R_1^e| \geq 4$. Let $b_1, b_2, b_3 \in m_1^* \text{ with } b_1 \neq b_2 \neq b_3$. Let $z_1 = (b_1, 0), z_2 = (b_2, 0), z_3 = (b_3, 0), z_4 = (u_1, 0), z_5 = (u_2, 0), z_6 = (u_3, 0), z_7 = (u_4, 0), z_8 = (b_1, 1), z_9 = (b_2, 1), z_{10} = (b_3, 1) \text{ and } z_{11} = (0, 1), \text{ where } u_i \in R_1^e, 1 \leq i \leq 4 \text{ and } u_i \neq u_j \text{ for } i \neq j$. Consider $S' = \{z_1, \ldots, z_{11}\}$. Then $K_{4, 7}$ is a subgraph of $\langle S' \rangle$ in $\Gamma_N(R)$ and so $\gamma(\Gamma_N(R)) \geq 2$, a contradiction.

Our next claim is that $|F_1| \leq 4$. Suppose $|F_1| \geq 5$. Note that $|R_1| \geq 4$ and $|m_1^*| \geq 1$. Let $z \in m_1^*$. Consider $y_1 = (z, 0), y_2 = (u_1, 0), y_3 = (u_2, 0), y_4 = (0, v_1), y_5 = (0, v_2), y_6 = (0, v_3), y_7 = (0, v_4), y_8 = (z, v_4), y_9 = (z, v_1), y_{10} = (z, v_2)$ where $u_1, u_2$ are distinct units in $R_1$ and $v_i \in F_1^e$ for all $i$. Consider $\Omega' = \{y_1, \ldots, y_{10}\}$. Then $K_{3, 7}$ is a subgraph of $\langle \Omega' \rangle$ in $\Gamma_N(R)$ and by Lemma 1.9, $\gamma(\Gamma_N(R)) \geq 2$, a contradiction. Hence $|F_1| \leq 4$.

Fig. 4. Embedding of $\Gamma_N(Z_4 \times Z_3) \cong \Gamma_N(\frac{\phi_3(x)}{\phi_3(x)} \times Z_3)$ in $S_1$.
Nilpotent graphs of genus one

Fig. 5. A planar embedding of $\Gamma_N(\mathbb{Z}_4 \times F_4) \cong \Gamma_N(\mathbb{Z}_{\frac{4}{x^2}} \times F_4)$ in $S_1$.

Fig. 6. $G$ with $\gamma(G) = 2 \ [13]$.

Hence in both the cases, we get $|m_i^1| = 1$ and so by Example 1.5, $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{(x^2)}$. Since $|F_1| \leq 4$ and by Theorem 3.2, $F_1 = \mathbb{Z}_4$ or $F_1 = \mathbb{F}_4$ and so $R \cong \mathbb{Z}_4 \times \mathbb{Z}_3$, $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_3$, $\frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{F}_4$ or $\mathbb{Z}_4 \times \mathbb{F}_4$ (see Figs. 5 and 6).

Theorem 4.5. Let $R \cong R_1 \times \cdots \times R_n$ be a finite commutative ring with identity, where each $(R_i, m_i)$ is a local ring and $n \geq 2$. Then $\gamma(\Gamma_N(R)) \geq 2$.

Proof. Note that $\mathbb{Z}_N(R^*) = R^*$. Clearly $|R_i| \geq 4$, $|m_i^1| \geq 1$ and $|R_i^x| \geq 2$ for all $i$, $1 \leq i \leq n$. Let $a_1 \in m_i^1$ and $a_2 \in m_i^2$. Let $x_1 = (a_1, 0, \ldots, 0), x_2 = (u_1, 0, \ldots, 0), x_3 = (u_2, 0, \ldots, 0), x_4 = (u_1, a_2, 0, \ldots, 0), x_5 = (u_2, a_2, 0, \ldots, 0), y_1 = (0, a_2, 0, \ldots, 0), y_2 = (0, v_1, 0, \ldots, 0), y_3 = (0, v_2, 0, \ldots, 0), y_4 = (a_1, a_2, 0, \ldots, 0)$ and $y_5 = (a_1, v_1, 0, \ldots, 0)$, where $u_1, u_2$ are distinct units in $R_1$ and $v_1, v_2$ are distinct units in $R_2$. Consider $\Omega = \{x_1, \ldots, x_4, y_1, \ldots, y_5\}$. Then $K_{4,5}$ is a subgraph of $\Omega$ in $\Gamma_N(R)$ and by Lemma 1.9, $\gamma(\Gamma_N(R)) \geq 2$. \qed

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(Continued)
A TYPICAL GRAPH STRUCTURE OF A RING

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Abstract. The zero-divisor graph of a commutative ring $R$ with respect to nilpotent elements is a simple undirected graph $\Gamma_N^*(R)$ with vertex set $\mathcal{Z}_N(R)^*$, and two vertices $x$ and $y$ are adjacent if and only if $xy$ is nilpotent and $xy \neq 0$, where $\mathcal{Z}_N(R) = \{x \in R : xy$ is nilpotent, for some $y \in R^*\}$. In this paper, we investigate the basic properties of $\Gamma_N^*(R)$. We discuss when it will be Eulerian and Hamiltonian. We further determine the genus of $\Gamma_N^*(R)$.

1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. In 1988, I. Beck began to investigate the possibility of coloring a commutative ring $R$ by associating to the ring a zero-divisor graph, defined as a simple graph, the vertices of which are the elements of the ring $R$, with two distinct elements $x$ and $y$ being adjacent if and only if $xy = 0$ [4]. Retaining the original definition, the next decade brought little progress. However, in 1999, D. F. Anderson and P. S. Livingston [2] modified and studied the zero-divisor graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of the commutative ring $R$. Note that $\Gamma(R)^c$ is the complement of the zero-divisor graph of $R$. In [7], Chen defined a kind of graph structure of rings. He let all the elements of ring $R$ be the vertices of the graph and two vertices $x$ and $y$ are adjacent if and only if $xy$ is nilpotent. However, in 2010, A. Li and Q. Li modified and studied a kind of new undirected graph $\Gamma_N(R)$ whose vertices are non zero elements of the set $\mathcal{Z}_N(R)$, and two vertices $x$ and $y$ are adjacent if and only if $xy$ is nilpotent, where $\mathcal{Z}_N(R) = \{x \in R : xy$ is nilpotent, for some $y \in R^*\}$. For any set $X$, let $X^*$ denote the nonzero


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elements of $X$. In this paper, we construct a graph called zero-divisor graph of a commutative ring $R$ with respect to nilpotent elements as a simple undirected graph $\Gamma_N^*(R)$ with vertex set $Z_N(R)^*$, and two vertices $x$ and $y$ are adjacent if and only if $xy$ is nilpotent and $xy \neq 0$. We investigate the interplay between the graph theoretic properties of $\Gamma_N^*(R)$ and the ring theoretic properties of $R$. We denote the ring of integers modulo $n$ by $\mathbb{Z}_n$, the field with $q$ elements by $\mathbb{F}_q$ and the set of all nilpotent elements in $R$ by $N(R)$. Note that $R^x$ be the set of all units in $R$ and $J(R)$ be the Jacobson radical of $R$. For basic definitions on rings, one may refer [3].

Let $G = (V, E)$ be a simple connected graph. The girth of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth. The closure of a graph $G$ is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $|V(G)|$ until no such pair remains. Note that a connected graph is Hamiltonian if and only if its closure is Hamiltonian. For basic definitions on graphs, one may refer [5]. The following results are listed for ready reference.

**Theorem 1.1.** [5] If $R$ is a finite local ring, then $|R| = p^n$ for some prime $p$ and some positive integer $n \geq 1$.

**Theorem 1.2.** [3] If $R$ is a finite commutative ring, then $R \cong R_1 \times \cdots \times R_n$, where each $R_i$ is a local ring.

**Example 1.3.** [1, 11] Let $(R, m)$ be a finite local ring and $\Gamma(R)$ be the zero-divisor graph of $R$. Then

| $|Z(R)^*|$ | $R$ | $|R|$ | $\Gamma(R)$ |
|----------|-----|------|---------|
| 1        | $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$ | 4 | $K_1$ |
| 2        | $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$ | 9 | $K_2$ |
| 3        | $\mathbb{Z}_8$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_2[x]/(2x, x^2-2)$ | 8 | $K_{1,2}$ |
|          | $\mathbb{Z}_4[x]/(2x, y)$, $\mathbb{Z}_2[x, y]/(x, y)$ | 8 | $K_3$ |
|          | $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_2[x]/(x^2+x+1)$ | 16 | $K_3$ |
| 4        | $\mathbb{Z}_{25}$, $\mathbb{Z}_5[x]/(x^2)$ | 25 | $K_4$ |

**Theorem 1.4.** [2] Let $R$ be a finite commutative ring. If $\Gamma(R)$ is complete, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R$ is a local ring with char $R = p$ or $p^2$ and $|\Gamma(R)| = p^n - 1$, where $p$ is prime and $n \geq 1$.

**Lemma 1.5.** $g(K_n) = \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor$ if $n \geq 3$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.

**Lemma 1.6.** $g(K_{m,n}) = \left\lfloor \frac{(m-2)(n-2)}{4} \right\rfloor$ if $m, n \geq 2$. In particular, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also $g(K_{5,4}) = g(K_{6,4}) = g(K_{5,m}) = 2$ if $m = 7, 8, 9, 10$.

2. Basic Properties of $\Gamma_N^*(R)$

**Remark 2.1.** Let $R$ be a reduced ring. Then $Z_N(R)^* = Z(R)^*$, $\Gamma_N(R) \cong \Gamma(R)$ and by definition, $\Gamma_N^*(R)$ is an empty graph.

**Remark 2.2.** Let $R$ be a local ring, but not a field. Then by the definition, $\Gamma_N^*(R)$ is connected and $\text{diam}(\Gamma_N^*(R)) = 2$. 

Theorem 2.3. Let \( R \cong R_1 \times \cdots \times R_n \) be a finite commutative ring with identity, where each \( R_i \) is a local ring and \( n \geq 2 \). Then \( \Gamma_N^*(R) \) is connected if and only if \( R_i \) is not a field for every \( i \). Further \( \text{diam}(\Gamma_N^*(R)) = 2 \).

Proof. Suppose \( R_i \) is not field for every \( i \). Then \( Z(R_i) \neq \{0\} \) for every \( i \). Let \( x, y \in Z_N(R)^* \) and \( x \neq y \). Note that \( J(R) = N(R) \).

If \( N(R)^* = \emptyset \), then \( N(R) = 0 \) which is a contradiction as \( Z(R_i) \neq \{0\} \) for every \( i \). Therefore \( N(R)^* \neq \emptyset \).

Case 1. \( x, y \in R^\times \)
Then there exists \( a \in N(R)^* \) such that \( xa, ya \in N(R)^* \). Thus \( x - a - y \) is a path in \( \Gamma_N^*(R) \).

Case 2. \( x \in R^\times \) and \( y \in N(R)^* \)
Then \( xy \in N(R)^* \) and so \( x \) and \( y \) are adjacent in \( \Gamma_N^*(R) \).

Case 3. \( x \in R^\times \) and \( y \in Z(R) - N(R) \)
Then there exists \( b \in N(R)^* \) such that \( xb, yb \in N(R)^* \). Thus \( x - b - y \) is a path in \( \Gamma_N^*(R) \).

Case 4. \( x, y \in Z(R) - N(R) \)
Then there exists \( c \in N(R)^* \) such that \( xc, yc \in N(R)^* \). Thus \( x - c - y \) is a path in \( \Gamma_N^*(R) \).

Case 5. \( x, y \in N(R)^* \)
Then there exists \( u \in R^\times \) such that \( xu, yu \in N(R)^* \) and so \( x - u - y \) is a path in \( \Gamma_N^*(R) \).

Case 6. \( x \in Z(R) - N(R) \) and \( y \in N(R)^* \)
If \( xy \neq \{0\} \), then \( xy \in N(R)^* \) and so \( x \) and \( y \) are adjacent in \( \Gamma_N^*(R) \). If \( xy = 0 \), then there exists \( z \in N(R)^* \) such that \( xz, yz \in N(R)^* \). Thus \( x - z - y \) is a path in \( \Gamma_N^*(R) \). Hence \( \Gamma_N^*(R) \) is connected.

Conversely, let \( \Gamma_N^*(R) \) be a connected graph. Suppose \( R_i \) is a field for some \( i \). Then there exists an element \( x = (0, \ldots, 0, 1, 0, \ldots, 0) \in Z_N(R)^* \), with 1 in the \( i \)th place of \( x \), such that \( x \) is not adjacent to any other vertex of \( \Gamma_N^*(R) \), a contradiction. \( \square \)

Theorem 2.4. Let \( R \) be a finite commutative ring with identity such that \( \Gamma_N^*(R) \) is connected. Then \( \Gamma(R)^c \) is a subgraph of \( \Gamma_N^*(R) \) if and only if \( R \) is a local ring.

Proof. By Theorems 1.2 and 2.3, \( R \cong R_1 \times \cdots \times R_n \), where each \( (R_i, m_i) \) is a local ring. Suppose \( R \) is a local ring. Then by definition of \( Z_N(R)^* \), \( Z_N(R)^* = R^* \) and so \( Z(R)^* \subset Z_N(R)^* \). Also \( R \cong R_1 \) and so \( x \) is nilpotent for all \( x \in m_1^\times \). Let \( x, y \in m_1^\times \) with \( x \neq y \). If \( x \) and \( y \) are adjacent in \( \Gamma(R) \), then \( xy = 0 \), and so \( x \) and \( y \) are non-adjacent in \( \Gamma_N^*(R) \). If \( x \) and \( y \) are non-adjacent in \( \Gamma(R) \), then \( xy \neq 0 \), \( xy \in N(R)^* \) and so \( x \) and \( y \) are adjacent in \( \Gamma_N^*(R) \). Hence \( \Gamma(R)^c \) is a subgraph of \( \Gamma_N^*(R) \).

Conversely, let \( \Gamma(R)^c \) be a subgraph of \( \Gamma_N^*(R) \). Suppose \( R \) is a non-local ring and \( n \geq 2 \). Let \( a = (1, 0, \ldots, 0), b = (u, 0, \ldots, 0) \in Z(R)^* \subset Z_N(R) \), where \( u \) is a unit in \( R_1 \). Then \( ab \neq 0 \) in \( R \) and so \( a \) and \( b \) are non-adjacent in \( \Gamma(R) \). Clearly \( ab \notin N(R) \) and so \( a \) and \( b \) are non-adjacent in \( \Gamma_N^*(R) \), a contradiction. \( \square \)

Theorem 2.5. Let \( R \) be a finite commutative ring with identity and \( \Gamma_N^*(R) \) be a connected graph. Then \( \Gamma_N^*(R) \) is a complete bipartite graph if and only if \( \Gamma(R) \) is a complete graph.
Then the following are equivalents:

\( a \) contradiction. Hence \( \Gamma(R) \) is complete. Then by Theorems 1.4 and 2.3, \( R \) is a local ring with unique maximal ideal \( m \) and \( xy = 0 \) for all \( x, y \in m^* \), \( x \neq y \). Note that \( Z_N(R)^* = R^* = m^* \cup (R \setminus m) \). By the definition of \( \Gamma_N(R) \), \( m^* \) and \( R \setminus m \) are independent sets of \( \Gamma_N(R) \). Also each edge in \( \Gamma_N(R) \) has one end in \( m^* \) and other end in \( R \setminus m \). Hence \( \Gamma_N(R) \) is a complete bipartite graph with bipartition \( (m^*, R \setminus m) \).

Conversely, suppose \( \Gamma_N(R) \) is complete bipartite graph. Note that \( R \cong R_1 \times \cdots \times R_n \), where each \( (R_i, m_i) \) is a local ring. If \( R \) is non-local ring, then \( n > 2 \). Let \( a \in m_{1}^*, b \in m_{2}^* \). Then \( (a, b, \ldots, 0) - (1,0,\ldots,0) - (1, 0, \ldots, 0) = (a, b, 0, \ldots, 0) - (a, b, 0, \ldots, 0) \) is a cycle of length 3 in \( \Gamma_N(R) \), a contradiction. Hence \( R \cong R_1 \) is a local ring. Suppose \( \Gamma(R) \) is not complete. Then there exists elements \( x_1, y_1 \in Z(R)^* \) such that \( x_1 y_1 \neq 0 \) and \( x_1 y_1 \in N(R)^* \). Thus \( x_1 - u - y_1 - x_1 \) is a cycle of length 3 in \( \Gamma_N(R) \), where \( u \in R^\times \), a contradiction. Hence \( \Gamma(R) \) is complete. \( \square \)

In view of Theorems 1.4 and 2.5, if \( \Gamma_N(R) \) is connected then we have \( \Gamma_N(R) \) is complete bipartite if and only if \( R \) is a local ring with \( \operatorname{char} R = p \) or \( p^2 \).

**Theorem 2.6.** Let \( R \) be a finite commutative ring with identity and \( \Gamma_N(R) \) be a connected graph. Then the following are equivalents:

(i) \( \Gamma_N(R) \) is a star

(ii) \( \Gamma_N(R) \) is a tree

(iii) \( R \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_2^{[x]} (x^2) \).

**Proof.** (i) \( \Rightarrow \) (ii) follows from the definition of tree.

(ii) \( \Rightarrow \) (iii) Suppose \( \Gamma_N(R) \) is tree. Then \( \Gamma_N(R) \) contains no cycle. Since \( R \) is connected, by Remark 2.2 and Theorem 2.3, \( R \cong R_1 \times \cdots \times R_n \) where each \( (R_i, m_i) \) is a local ring, but not a field. If \( n \geq 2 \), then \( (a, b, \ldots, 0) - (1, 0, \ldots, 0) = (a, 1, 0, \ldots, 0) - (a, b, 0, \ldots, 0) \) is a cycle in \( \Gamma_N(R) \), where \( a \in m_{1}^*, b \in m_{2}^* \), a contradiction. Thus \( n = 1 \) and so \( R \) is local. Suppose \( |m_{1}^*| \geq 2 \). Then \( |R^\times| \geq 3 \). Let \( x, y \in m_{1}^* \) with \( xy = 0 \) and \( u, v \in R^\times \). Then \( x - u - y - v - x \) is a cycle in \( \Gamma_N(R) \), a contradiction. Hence \( |m_{1}^*| = 1 \) and so \( R \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_2^{[x]} (x^2) \).

(iii) \( \Rightarrow \) (i) If \( R \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_2^{[x]} (x^2) \), then \( \Gamma_N(R) \cong K_{1,2} \). \( \square \)

**Theorem 2.7.** Let \( R \) be a finite commutative ring with identity and \( \Gamma_N(R) \) be a connected graph.

(i) \( \operatorname{gr}(\Gamma_N(R)) = \infty \) if and only if \( R \cong \mathbb{Z}_4 \) and \( \frac{Z_2[x]}{(x^2)} \).

(ii) \( \operatorname{gr}(\Gamma_N(R)) = 4 \) if and only if \( R \) is local with \( Z(R)^2 = 0 \) and \( |Z(R)^*| \geq 2 \).

(iii) \( \operatorname{gr}(\Gamma_N(R)) = 3 \) if and only if \( R \cong \mathbb{Z}_4, \frac{Z_2[x]}{(x^2)} \) and \( R \) is not a local ring with \( Z(R)^2 = 0 \) and \( |Z(R)^*| \geq 2 \).

**Proof.** (ii) follows from Theorem 2.6.

(ii) Suppose \( R \) is local with \( \operatorname{Char}(R) = p^2 \) and \( |Z(R)^*| \geq 2 \). Then by Theorems 1.4 and 2.5, \( \Gamma_N(R) \) is a complete bipartite graph and so \( \operatorname{gr}(\Gamma_N(R)) = 4 \).

Conversely, let \( \operatorname{gr}(\Gamma_N(R)) = 4 \). Then \( \Gamma_N(R) \) does not contain a cycle of length 3 and \( |Z(R)^*| \geq 4 \). Since \( R \) is finite, \( R \cong R_1 \times \cdots \times R_n \), where each \( (R_i, m_i) \) is a local ring. Since \( \Gamma_N(R) \) is connected, \( m_i \neq \{0\} \) for all \( i \). Suppose \( n \geq 2 \). Let \( x_1 \in m_{1}^* \) and \( x_2 \in m_{2}^* \). Then \( (x_1, 0, \ldots, 0) - (0, x_2, 0, \ldots, 0) - (1, 0, \ldots, 0) - (x_1, 0, \ldots, 0) \) is a cycle in \( \Gamma_N(R) \), a contradiction. Thus \( R \) is local. If \( Z(R)^2 \neq 0 \), then
there exists \( x, y \in \mathbb{Z}(R)^* \) such that \( xy \neq 0 \) and so \( x - u - y - x \) is a cycle in \( \Gamma_N^*(R) \), a contradiction. (iii) follows from (i) and (ii).

\[ \square \]

**Theorem 2.8.** [8, S. Földes, P. L. Hammer] Let \( G \) be a connected graph. Then \( G \) is a split graph if and only if \( G \) contains no induced subgraph isomorphic to \( 2K_2, C_4, C_5 \), where \( C_4 \) is a cycle of length 4.

**Theorem 2.9.** Let \( R \) be a finite commutative ring with identity and \( \Gamma_N^*(R) \) be a connected graph. Then \( \Gamma_N^*(R) \) is split if and only if \( R \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_2[\mathbb{Z}_2] \).

**Proof.** Suppose \( \Gamma_N^*(R) \) is split. Since \( R \) is finite, \( R \cong R_1 \times \cdots \times R_n \), where each \( (R_i, m_i) \) is a local ring. If \( n \geq 2 \), then \( (1, \ldots, 1) - (a_1, 0, \ldots, 0) - (u_1, u_2, \ldots, u_n) - (0, a_2, 0, \ldots, 0) \) is cycle of length 4 in \( \Gamma_N^*(R) \), where \( a_1 \in m_1, a_2 \in m_2, u_i \in R_i^\times \). By Theorem 2.8, \( \Gamma_N^*(R) \) is not split, a contradiction. Hence \( R \) is local. If \( |m_1| \geq 2 \), then \( C_4 \) is a subgraph of \( \Gamma_N^*(R) \), a contradiction. Hence \( |m_1| = 2 \) and so \( R \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_2[\mathbb{Z}_2] \).

Conversely, if \( R \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_2[\mathbb{Z}_2] \), then by Theorem 2.6(i), \( \Gamma_N^*(R) \) is star and so \( \Gamma_N^*(R) \) is split. \[ \square \]

3. Eulerian and Hamiltonian Nature of \( \Gamma_N^*(R) \)

In this section, we are interested in the Eulerian and Hamiltonian nature of \( \Gamma_N^*(R) \).

**Theorem 3.1.** Let \( (R, m) \) be a finite local ring but not a field and \( |R| = p^n \), where \( p \) is prime and \( n > 1 \). If \( \Gamma_N^*(R) \) is a connected graph with \( |m^*| \geq 2 \), then \( \Gamma_N^*(R) \) is Eulerian if and only if \( |R| \) is odd and \( x^2 = 0 \) for all \( x \in m^* \).

**Proof.** Note that \( |m| = p^k \) for some \( k < n \) and \( |R^\times| = p^k(p^{n-k} - 1) \). Suppose \( \Gamma_N^*(R) \) is Eulerian. Then \( \deg_{\Gamma_N^*(R)}(v) \) is even for all \( v \in \mathbb{Z}_N(R)^* \). Suppose \( |R| \) is even. Then \( |m^*| \) is odd and so \( \deg_{\Gamma_N^*(R)}(u) = p^k - 1 \) is odd for all \( u \in R^\times \), a contradiction. Hence \( |R| \) is odd and so \( |m^*|, \; |R^\times| \) are even. If \( x^2 \neq 0 \) for some \( x \in m^* \), then \( |ann(x)| = p^\ell \) for some \( \ell < n \) and so \( \deg_{\Gamma_N^*(R)}(x) = |R^\times| + |m| - p^\ell - 1 \) is odd, a contradiction. Hence \( x^2 = 0 \) for all \( x \in m^* \).

Conversely, let \( |R| \) be an odd integer such that \( x^2 = 0 \) for all \( x \in m^* \). Then \( \deg_{\Gamma_N^*(R)}(u) = p^k - 1 \) is even for all \( u \in R^\times \) and \( \deg_{\Gamma_N^*(R)}(z) = |R^\times| \) is even for all \( z \in m^* \). Hence \( \Gamma_N^*(R) \) is Eulerian. \[ \square \]

**Theorem 3.2.** Let \( R \) be a finite commutative nonlocal ring with identity and \( \Gamma_N^*(R) \) be a connected graph. Then \( \Gamma_N^*(R) \) is Eulerian if and only if \( |R| \) is odd and \( x^2 = 0 \) for all \( x \in J(R)^* \).

**Proof.** Suppose \( \Gamma_N^*(R) \) is Eulerian. Then \( \deg_{\Gamma_N^*(R)}(x) \) is even for all \( x \in \mathbb{Z}_N(R)^* \). Since \( R \) is finite, \( R \cong R_1 \times \cdots \times R_n \) where each \( (R_i, m_i) \) is a local ring and \( n \geq 2 \). If \( |R| \) is even, then \( |R_i| \) is even for some \( i \), \( |m_i| \) is even and so \( \deg_{\Gamma_N^*(R)}(u) = (\Pi_{i=1}^n |m_i|) - 1 \) is odd for all \( u \in R^\times \), a contradiction. Hence \( |R| \) is odd and so \( |m_i| \) is odd.

If \( x^2 \neq 0 \) for some \( x \in J(R)^* \), then \( \deg_{\Gamma_N^*(R)}(x) = |R| - (|ann(x)| + 1) \) is odd, a contradiction. Hence \( x^2 = 0 \) for all \( x \in J(R)^* \).
Conversely, let $|R|$ be an odd integer and $x^2 = 0$ for all $x \in J(R)^*$. Then $|R_i|$ is odd for all $i$ and so $|m_i|$ is odd. Let $y = (y_1, \ldots, y_n) \in \mathcal{Z}_N(R)^*$. Then

$$\deg_{\Gamma^*_N(R)}(y) = \begin{cases} |R| - |\text{ann}(y)| & \text{if } y \in J(R)^* \\ (\prod_{i=1}^n |m_i|) - 1 & \text{if } y \in R^x \end{cases}$$

In equation 3.1, $\deg_{\Gamma^*_N(R)}(y)$ is even. If $y \in \mathcal{Z}(R) \setminus J(R)$, then $\deg_{\Gamma^*_N(R)}(y) = |\mathcal{Z}(R)| - |\text{ann}(y)| - 1$ is even. Hence $\deg_{\Gamma^*_N(R)}(a)$ is even for all $a \in \mathcal{Z}_N(R)^*$ and so $\Gamma^*_N(R)$ is Eulerian. \hfill $\Box$

**Theorem 3.3.** Let $(R, m)$ be a finite local ring but not a field and $|R| = p^n \geq 4$, where $p$ is prime and $n > 1$. Then $\Gamma^*_N(R)$ has a Hamiltonian path if and only if $R/m \cong \mathbb{Z}_2$. Hence $|R|$ is even.

**Proof.** Note that $|m| = p^k$ for some $k < n$ and $|R^x| = p^k(p^{n-k} - 1) = t$. Clearly $K_{p^{k-1}, t}$ is a subgraph of $\Gamma^*_N(R)$ and $R^x$ is an independent subset of $\Gamma^*_N(R)$. Suppose $R/m \cong \mathbb{Z}_2$. Then $|m| = |R^x|$ and $|\mathcal{Z}_N(R)| = |m^*| + |R^x|$. Note that $K_{p^{k-1}, p^k}$ is a subgraph of $\Gamma^*_N(R)$. From this, we get

$$u_1 - z_1 - u_2 - z_2 - u_3 - \cdots - u_{p^k-1} - z_{p^k-1} - u_{p^k}$$

is a Hamiltonian path in $\Gamma^*_N(R)$, where $u_i \in R^x$ and $z_j \in m^*$. Conversely, suppose $\Gamma^*_N(R)$ has a Hamiltonian path. Suppose $|R/m| > 2$. Then $|m| < |R^x|$ and so $|R^x - m| > 2$. Note that the closure of $\Gamma^*_N(R)$ is $K_{p^{k-1}, p^k} + \overline{K}_{p^k(p^{n-k} - 1)}$. Clearly $K_{p^{k-1}, p^k} + \overline{K}_{p^k(p^{n-k} - 1)}$ is not Hamiltonian and so $\Gamma^*_N(R)$ is not Hamiltonian. Hence $|R/m| = 2$ and so $R/m \cong \mathbb{Z}_2$. \hfill $\Box$

From the above theorem, one can observe the following corollary.

**Corollary 3.4.** Let $(R, m)$ be a finite local ring but not a field. Then $\Gamma^*_N(R)$ is nonHamiltonian.

4. Genus of $\Gamma^*_N(R)$

In this section, we characterize the class of rings for which $\Gamma^*_N(R)$ is planar. Also we determine all isomorphism classes of finite commutative rings with identity whose $\Gamma^*_N(R)$ has genus one.

Let $S_k$ denote the sphere with $k$ handles, where $k$ is a non-negative integer. The genus of any graph $G$, denoted $g(G)$, is the minimal integer $\ell$ such that the graph can be embedded in $S_\ell$. A genus 0 graph is called a planar graph and a genus 1 graph is called a toroidal graph. For details on embedding a graph in a surface, see [13].

**Theorem 4.1.** Let $R$ be a finite commutative ring with identity and $\Gamma^*_N(R)$ be a connected graph. Then $\Gamma^*_N(R)$ is planar if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\mathbb{Z}_2[x] \langle x^4 \rangle$, $\mathbb{Z}_9$, or $\mathbb{Z}_3[x] \langle x^2 \rangle$.

**Proof.** Suppose $\Gamma^*_N(R)$ is planar. Clearly if $R$ is a non-local ring, then $K_{3, 3}$ is a subgraph of $\Gamma^*_N(R)$. Hence $(R, m)$ is a local ring with $|R| = p^n$ where $p$ is prime and $n > 1$. Note that $|m| = p^k$ for some $k < n$ and $|R^x| = |R \setminus m| = p^k(p^{n-k} - 1)$. If $|m^*| \geq 3$, then $|R^x| \geq 4$ and so $K_{3, 4}$ is a subgraph of $\Gamma^*_N(R)$, a contradiction. Hence $|m^*| \leq 2$ and so $R \cong \mathbb{Z}_4$, $\mathbb{Z}_2[x] \langle x^4 \rangle$, $\mathbb{Z}_9$, or $\mathbb{Z}_3[x] \langle x^2 \rangle$. Converse is obvious. \hfill $\Box$
Thus $m(\Gamma_N(R)) = 3$ and by Example 1.3, $\Gamma_N(R)$ is non-planar. Since $Γ$ is finite, $R \cong R_1 \times \cdots \times R_n$, where each $(R_i, m_i)$ is a local ring. Since $Γ_N(R)$ is connected, $m_i \neq \{0\}$ for every $i$. Suppose $n \geq 2$. Let $a_1 \in m_1$, $a_2 \in m_2$ with $a_1^2 = 0$ and $a_2^2 = 0$.

Consider $x_1 = (0, a_2, 0, \ldots, 0)$, $x_2 = (a_1, 0, \ldots, 0)$, $x_3 = (1, 1, 0, \ldots, 0)$, $x_4 = (1, u_2, 0, \ldots, 0)$, $x_5 = (u_1, 1, 0, \ldots, 0)$, $x_6 = (a_1, a_2, 0, \ldots, 0)$, $x_7 = (u_1, a_2, 0, \ldots, 0)$, $x_8 = (1, a_2, 0, \ldots, 0)$, $x_9 = (0, u_2, 0, \ldots, 0)$, $x_{10} = (a_1, 1, 0, 0, \ldots, 0)$ and $x_{11} = (u_1, u_2, 0, \ldots, 0) \in Z_N(\Gamma^*)$, where $u_1 \in R_1^*$, $u_2 \in R_2^*$. Let $Ω = \{x_1, \ldots, x_{11}\}$. Then $G$ is a subgraph of $(Ω)$ in $Γ^*(R)$ (see, Fig. 2.1). Note that $g(G) = 2[12, C. Wickham]$. Therefore $g(Γ^*(R)) \geq 2$, a contradiction. Hence $(R, m)$ is local.

Since $Γ^*(R)$ is non-planar, by Theorem 4.1, $|m^*| \geq 3$. If $|m^*| = 4$, then by Example 1.3, $R \cong Z_{25}$ or $\frac{Z_5[x]}{(x^2)}$, $|R^*| = 20$ and so $K_{4,20}$ is a subgraph of $Γ^*(R)$. By Lemma 1.6, $g(Γ^*(R)) > 1$, a contradiction. If $|m^*| \geq 5$, then $|R^*| \geq 6$ and so $K_{5,6}$ is a subgraph of $Γ^*(R)$, now $g(Γ^*(R)) > 1$, a contradiction. Thus $|m^*| = 3$ and by Example 1.3, $R$ is isomorphic to one of the following rings: $Z_8$, $\frac{Z_2[x]}{(x^2)}$, $\frac{Z_4[x]}{(2x,x^2-2)}$,

$$
\frac{Z_2[x,y]}{(x,y)^2}, \frac{Z_4[x]}{(2x)^2}, \frac{Z_4[x]}{(x^2+1)}.
$$

Suppose $R \cong \frac{Z_4[x]}{(x^2)}$ or $\frac{Z_4[x]}{(x^2+1)}$. Then by Example 1.3, $|R| = 16$, $|R^*| = 12$ and so $K_{3,12}$ is a subgraph of $Γ_N(R)$ and by Lemma 1.6, $g(Γ_N(R)) \geq 2$, a contradiction.
Since $\Gamma^*_N(Z_8) \cong \Gamma^*_N\left(\frac{Z_4[x]}{(x^2)}\right) \cong \Gamma^*_N\left(\frac{Z_4[x]}{(2x,x^2-2)}\right) \cong K_{3,4}$ and Fig. 2.2(b), $R$ is isomorphic to one of the following rings: $Z_8$, $\frac{Z_4[x]}{(x^3)}$, $\frac{Z_4[x,y]}{(x,y^2)}$ or $\frac{Z_4[x]}{(2x)}$. □

**Theorem 4.3.** There exists no finite local ring $R$ with $g(\Gamma^*_N(R)) = 2$

**Proof.** Let $(R, m)$ be a finite local ring with $|R| = p^n$, where $p$ is prime and $n > 1$. Then $|m| = p^k$ for some $k < n$ and $|R^*| = p^{k(p^n-k-1)} > |m^*| = p^k - 1$. By Theorems 4.1 and 4.2, $|m| \geq 3$.

If $|m| = 3$, then by Example 1.3 and Theorem 4.2, $R \cong \frac{Z_4[x]}{(x^2)}$ or $\frac{Z_4[x]}{(x^2+x+1)}$. In this case, $\Gamma^*_N(R) \cong K_{3,12}$ and by Lemma 1.6, $g(\Gamma^*_N(R)) = 5$.

If $|m^*| \geq 4$, then $H = K_{p^k-1,p^k(p^n-k-1)}$ is a subgraph of $\Gamma^*_N(R)$ and so $g(\Gamma^*_N(R)) \geq g(H) > 2$. Hence there exists no finite local ring $R$ with $g(\Gamma^*_N(R)) = 2$. □

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