Chapter 7

Sum annihilating ideal graph of commutative rings

In this Chapter, we study some fundamental properties of $\Omega(R)$. Especially we identify when the annihilating ideal graph is isomorphic to some well-known graphs. Further, we discuss about the Hamiltonian property of $\Omega(R)$. Finally, we characterize all commutative rings $R$ for which $\Omega(R)$ is planar. Also we determine all isomorphism classes of finite commutative rings with identity whose $\Omega(R)$ has genus one. The results obtained in this direction are communicated to Journal of algebra and discrete Mathematics.
7.1 Introduction

In [46], S. Visweswaran, and H. D. Patel have introduced and investigated the graph $\Omega(R)$ of a commutative ring $R$. For a non-domain commutative ring $R$, let $A(R)^*$ be the set of non-zero ideals with non-zero annihilators. A graph associated with the set of all nonzero annihilating-ideals of a commutative ring $R$ which is not an integral domain is defined as follows. The vertex set of this graph is $A(R)^*$ the set of all nonzero annihilating-ideals of $R$ and for distinct $I, J \in A(R)^*$, the vertices $I$ and $J$ are joined by an edge in this graph if and only if $I + J \in A(R)^*$. For convenience we denote this graph by $\Omega(R)$. The main aim of this chapter is to study some of the properties of $\Omega(R)$.

7.2 Definitions and Examples

Definition 7.2.1. A graph $\Omega(R)$ associated with the set of all nonzero annihilating-ideals of a commutative ring $R$ which is not an integral domain is defined as follows. The vertex set of $\Omega(R)$ is $A(R)^*$ the set of all non-zero annihilating ideals of $R$ and for distinct $I, J \in A(R)^*$, the vertices $I$ and $J$ are joined by an edge if and only if $I + J \in A(R)^*$.

Remark 7.2.2. By the definition of $\Omega(R)$, if $R$ is an integral domain, then $\Omega(R)$ is an empty graph.
Remark 7.2.3. Let $R$ be a finite commutative ring but not a field. Then every non-zero proper ideal is an annihilating ideal of $R$.

Example 7.2.4. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_8$. Then $V(\mathcal{G}(R)) = \{(0) \times \mathbb{Z}_8, (0) \times (2), (0) \times (4), (2) \times \mathbb{Z}_8, (2) \times (0), (2) \times (2), (2) \times (4), \mathbb{Z}_4 \times (0), \mathbb{Z}_4 \times (2), \mathbb{Z}_4 \times (4)\}$. The comaximal ideal graph $\Omega(\mathbb{Z}_4 \times \mathbb{Z}_8)$ of $R$ is given below:

![Diagram of the comaximal ideal graph Ω(ℤ₄ × ℤ₈) of R.](image-url)
7.3 Basic Properties of $\Omega(R)$

In this section, we study some fundamental properties of $\Omega(R)$. Especially we identify when the graph $\Omega(R)$ is isomorphic to some well-known graphs.

**Theorem 7.3.1.** Let $R$ be a finite commutative ring. Then $R$ is a local ring if and only if $\Omega(R)$ is a complete graph.

**Proof.** Suppose that $R$ is a local ring. Then $R$ has a unique maximal ideal, say, $m$. Note that any non-zero proper ideal of $R$ is an annihilating ideal of $R$. For any two non-zero proper ideals $I, J$ in $R$, $I + J \subseteq m$ and so $I + J$ is an annihilating ideal in $R$. By definition of $\Omega(R)$, $I$ and $J$ are adjacent in $\Omega(R)$ for all non-zero proper ideals $I, J$ in $R$ and hence $\Omega(R)$ is complete.

Conversely, assume that $\Omega(R)$ is complete. Suppose that $R$ is not a local ring. Then $R$ has at least two maximal ideals, say, $M_1$ and $M_2$. Note that $M_1 + M_2 = R$. By definition of $\Omega(R)$, $M_1 + M_2$ is not an annihilating ideal of $R$ and so $M_1$ and $M_2$ are nonadjacent in $\Omega(R)$, a contradiction. Hence $R$ is a local ring. \[\Box\]

**Theorem 7.3.2.** Let $R$ be a finite commutative non-local ring. Then $\Omega(R)$ is totally disconnected if and only if $R \cong F_1 \times F_2$ where $F_1$ and $F_2$ are fields.
Proof. Suppose that $\Omega(R)$ is totally disconnected. Then $\Omega(R)$ has no edge. Since $R$ is a finite non-local ring, $R \cong R_1 \times \cdots \times R_n$, where $(R_i, m_i)$ is a local ring and $n \geq 2$. If $n \geq 3$, then $(0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$ and $(0) \times R_2 \times R_3 \times (0) \times \cdots \times (0)$ are adjacent in $\Omega(R)$, a contradiction. Hence $n = 2$.

Suppose $m_1 \neq (0)$. Then $(0) \times R_2$ and $m_1 \times (0)$ are adjacent in $\Omega(R)$, a contradiction. Hence $R_1$ and $R_2$ are fields.

Conversely, if $R \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields, then $\Omega(R) \cong K_2$ and hence $\Omega(R)$ is totally disconnected. □

Remark 7.3.3. Let $(R, m)$ be a finite local ring. Then $\Omega(R)$ is totally disconnected if and only if $m$ is the only non-zero proper ideal of $R$. Hence in this case $\text{diam}(\Omega(R)) = \infty$.

Corollary 7.3.4. Let $R$ be a finite commutative non-local ring. Then $\text{diam}(\Omega(R)) = \infty$ if and only if $R \cong F_1 \times F_2$ where $F_1$ and $F_2$ are fields.

Proof. If $R \cong F_1 \times F_2$, where $F_1$ and $F_2$ are fields, then $\Omega(R) \cong K_2$ and hence $\text{diam}(\Omega(R)) = \infty$.

Suppose that $\text{diam}(\Omega(R)) = \infty$. Since $R$ is a finite non-local ring, $R \cong R_1 \times \cdots \times R_n$, where $(R_i, m_i)$ is a local ring and $n \geq 2$. If $n \geq 3$, then $\Omega(R)$ is connected, a contradiction. Hence $n = 2$ and $R = R_1 \times R_2$.

If $m_i \neq (0)$ for some $i$, then $\Omega(R)$ is connected, a contradiction. Hence $R_1$ and $R_2$ are fields. □
Theorem 7.3.5. Let $R$ be a finite commutative ring and $|\Omega(R)| \geq 3$. Then $\Omega(R)$ is unicyclic if and only if

(i) $R$ is a local ring which contains three non-zero proper ideals

(ii) $R = R_1 \times R_2$, where $(R_1, m_1)$ is a local ring with $m_1$ as the only non-zero proper ideal in $R_1$ and $R_2$ is a field.

Proof. Suppose that $\Omega(R)$ is unicyclic. Since $R$ is finite, $R = R_1 \times \cdots \times R_n$, where $(R_i, m_i)$ is a local ring. If $n \geq 3$, then $(0) \times R_2 \times (0) \times (0) \times \cdots \times (0) - R_1 \times (0) \times (0) \times \cdots \times (0) - (0) \times (0) \times R_3 \times (0) \times \cdots \times (0) - (0) \times R_2 \times (0) \\
R_2 \times (0) \times \cdots \times (0)$ and $R_1 \times R_2 \times (0) \times (0) \times \cdots \times (0) - (0) \times R_2 \times (0) \times \\
(0) \times \cdots \times (0) - R_1 \times (0) \times (0) \times \cdots \times (0) - R_1 \times R_2 \times (0) \times \cdots \times (0)$ are two distinct cycles in $\Omega(R)$, a contradiction. Hence $n \leq 2$.

If $n = 1$, then by Theorem 7.3.1, $\Omega(R)$ is complete. Since $\Omega(R)$ is unicyclic and $|\Omega(R)| \geq 3$, $R$ contains three non-zero proper ideals.

Suppose that $n = 2$. Then $R = R_1 \times R_2$. If $m_i \neq (0)$ for $i = 1, 2$, then $m_1 \times (0) - m_1 \times m_2 - (0) \times m_2 - m_1 \times (0)$ and $m_1 \times (0) - (0) \times \\
m_2 - m_1 \times R_2 - m_1 \times (0)$ are two distinct cycles in $\Omega(R)$, a contradiction. Hence $m_i = (0)$ for some $i$. 

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Without loss of generality, we assume that $m_2 = (0)$. Then $R_2$ is a field. Since $|\Omega(R)| \geq 3$, by Corollary 7.3.4, $\Omega(R)$ is connected and so $R_1$ is not a field. Suppose that $I$ is any non-zero proper ideal of $R_1$ with $I \neq m_1$. Then $I \times (0) - m_1 \times (0) - (0) \times R_2 - I \times (0)$ and $I \times (0) - (0) \times R_2 - m_1 \times R_2 - I \times (0)$ are two distinct cycles in $\Omega(R)$, a contradiction. Hence $m_1$ is the only non-zero proper ideal in $R_1$.

Conversely, suppose that $(i)$ and $(ii)$ holds. Then $\Omega(R) \cong K_3$ or $\Omega(R)$ is isomorphic to graph given in Fig. 6.1.

**Theorem 7.3.6.** Let $R$ be a finite commutative ring. If $\Omega(R)$ is connected, then $\Omega(R)$ is a tree if and only if $R$ is a local ring which contains two non-zero proper ideals.

**Proof.** Suppose that $R$ is a local ring which contains two non-zero proper ideals. Then by Theorem 7.3.1, $\Omega(R) \cong K_2$.

Conversely, assume that $\Omega(R)$ is a tree. Suppose $R$ is a non-local ring. Then $R = R_1 \times \cdots \times R_n$, where $(R_i, m_i)$ is a local ring and $n \geq 2$. If
If $n \geq 3$ then $R$ contains a cycle, a contradiction. Hence $n = 2$. Since $\Omega(R)$ is connected, $R_1$ and $R_2$ are not fields and so $R_i$ is not a field for some $i$. Since Fig. 6.1 is a subgraph of $\Omega(R_1 \times R_2)$, $\Omega(R)$ contains a cycle, a contradiction. Hence $R$ is a local ring and by Theorem 7.3.1, $\Omega(R)$ is complete so that $R$ contains two non-zero proper ideals. 

\[ \Box \]

### 7.4 Hamiltonian nature of $\Omega(R)$

In this section, we discuss about the Hamiltonian property of $\Omega(R)$. In view of Theorem 7.3.1, $\Omega(R)$ is Hamiltonian when $R$ is a local ring which contains at least three non-zero proper ideals.

If $R$ is finite, then $R = R_1 \times \cdots \times R_n$, where $(R_i, m_i)$ is a local ring and $n \geq 3$. Let $Max(R) = \{ M_i : M_i = R_2 \times \cdots \times R_{i-1} \times m_i \times R_{i+1} \times \cdots \times R_n, 1 \leq i \leq n \}$ be the set of all maximal ideals in $R$ and $J(R)$ be the Jacobson radical of $R$.

**Theorem 7.4.1.** Let $R$ be a finite commutative ring and $|Max(R)| \geq 3$. Then $\Omega(R)$ is Hamiltonian.

**Proof.** Let $A_i = \{ I \subseteq M_i : I$ is a non-zero proper ideal in $R \}$ for $1 \leq i \leq n$. Then $A_i \cap A_j \neq \emptyset$ for all $i \neq j$ and $V(\Omega(R)) = \bigcup_{i=1}^{n} A_i$. Clearly the subgraph $\langle A_i \rangle$ induced by $A_i$ is a complete subgraph of $\Omega(R)$ and also $\langle A_i \cap A_j \rangle$ is a complete subgraph of $\Omega(R)$. Let $I_{i(i+1)} \in A_i \cap A_{i+1}$
for \(1 \leq i \leq n - 1\) and \(I_{n1} \in A_n \cap A_1\). Now we start with the vertex \(M_1\), traverse all vertices in \(\langle A_1 - \{I_{i(i+1)}, I_{n1} : 1 \leq i \leq n - 1\} \rangle\) through a spanning path in \(\langle A_1 - \{I_{i(i+1)}, I_{n1} : 1 \leq i \leq n - 1\} \rangle\), pass on to \(I_{12}\), traverse vertices in \(\langle A_2 - \{I_{i(i+1)}, I_{n1} : 2 \leq i \leq n - 1\} \rangle\) through a spanning path in \(\langle A_2 - \{I_{i(i+1)}, I_{n1} : 2 \leq i \leq n - 1\} \rangle\) and pass on to \(I_{23}\). Continuing this process through \(\langle A_3 - \{I_{i(i+1)}, I_{n1} : 3 \leq i \leq n - 1\} \rangle\), \(\langle A_3 - \{I_{i(i+1)}, I_{n1} : 3 \leq i \leq n - 1\} \rangle\), \(\langle A_4 - \{I_{i(i+1)}, I_{n1} : 4 \leq i \leq n - 1\} \rangle\), \ldots, \(\langle A_n - \{I_{n1}\} \rangle\) we get a Hamiltonian path at \(I_{n1}\). From this Hamiltonian path together with the edge joining \(M_1\) and \(I_{n1}\) we get the required Hamiltonian cycle in \(\Omega(R)\). Hence \(\Omega(R)\) is Hamiltonian. \(\square\)

**Corollary 7.4.2.** Let \(R\) be a finite commutative ring and \(|\text{Max}(R)| = 2\). If the condition (ii) in Theorem 7.3.5 is not holds, then \(\Omega(R)\) is Hamiltonian.

**Proof.** Analogous to the proof of Theorem 7.4.1 \(\square\)

### 7.5 Genus of \(\Omega(R)\)

In this section, we characterize all commutative rings \(R\) for which \(\Omega(R)\) is planar. Also we determine all isomorphism classes of finite commutative rings with identity whose \(\Omega(R)\) has genus one.
First let us characterize finite commutative rings $R$ for which genus of $AG(R)$ is zero.

**Theorem 7.5.1.** Let $R \cong R_1 \times \cdots \times R_n$ be a finite commutative ring with identity, where each $(R_i, m_i)$ is a local ring but not a field and $n \geq 1$. Then $\Omega(R)$ is planar if and only if $R$ is a local ring and $R$ contains at most four non-zero proper ideals.

**Proof.** Assume that $\Omega(R)$ is planar. Suppose $n \geq 2$. Let $A = \{m_1 \times 0, 0 \times m_2, m_1 \times m_2, R_1 \times 0, m_1 \times R_2, R_1 \times m_2\} \subseteq V(\Omega(R))$. Then the subgraph induced by $A$ in $\Omega(R)$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence $n = 1$, $R$ is local and by Theorem 7.3.1 , $\Omega(R)$ is complete. Since $\Omega(R)$ is planar, $R$ contains at most four non-zero proper ideals.

Conversely, suppose $R$ is a local ring which contains at most four non-zero proper ideals. Then by Theorem 7.3.1, $\Omega(R) \cong K_n$, where $1 \leq n \leq 4$ and hence $\Omega(R)$ is planar. \qed

**Theorem 7.5.2.** Let $R \cong F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where each $F_i$ is a field and $n \geq 2$. Then $\Omega(R)$ is planar if and only if $n = 2$ or $3$.

**Proof.** Suppose $\Omega(R)$ is planar. Suppose $n \geq 4$. Let $A = \{0 \times F_2 \times F_3 \times \cdots \times F_n, 0 \times 0 \times F_3 \times \cdots \times F_n, 0 \times F_2 \times 0 \times \cdots \times F_n, 0 \times F_2 \times F_3 \times$
\( \cdots \times F_n, 0 \times 0 \times 0 \times F_4 \times \cdots \times F_n \subseteq V(\Omega(R)) \). Then the subgraph induced by \( A \) in \( \Omega(R) \) contains \( K_5 \) as a subgraph, a contradiction. Hence \( n \leq 3 \).

Suppose \( n = 2 \). Then \( R \cong F_1 \times F_2 \) and by Theorem 7.3.1, \( \Omega(R) = \overline{K}_{2} \).

Suppose \( n = 3 \). Then \( R \cong F_1 \times F_2 \times F_3 \). Then \( V(\Omega(R)) = \{ 0 \times F_2 \times F_3, F_1 \times 0 \times F_3, F_1 \times F_2 \times 0, 0 \times 0 \times F_3, 0 \times F_2 \times 0, F_1 \times 0 \times 0 \} \).

![Figure 6.2: \( \Omega(F_1 \times F_2 \times F_3) \)](image)

Converse follows from Fig. 6.2. \( \square \)

**Theorem 7.5.3.** Let \( R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m \) be a finite commutative ring with identity but not a field, where each \((R_i, m_i)\) is a local ring and \( F_j \) is a field. Then \( \Omega(R) \) is planar if and only if \( n = 1, m = 1 \) and \( R_1 \) contains exactly one proper ideal.

**Proof.** Assume that \( \Omega(R) \) is planar. Suppose \( n \geq 2 \). Then by Theorem 7.5.1, \( \Omega(R) \) is non-planar, a contradiction. Hence \( n = 1 \). Suppose \( m \geq 2 \). Let \( A = \{ I \subseteq m_1 \times F_1 \times \cdots \times F_m : I \neq (0), I \text{ is an ideal } \} \). Then \( |A| \geq 7 \) and so the subgraph induced by \( A \) in \( \Omega(R) \) contains \( K_7 \) as a subgraph, a contradiction. Hence \( m = 1 \) and \( R = R_1 \times F_1 \).
Suppose $R_1$ contains two proper ideals. Let $I_1, m_1$ be two proper ideals with $m_1 \neq I_1$. Let $B = \{I \subseteq m_1 \times F_1, I \neq 0, I$ is an ideal $\}$. Then $|B| \geq 5$ and $\langle B \rangle \cong K_5$ so that $\Omega(R)$ contains $K_5$ as a subgraph, a contradiction. Hence $R_1$ contains a unique proper ideal.

Conversely, suppose $n = m = 1$ and $R_1$ contains unique proper ideal $m_1$. Then $V(\Omega(R)) = \{m_1 \times F_1, 0 \times F_1, m_1 \times 0, R_1 \times 0\}$ and hence $\Omega(R)$ is isomorphic to Fig 6.3.

\[\begin{array}{c}
m_1 \times F_1 \\
\downarrow \\
m_1 \times (0) \\
\downarrow \\
R_1 \times (0) \\
\end{array}\]

Fig. 6.3: $\Omega(R_1 \times F_1)$

\textbf{Theorem 7.5.4.} Let $R$ be a finite local ring but not a field. Then $g(\Omega(R)) = 1$ if and only if $R$ contains at most $n$ non-zero proper ideals, where $5 \leq n \leq 7$.

\textbf{Proof.} Assume that $g(\Omega(R)) = 1$. Then by Theorem 7.5.1, $R$ contains at least 5 proper ideals. Since $R$ is local, by Theorem 7.3.1, $\Omega(R)$ is complete and hence $R$ contains at most $n$ non-zero proper ideals, where $5 \leq n \leq 7$.

Conversely, suppose $R$ contains at most $n$ non-zero proper ideals, where $5 \leq n \leq 7$. Note that $\Omega(R)$ is complete so that $g(\Omega(R)) = 1$. \qed
Theorem 7.5.5. Let $R \cong R_1 \times \cdots \times R_n$ be a finite commutative ring with identity, where each $(R_i, m_i)$ is a local ring but not a field and $n \geq 2$. Then $g(\Omega(R)) = 1$ if and only if $n = 2$ and each $R_i$ contains exactly one non-zero proper ideal.

Proof. Assume that $g(\Omega(R)) = 1$. Suppose $n \geq 3$. Let $A = \{ I \subseteq M_1 : I \neq 0, I$ is an ideal $\} \subseteq V(\Omega(R))$. Then $|A| \geq 17$ and so the subgraph induced by $A$ in $\Omega(R)$ contains $K_{17}$ as a subgraph so that $g(\Omega(R)) \geq 4$, a contradiction. Hence $n = 2$ and so $R \cong R_1 \times R_2$.

Suppose $R_1$ contains two proper ideals. Let $I, m_1$ be two non-zero proper ideals of $R_1$ with $I \neq m_1$. Let $B = \{ I \subseteq M_1; I \neq 0, I$ is an ideal $\}$. Then $|B| \geq 8$ and so the subgraph induced by $B$ in $\Omega(R)$ contains $K_8$ as a subgraph. Thus so $g(\Omega(R)) \geq 2$, a contradiction. Hence each $R_i$ contains exactly one non-zero proper ideal.

Conversely, assume that $n = 2$ and each $R_i$ contains exactly one non-zero proper ideal. Then $|V(\Omega(R))| = 7$ and so $\Omega(R)$ is a subgraph of $K_7$. Since $g(K_7) = 1$, $g(\Omega(R)) = 1$. \hfill \Box

Theorem 7.5.6. Let $R \cong F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where each $F_i$ is a field and $n \geq 4$. Then $g(\Omega(R)) > 1$.

Proof. As in the proof of Theorem 7.5.2, $\Omega(R)$ is non-planar and so $g(\Omega(R)) \geq 1$. Suppose $n \geq 5$. Let $A = \{ I \subseteq 0 \times F_2 \times \cdots \times F_n : I \neq 0, I$ is an ideal $\}$. Then $|A| \geq 8$ and so the subgraph induced by
A in $\Omega(R)$ contains $K_8$ as a subgraph so that $g(\Omega(R)) \geq 2$. Hence $g(\Omega(R)) > 1$.

Suppose $n = 4$. Let $B = \{ F_1 \times F_2 \times F_3 \times 0, F_1 \times F_2 \times 0 \times 0, 0 \times F_2 \times F_3 \times 0, F_1 \times 0 \times F_3 \times 0, 0 \times 0 \times F_3 \times F_4, 0 \times 0 \times 0 \times F_4, 0 \times F_2 \times 0 \times F_4, 0 \times F_2 \times F_3 \times F_4, \} \subseteq V(\Omega(R))$. Then the graph induced by $B$ in $\Omega(R)$ contains $H$ as a subgraph, where $H = 2K_4 + K_1$. Since $g(H) > 1$, $g(\Omega(R)) > 1$.

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$\blacksquare$
\end{flushright}

Theorem 7.5.7. Let $R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity but not a field, where each $(R_i, m_i)$ is a local ring, $F_j$ is a field and $n, m \geq 1$. Then $g(\Omega(R)) = 1$ if and only if $n = m = 1$ and $R_1$ contains $k$ non-zero proper ideals, where $k = 2, 3$.

Proof. Assume that $g(\Omega(R)) = 1$. Suppose $n \geq 2$. Let $A = \{ I \subseteq M_1 : I \neq 0, I$ is an ideal $\}$. Then the subgraph induced by $A$ in $\Omega(R)$ contains $K_{11}$ as a subgraph and so $g(\Omega(R)) > 1$, a contradiction. Hence $n = 1$.

Suppose $n = 1$ and $m \geq 3$. Then $n + m \geq 4$. Clearly $\Omega(F_1 \times F_2 \times F_3 \times F_4)$ is a subgraph of $\Omega(R)$. But by Theorem 7.5.6, $g(\Omega(F_1 \times F_2 \times F_3 \times F_4)) > 1, g(\Omega(R)) > 1$, a contradiction. Hence $m = 1$ or 2.
Suppose \( m = 2 \). Then \( R = R_1 \times F_1 \times F_2 \). Let \( B = \{ I \subseteq R : I \neq 0 \text{ and } I \neq R, I \text{ is an ideal} \} \subseteq V(\Omega(R)) \). Then \( |B| \geq 10 \), the subgraph induced by \( B \) in \( \Omega(R) \) contains \( K_{10} \) as a subgraph and so \( g(\Omega(R)) > 1 \), a contradiction. Hence \( m = 1 \) and so \( R = R_1 \times F_1 \).

Suppose \( R_1 \) contains at least 4 proper ideals. Let \( m_1, I_1, I_2, I_3 \) be four proper ideals in \( R_1 \) with \( m_1 \neq I_1 \neq I_2 \neq I_3 \). Let \( C = \{ I \subseteq M_1 : I \neq 0, I \text{ is an ideal} \} \). Then \( |C| \geq 9 \), the subgraph induced by \( C \) in \( \Omega(R) \) contains \( K_9 \) as a subgraph and so \( g(\Omega(R)) > 1 \), a contradiction.
Theorem 7.5.3, $R_1$ contains $k$ non-zero proper ideals, where $k = 2, 3$.

Conversely, suppose $R_1$ contains two non-zero proper ideals. Then $V(\Omega(R)) = \{x_1 = 0 \times F_1, x_2 = R_1 \times 0, x_3 = m_1 \times 0, x_4 = I \times 0, x_5 = m_1 \times F_1, x_6 = I \times F_1\}$, $K_5$ is a subgraph of $\Omega(R)$ and so $g(\Omega(R)) \geq 1$.

However, we can draw $\Omega(R)$ on the surface of a torus, see Fig. 6.4. Hence $g(\Omega(R)) = 1$.

Suppose $R_1$ contains three non-zero proper ideals. Then $V(\Omega(R)) = \{x_1 = 0 \times F_1, x_2 = m_1 \times 0, x_3 = I_1 \times 0, x_4 = I_2 \times 0, x_5 = I_2 \times F_1, x_6 = I_1 \times F_1, x_7 = m_1 \times F_1, x_8 = R_1 \times 0\}$ and by Theorem 7.5.3, $g(\Omega(R)) \geq 1$.

However, we can draw $\Omega(R)$ on the surface of a torus, see Fig. 6.5. Hence $g(\Omega(R)) = 1$. \hfill \square

### 7.6 Isomorphism Properties of $\Omega(R)$

Consider the question: If $R$ and $S$ are two rings with $\Omega(R) \cong \Omega(S)$, then do we have $R \cong S$? The following example shows that the above question is not valid in general.

**Example 7.6.1.** Let $R = \mathbb{Z}_{25} \times \mathbb{Z}_{13}$ and $S = \mathbb{Z}_9 \times \mathbb{Z}_{29}$. Then $\Omega(R) \cong \Omega(S)$(see. Fig. 6.6). But $R$ and $S$ are not isomorphic.
Theorem 7.6.2. Let $R = \prod_{i=1}^{n} R_i \times \prod_{j=1}^{m} F_j$ and $S = \prod_{i=1}^{n} R'_i \times \prod_{j=1}^{m} F'_j$ be finite commutative rings with $n+m \geq 2$, where each $(R_i, m_i)$ and $(R'_i, m'_i)$ are local rings which are not fields and $F_i$ and $F'_j$ are fields. Let $k_i$ be the number of ideals in $R_i$ and $k'_i$ be the number of ideals in $R'_i$. Then $\Omega(R) \cong \Omega(S)$ if and only if $k_i = k'_i$ for all $i$, $1 \leq i \leq n$.

Proof. If $R \cong S$, then the result is obvious. Assume that $R \not\cong S$. Suppose $k_i = k'_i$ for all $i$, $1 \leq i \leq n$. Then $|V(\Omega(R))| = |V(\Omega(S))|$. Let $\mathbb{I}_j(R_j) = \{I_{1j} = (0), I_{2j} = m_j, I_{3j}, \ldots, I_{k_j} = R_j\}$ be the set of ideals in $R_j$ and $\mathbb{I}'_j(R'_j) = \{I'_{1j} = (0), I'_{2j} = m_j, I'_{3j}, \ldots, I'_{k'_j} = R'_j\}$ be the set of ideals in $R'_j$. Then the map $I_{ij} \mapsto I'_{ij}$ is a bijection from $\mathbb{I}_j(R_j)$ onto $\mathbb{I}'_j(R'_j)$. Define $\psi : V(\Omega(R)) \longrightarrow V(\Omega(S))$ by $\psi(\prod_{i=1}^{n} I_{ii} \times \prod_{j=1}^{m} J_j) = \prod_{i=1}^{n} I'_{ti} \times \prod_{j=1}^{m} J'_j$ where

$$J'_j = \begin{cases} F'_j & \text{if } J_j = F_j \\ (0) & \text{if } J_j = (0) \end{cases}$$
Then $\psi$ is well-defined and bijective. Let $I = \prod_{i=1}^n I_i \times \prod_{j=1}^m J_j$ and $J = \prod_{i=1}^n A_i \times \prod_{j=1}^m B_j$ be two non-zero ideals in $R$. Suppose $I$ and $J$ are adjacent in $\Omega(R)$. Then $I + J$ is an annihilating ideal of $R$ and so $I_i + A_i \subseteq m_i$ for some $i$ or $J_j + B_j = (0)$ for some $j$. From this, $I_i, A_i \subseteq m_i$ or $J_j = (0)$ and $B_j = (0)$.

Let $\psi(I) = \prod_{i=1}^n I'_i \times \prod_{j=1}^m J'_j$ and $\psi(J) = \prod_{i=1}^n A'_i \times \prod_{j=1}^m B'_j$. By definition of $\psi$, $I'_i + A'_i \subseteq m'_i$ for some $i$ or $J'_j + B'_j = (0)$ for some $j$ and so $\psi(I) + \psi(J) \neq S$. Hence $\psi(I)$ and $\psi(J)$ are adjacent in $\Omega(S)$. Similarly one can prove that $\psi$ preserves non-adjacency also. Hence $\Omega(R) \cong \Omega(S)$.

Conversely, assume that $\Omega(R) \cong \Omega(S)$. Suppose $k_i \neq k'_i$ for some $i$. Then $|V(\Omega(R))| \neq |V(\Omega(S))|$, a contradiction. Hence $k_i = k'_i$ for all $i$.

$\square$

**Corollary 7.6.3.** Let $R_1 = \prod_{i=1}^n F_i$ and $R_2 = \prod_{j=1}^m F'_j$, where $F_i$ and $F'_j$ are fields and $n \geq 2$. Then $\Omega(R_1) \cong \Omega(R_2)$.

**Corollary 7.6.4.** Let $R = \prod_{i=1}^n R_i$ and $S = \prod_{i=1}^n R'_i$ be finite commutative rings with $n \geq 2$, where each $(R_i, m_i)$ and $(R'_i, m'_i)$ are local rings which are not field. Let $k_i$ be the number of ideals in $R_i$ and $k'_i$ be the number of ideals in $R'_i$. Then $\Omega(R) \cong \Omega(S)$ if and only if $k_i = k'_i$ for all $i$, $1 \leq i \leq n$. 

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