Chapter 6

Comaximal ideal graphs of commutative rings

In this Chapter, A dominating set of the comaximal ideal graph $\mathcal{I}(R)$ is constructed using elements of the center when $R$ is a finite commutative ring. We find the radius, center and median of the comaximal ideal graph $\mathcal{I}(R)$ and we prove that the domination number of $\mathcal{I}(R)$ is equal to the number of factors in the Artinian decomposition of $R$. Also, we characterize all finite commutative rings(non-local rings) with identity for which $\mathcal{I}(R)$ is planar. The results obtained in this direction are communicated to Discrete Mathematics, Algorithms and Applications.
6.1 Definitions and Examples

Definition 6.1.1. Let \( R \) be a commutative ring with identity. The comaximal ideal graph \( \mathcal{G}(R) \) of \( R \) is a simple graph with its vertices are the proper ideals of \( R \) which are not contained in the Jacobson radical of \( R \), and two vertices \( I_1 \) and \( I_2 \) are adjacent if and only if \( I_1 + I_2 = R \).

Remark 6.1.2. Let \( R \) be a ring. Then \( \mathcal{G}(R) \) is the empty graph if and only if \( R \) is a local ring.

Remark 6.1.3. Let \( R = F_1 \times F_2 \), where \( F_1 \) and \( F_2 \) are fields. Then \( \mathcal{G}(R) \cong K_2 \).

Example 6.1.4. Let \( R = \mathbb{Z}_4 \times \mathbb{Z}_8 \). Then \( V(\mathcal{G}(R)) = \{ (0) \times \mathbb{Z}_8, \mathbb{Z}_4 \times (0), (2) \times \mathbb{Z}_8, \mathbb{Z}_4 \times (2), \mathbb{Z}_4 \times (4) \} \). The comaximal ideal graph \( \mathcal{G}(\mathbb{Z}_4 \times \mathbb{Z}_8) \) of \( R \) is given below:

![Diagram of the comaximal ideal graph](image_url)
6.2 Central sets in $\mathcal{I}(R)$

In this section, we find certain central sets in the comaximal ideal graph and use the same to obtain the value of certain domination parameters of the comaximal ideal graph. By Theorem 1.3.14, if $|Max(R)| = 2$, then the radius of $\mathcal{I}(R)$ is either one or two. Hence in this section, we assume that $R$ is a finite commutative ring with $|Max(R)| \geq 3$.

Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with $|Max(R)| \geq 3$, where each $(R_i, m_i)$ is a local ring but not a field and $F_j$ is a field. Then $Max(R) = \{M_1, \ldots, M_n, M'_1, \ldots, M'_m\}$, where $M_i = R_1 \times \cdots \times R_{i-1} \times m_i \times R_{i+1} \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ and $M'_k = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_{k-1} \times (0) \times F_{k+1} \times \cdots \times F_m$ for $1 \leq i \leq n$ and $1 \leq k \leq m$.

**Remark 6.2.1.** Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with $|Max(R)| \geq 3$, where each $(R_i, m_i)$ is a local ring but not a field and $F_j$ is a field. Then $\Delta(\mathcal{I}(R)) = deg_{\mathcal{I}(R)}(M)$ for some $M \in Max(R)$.

**Theorem 6.2.2.** Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with $|Max(R)| \geq 3$, where each $(R_i, m_i)$ is a local ring but not a field and $F_j$ is a field. Then the radius of $\mathcal{I}(R)$ is 2 and the
center of \( \mathcal{I}(R) \) is \( \text{Max}(R) \).

**Proof.** Let \( I = \prod_{i=1}^{n} I_i \times \prod_{k=1}^{m} I'_k \) be any ideal of \( R \) with \( I \not\subset \mathcal{I}(R) \), where \( I_i \) is an ideal in \( R_i \) and \( I'_k \) is an ideal in \( F_k \).

**Case 1.** \( m = 0 \).

Then \( I = \prod_{i=1}^{n} I_i \). By the assumption that \( |\text{Max}(R)| \geq 3 \), \( n \geq 3 \). For any ideal \( K \) in \( \mathcal{I}(R) \), \( K \) is adjacent to some maximal ideal in \( R \) and \( M_i + M_j = R \) for \( i \neq j \).

Suppose \( I \) is maximal. Then \( I = M_i \) for some \( i \). Note that each \( M_i \) is non-adjacent only to \( J = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_n \) for every ideal \( I_i \subset \mathfrak{m}_i \). Then by definition, \( J + M_k = R \) for all \( k \neq i \) and so \( I - M_k - J \) is a path of length 2 in \( \mathcal{I}(R) \). Hence \( e(I) = 2 \) and so \( e(I) = 2 \) for all \( I \in \text{Max}(R) \).

Suppose \( I \) is not maximal. Then \( I \subset M_i \) for some \( i \) and so \( M_i + I \neq R \). If \( I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_n \) for \( I_i \subset \mathfrak{m}_i \), then there exist an ideal \( I' = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_{j-1} \times I_j \times R_{j+1} \times R_n (i \neq j) \) for some \( I_j \subset \mathfrak{m}_j \) such that \( I' + M_j \neq R \), \( I + I' \neq R \) and \( I + M_j = R \). Since \( I' + M_k = R \) for some \( k \neq i,k \), \( I - M_j - M_k - I' \) is a path of length 3 and hence \( e(I) = 3 \). From this, we have \( e(I) = 3 \) for every ideal \( I \not\subset \mathcal{I}(R) \) and \( I \notin \text{Max}(R) \).

**Case 2.** \( n = 0 \).

Then \( m \geq 3 \) and \( I = \prod_{k=1}^{m} I'_k \). Suppose \( I \) is maximal. Then \( I = M'_k \) for some \( k \) and \( I \) is not adjacent to \( J \) for all \( J = \prod_{k=1}^{m} J'_k \) with \( J'_k = (0) \),
$J \neq M_k'$ and $J \notin \mathcal{J}(R)$. Since $J + M_t' = R$ for some $t$, $I - M_t' - J$ is a path of length 2, $e(I) = 2$ and hence $e(I) = 2$ for all $I \in Max(R)$.

If $I$ is not maximal, then $I_i' = (0)$ and $I_i' = (0)$ for some $i \neq t$ and so $I + M_i' \neq R$, $I + M_t' \neq R$. Since $m \geq 3$, there exist an ideal $I' = \prod_{i=1}^{m} J_i' \notin \mathcal{J}(R)$ with $J_i' = (0)$ such that $I + I' \neq R$, $I' + M_i \neq R$ and $I' + M_t' = R$. Since $I' + M_j' = R$ for $j \neq i, t$, $I - M_j - M_t' - J$ is a path of length 3 and so $e(I) = 3$. From this, we have $e(I) = 3$ for every ideals $I \notin \mathcal{J}(R)$ and $I \notin Max(R)$.

**Case 3.** $n \geq 1$ and $m \geq 1$.

Then $n + m \geq 3$. Let $I$ be any nonzero ideal of $R$ with $I \notin \mathcal{J}(R)$. Suppose $I$ is maximal. Note that any ideal is adjacent to some maximal ideal. If $J$ is an ideal not adjacent to $I$, then $J + M = R$ for some maximal ideal $M$ in $R$, $M \neq I$ and so $I - M - J$ is a path of length 2. Hence $e(I) = 2$ for all $I \in Max(R)$.

Suppose $I$ is not maximal. Then $I \subset M$ for some $M \in Max(R)$. As in the proof of case 1 and 2, we can find ideals $I' \notin \mathcal{J}(R)$ and $M', M'' \in Max(R)$ such that $I'$ is not maximal, $I - M' - M'' - I'$ is a path of length 3 so that $e(I) = 3$. From this, we have $e(I) = 3$ for all ideals $I \notin \mathcal{J}(R)$ and $I \notin Max(R)$.

Hence in all cases, the center of $\mathcal{J}(R)$ is $Max(R)$. \qed

**Theorem 6.2.3.** Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with $|Max(R)| \geq 3$, where each $(R_i, m_i)$ is a local ring.
but not a field and $F_j$ is a field. Then the median is a subset of the center of $\mathcal{I}(R)$.

**Proof.** By Theorem 6.2.2, the radius of $\mathcal{I}(R)$ is 2 and the center of $\mathcal{I}(R)$ is $Max(R)$. Let $k$ be the number of proper ideals in $\mathcal{I}(R)$. Let $I$ be any ideal in $R$ with $I \not\subseteq J(R)$. Suppose $I$ is maximal. Then

\[ s(I) = deg_{\mathcal{I}(R)}(I) + 2(k - 1 - deg_{\mathcal{I}(R)}(I)) = 2k - deg_{\mathcal{I}(R)}(I) - 2 \quad (6.1) \]

Note that equation 6.1 implies that all the vertices of the median must have the same degree. If $J$ is any ideal in $\mathcal{I}(R)$ that is not maximal, then there exists an ideal $J'$ such that $d(J, J') = 3$ and so

\[ s(J) > deg_{\mathcal{I}(R)}(J) + 2(k - 1 - deg_{\mathcal{I}(R)}(J)) = 2k - deg_{\mathcal{I}(R)}(J) - 2 \quad (6.2) \]

Thus there is a maximal ideal $I$ with $s(I) < s(J)$ for $J \not\subseteq Max(R)$ and so any ideal not in the center of $\mathcal{I}(R)$ cannot be in the median of $\mathcal{I}(R)$. Hence the median is a subset of the center of $\mathcal{I}(R)$. \hfill \Box

**Corollary 6.2.4.** Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with $|Max(R)| \geq 3$, where each $(R_i, m_i)$ is a local ring but not a field and $F_j$ is a field. Then the median of $\mathcal{I}(R)$ is $\{M'_j : 1 \leq j \leq m\}$.

**Proof.** Let $k_i$ be the number of ideals in $R_i$ for $1 \leq i \leq n$. Then
Clearly by the definition of \( \mathcal{G}(R) \), 
\[
\deg_{\mathcal{G}}(M_i) = 2^m \prod_{t=1 \atop t \neq i}^n k_t - 1,
\]
\[
\deg_{\mathcal{G}}(M_j') = 2^{m-1} \prod_{t=1 \atop t \neq i}^n k_t - 1
\]
and so \( \deg_{\mathcal{G}}(M_i) < \deg_{\mathcal{G}}(M_j') \). By equation 6.1, \( s(M_j') < s(M_i) \). Since \( \deg_{\mathcal{G}}(M_j') = \deg_{\mathcal{G}}(M_j) \) for all \( j \neq \ell \), \( s(M_j') = s(M_j) \) for all \( j \neq \ell \) and hence the median of \( \mathcal{G}(R) \) is \( \{M_j' : 1 \leq j \leq m\} \).

\[\square\]

**Corollary 6.2.5.** Let \( R = F_1 \times \cdots \times F_m \) be a finite commutative ring with \( |\text{Max}(R)| \geq 3 \), where each \( F_i \) is a field. Then the median and center of \( \mathcal{G}(R) \) are equal.

**Proof.** This follows from Corollary 6.2.3. \[\square\]

**Theorem 6.2.6.** Let \( R = R_1 \times \cdots \times R_n \) be a finite commutative ring with \( |\text{Max}(R)| \geq 3 \), where each \( (R_i, m_i) \) is a local ring but not a field. Let \( k_i \) be the number of ideals in \( R_i \) for \( 1 \leq i \leq n \). Then the median and center of \( \mathcal{G}(R) \) are equal if and only if \( k_i = k_j \) for all \( i \neq j \).

**Proof.** Suppose \( k_i = k_j \) for all \( i \neq j \). Then by definition of \( \mathcal{G}(R) \), 
\[
\deg_{\mathcal{G}(R)}(M_i) = \deg_{\mathcal{G}(R)}(M_j)
\]
for all \( i \neq j \). By Theorems 6.2.2 and 6.2.3, the median of \( \mathcal{G}(R) \) is \( \text{Max}(R) \).

Conversely, assume that the median and center of \( \mathcal{G}(R) \) are equal. As in the proof of Theorem 6.2.2, the median of \( \mathcal{G}(R) \) is \( \text{Max}(R) \). Suppose \( k_i \neq k_j \) for some \( i \neq j \). Without loss of generality, we assume that
\( k_i < k_j \). Then \( \deg_{G(R)}(M_j) < \deg_{G(R)}(M_i) \) and so \( s(M_i) < s(M_j) \), a contradiction.

The following result proved by Meng Ye et al.[54, Theorem 4.8] is used frequently.

**Theorem 6.2.7.** [54, Theorem 4.8] (1) For a ring \( R \), \( G(R) \) is the finite complete bipartite graph \( K_{n,m} \) (where \( n \) and \( m \) are finite integers) if and only if \( R \cong R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are artinian local rings with \( n + 1 \) and \( m + 1 \) ideals respectively.

(2) For a ring \( R \), \( G(R) \) is a finite star graph \( K_{1,n} \) if and only if \( R \cong F \times R_1 \), where \( F \) is a field and \( R_1 \) is an artinian local ring with exactly \( n + 1 \) ideals.

In view of Theorem 6.2.7(2), we have the following, \( \gamma(G(R)) = 1 \) if and only if \( R \cong F \times R_1 \), where \( F \) is a field and \( R_1 \) is an artinian local ring. Also \( \gamma(G(R)) = 2 \) if and only if \( R \cong R_1 \times R_2 \), where each \( R_i \) is an artinian local ring but not a field.

**Theorem 6.2.8.** Let \( R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m \) be a finite commutative ring with \( |\text{Max}(R)| \geq 3 \), where each \( (R_i, m_i) \) is a local ring but not a field and \( F_j \) is a field. Then \( \gamma(G(R)) = |\text{Max}(R)| \).

**Proof.** Let \( D = \{M_1, \ldots, M_n, M_1', \ldots, M_m'\} = \text{Max}(R) \). Let \( I \) be any ideal in \( G(R) \). Then by definition, \( I \) is adjacent to some maximal ideal
in \( R \). Hence \( D \) is a dominating set of \( \mathcal{G}(R) \) and so \( \gamma(\mathcal{G}(R)) \leq n + m \).

Suppose \( S \) is a dominating set for \( \mathcal{I}(R) \). Since \( \mathcal{I}(R) \) has no vertex adjacent to all others, \( |S| \geq 2 \). For each \( k = 1, 2, \ldots, n + m \), let \( A_k = \prod_{i=1}^{n+m} I_i \), where \( I_k = (0) \) and \( I_t = R_t \) or \( F_t \) for all \( t \neq k \). Then each \( A_k \) is a vertex of \( \mathcal{I}(R) \). For each \( k \), either \( A_k \in S \) or there is an element adjacent to \( A_k \) in \( S \). That is, for each \( k \), either \( A_k \in S \) or there is an element of the form \( (0) \times \cdots \times (0) \times J_k \times (0) \times \cdots \times (0) \in S \), where \( J_k = R_k \) or \( F_k \). Thus \( S \) contains at least \( n + m \) elements and so \( \gamma(\mathcal{I}(R)) = n + m \).

\[ \square \]

In view of Theorem 6.2.8, \( Max(R) \) is a \( \gamma \)-set of \( \mathcal{I}(R) \).

**Theorem 6.2.9.** Let \( R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m \) be a finite commutative ring with \( |Max(R)| \geq 3 \), where each \( (R_i, m_i) \) is a local ring but not a field and \( F_j \) is a field. Then \( i(\mathcal{G}(R)) \leq |V(\mathcal{I}(R))| - \Delta(\mathcal{I}(R)) \).

**Proof.** For each \( M \in Max(R) \), \( V(\mathcal{G}(R)) - N_{\mathcal{G}(R)}(M) \) is an independent set of \( \mathcal{G}(R) \). Let \( M \in Max(R) \) with \( \Delta(\mathcal{I}(R)) = deg_{\mathcal{I}(R)}(M) \). Then \( V(\mathcal{G}(R)) - N_{\mathcal{G}(R)}(M) \) is an independent dominating set of \( \mathcal{I}(R) \) and so \( i(\mathcal{I}(R)) \leq |V(\mathcal{I}(R))| - \Delta(\mathcal{I}(R)). \) \[ \square \]
6.3 Isomorphism Properties of $\mathcal{J}(R)$

Consider the question: If $R$ and $S$ are two rings with $\mathcal{J}(R) \cong \mathcal{J}(S)$, then do we have $R \cong S$? The following example shows that the above question is not valid in general.

Example 6.3.1. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ and $S = \mathbb{Z}_7 \times \mathbb{Z}_{11} \times \mathbb{Z}_{13}$. Then $\mathcal{J}(R) \cong \mathcal{J}(S)$ (see Fig. 4.1). But $R$ and $S$ are not isomorphic.

![Fig. 4.1. $\mathcal{J}(R) \cong \mathcal{J}(S)$](image_url)

Theorem 6.3.2. Let $R = \prod_{i=1}^{n} R_i \times \prod_{j=1}^{m} F_j$ and $S = \prod_{i=1}^{n} R_i' \times \prod_{j=1}^{m} F_j'$ be finite commutative rings with $n + m \geq 2$, where each $(R_i, m_i)$ and $(R_i', m_i')$ are local rings which are not fields and each $F_i$ and $F_j$ are field. Let $k_i$ be the number of ideals in $R_i$ and $k_i'$ be the number of ideals in $R_i'$. Then $\mathcal{J}(R) \cong \mathcal{J}(S)$ if and only if $k_i = k_i'$ for all $i, 1 \leq i \leq n$.

Proof. If $R \cong S$, then the result is obvious. Assume that $R \not\cong S$. Suppose $k_i = k_i'$ for all $i, 1 \leq i \leq n$. Then $|V(\mathcal{J}(R))| = |V(\mathcal{J}(S))|$. Let $\mathcal{J}_j(R_j) = \{I_{1j} = (0), I_{2j} = m_j, I_{3j}, \ldots, I_{k_jj} = R_j\}$ be the set of ideals in $R_j$ and $\mathcal{J}_j'(R_j') = \{I_{1j}' = (0), I_{2j}' = m_j, I_{3j}', \ldots, I_{k_jj}' = R_j'\}$ be the
set of ideals in $R'_j$. Then the map $I_{tj} \rightarrow I'_{tj}$ is a bijection from $\mathcal{I}_j(R_j)$ onto $\mathcal{I}'_j(R'_j)$. Define $\phi : V(\mathcal{I}(R)) \rightarrow V(\mathcal{I}(S))$ by $\phi(\prod_{i=1}^{n} I_i \times \prod_{j=1}^{m} J_j) = \prod_{i=1}^{n} I'_{ti} \times \prod_{j=1}^{m} J'_j$ where

$$J'_j = \begin{cases} F'_j & \text{if } J_j = F_j \\ (0) & \text{if } J_j = (0) \end{cases}$$

Then $\phi$ is well-defined and bijective. Let $I = \prod_{i=1}^{n} I_i \times \prod_{j=1}^{m} J_j$ and $J = \prod_{i=1}^{n} A_i \times \prod_{j=1}^{m} B_j$ be two non-zero ideals in $R$. Suppose $I$ and $J$ are adjacent in $\mathcal{I}(R)$. Then $I + J = R$ and so $I_i + A_i = R_i$, $J_j + B_j = F_j$ for all $i, j$. Let $\phi(I) = \prod_{i=1}^{n} I'_i \times \prod_{j=1}^{m} J'_j$ and $\phi(J) = \prod_{i=1}^{n} A'_i \times \prod_{j=1}^{m} B'_j$. By definition of $\phi$, $I'_i + A'_i = R'_i$ and $J'_j + B'_j = F_j$ for all $i, j$ and so $\phi(I) + \phi(J) = S$. Hence $\phi(I)$ and $\phi(J)$ are adjacent in $\mathcal{I}(S)$. Similarly one can prove that $\phi$ preserves non-adjacency also. Hence $\mathcal{I}(R) \cong \mathcal{I}(S)$.

Conversely, assume that $\mathcal{I}(R) \cong \mathcal{I}(S)$. Suppose $k_i \neq k'_i$ for some $i$. Then $|V(\mathcal{I}(R))| \neq |V(\mathcal{I}(S))|$, a contradiction. Hence $k_i = k'_i$ for all $i$.

Example 6.3.3. Let $R = Z_4 \times Z_2$ and $S = Z_9 \times Z_3$. Then $\mathcal{I}(R) \cong \mathcal{I}(S) \cong K_{1,2}$ (by Theorem 6.2.7). But $R$ and $S$ are not isomorphic.

Using Theorem 6.3.2, we have the following corollary.
Corollary 6.3.4. Let \( R_1 = \prod_{i=1}^{n} F_i \) and \( R_2 = \prod_{j=1}^{n} F'_j \), where each \( F_i \) and \( F'_j \) are fields and \( n \geq 2 \). Then \( \mathcal{G}(R_1) \cong \mathcal{G}(R_2) \).

Corollary 6.3.5. Let \( R = \prod_{i=1}^{n} R_i \) and \( S = \prod_{i=1}^{n} R'_i \) be finite commutative rings with \( n \geq 2 \), where each \( (R_i, m_i) \) and \( (R'_i, m'_i) \) are local rings which are not field. Let \( k_i \) be the number of ideals in \( R_i \) and \( k'_i \) be the number of ideals in \( R'_i \). Then \( \mathcal{G}(R) \cong \mathcal{G}(S) \) if and only if \( k_i = k'_i \) for all \( i, 1 \leq i \leq n \).

In view of the above, it is natural to consider the question that whether the comaximal-ideal graph is isomorphic to the zero-divisor graph or the annihilating ideal graph. In [54], it has been proved that for a finite commutative ring \( R = \prod_{i=1}^{n} F_i \), where \( F_i \) is field and \( n \geq 2 \), the co-maximal graph \( \mathcal{G}(R) \) of \( R \) is isomorphic to the zero-divisor graph of \( \mathbb{Z}_2^n \). In this section, we prove that the comaximal ideal graph of a particular ring is isomorphic to the annihilating-ideal graph of another ring.

Theorem 6.3.6. Let \( R_1 = \mathbb{Z}_2^n \) and \( R_2 = \prod_{k=1}^{n} F_k \) where each \( F_i \) is a field and \( n \geq 2 \). Let \( \Gamma(R_1) \) be the zero-divisor graph of \( R_1 \). Then \( \mathcal{G}(R_2) \cong A\mathcal{G}(R_2) \cong \Gamma(R_1) \).

Proof. Note that \( V(A\mathcal{G}(R_2)) = \{I = \prod_{i=1}^{n} I_i : I_i \in \{(0), F_i\}, \ 1 \leq i \leq n\} \setminus \{(0), R_2\} \), \( V(\Gamma(R_1)) = \{a = (a_1, a_2, \ldots, a_n) : a_i \in \{0, 1\}, \ 1 \leq i \leq n\} \setminus \{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\} \) and \( |V(A\mathcal{G}(R_2))| = |V(\Gamma(R_1))| = 97 \).
Define $f : V(\mathbb{A}G(R_2)) \rightarrow V(\Gamma(R_1))$ by $f(\prod_{i=1}^{n} I_i) = (a_1, a_2, \ldots, a_n)$ where

$$a_i = \begin{cases} 
1 & \text{if } I_i = F_i \\
0 & \text{if } I_i = (0) 
\end{cases}$$

Clearly $f$ is well-defined and bijective. Let $I = \prod_{i=1}^{n} I_i$ and $I' = \prod_{i=1}^{n} I'_i$ be two non-zero ideals in $R_2$. Suppose $I$ and $I'$ are adjacent in $\mathbb{A}G(R_2)$. Then $II' = (0)$ and so $I_iI'_i = (0)$ for all $i$. Hence $I_i = (0)$ or $I'_i = (0)$ for all $i$. Suppose $f(I) = (b_1, b_2, \ldots, b_n)$ and $f(I') = (c_1, c_2, \ldots, c_n)$. Then either $b_i = 0$ or $c_i = 0$ and so $b_ic_i = 0$ for all $i$, i.e., $f(I)f(I') = 0$ and so $f(I)$ and $f(I')$ are adjacent in $\Gamma(R_1)$. Similarly one can prove that $f$ preserves non-adjacency also. Hence $\mathbb{A}G(R_2) \cong \Gamma(R_1) \cong \mathcal{G}(R_2)$.  

$\square$
6.4 Planarity of $\mathcal{I}(R)$

In this section, we characterize all finite commutative rings (non-local rings) with identity for which $\mathcal{I}(R)$ is planar.

Remark 6.4.1. Note that if $n \geq 4$, then $\Gamma(\mathbb{Z}_2^n)$ is non-planar. Hence if $R = \prod_{i=1}^{n} F_i$ where $F_i$ is a field and $n \geq 2$, then $\mathcal{I}(R)$ is planar if and only if $R \cong F_1 \times F_2$ or $R \cong F_1 \times F_2 \times F_3$.

Theorem 6.4.2. Let $R = \prod_{i=1}^{n} R_i \times \prod_{j=1}^{m} F_j$ be a finite commutative ring with $n + m \geq 2$, where each $(R_i, m_i)$ is a local ring which not a field and each $F_i$ is a field. Then $\mathcal{I}(R)$ is planar if and only if $R$ satisfies any one of the following conditions:

(i) $F_1 \times F_2 \times F_3$ or $R_1 \times F_1 \times F_2$ and $m_1$ is the only nonzero proper ideal in $R_1$

(ii) $F_1 \times F_2$ or $R_1 \times F_1$

(iii) $R_1 \times R_2$ where $R_1$ has at most 3 nonzero ideals and $R_2$ has at most 2 nonzero ideals and $R_1$ has at most 2 nonzero ideals and $R_2$ has at most 3 nonzero ideals.

Proof. Suppose $\mathcal{I}(R)$ is planar. Note that $\Gamma(\mathbb{Z}_2^{n+m})$ is a subgraph of $\mathcal{I}(R)$. Suppose $n + m \geq 4$. Since $\Gamma(\mathbb{Z}_2^{n+m})$ is nonplanar, $\mathcal{I}(R)$ is nonplanar and hence $n + m \leq 3$. 

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Case 1. \( n + m = 3 \).

Subcase 1. \( n = 0 \) and \( m = 3 \). Then by Remark 6.4.1,
\[ R = F_1 \times F_2 \times F_3. \]

Subcase 2. \( m = 0 \) and \( n = 3 \). Let \( \Omega = \{ x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \} \) where \( x_1 = m_1 \times R_2 \times R_3, x_2 = R_1 \times m_2 \times R_3, x_3 = R_1 \times R_2 \times m_3, \)
\( y_1 = (0) \times R_2 \times R_3, y_2 = R_1 \times (0) \times R_3, y_3 = R_1 \times R_2 \times (0), z_1 = R_1 \times (0) \times (0), z_2 = (0) \times R_2 \times (0), z_3 = (0) \times (0) \times R_3. \) Then \( \langle \Omega \rangle \) is a subgraph of \( \mathcal{I}(R) \), \( \langle \Omega \rangle \) contains a subdivision of \( K_{3,3} \) (see Fig. 4.2(a)) and hence \( \mathcal{I}(R) \) is nonplanar.

Subcase 3. \( n = 2 \) and \( n = 1 \). Then \( R = R_1 \times R_2 \times F_1. \)

Let \( \Omega' = \{ a_1, a_2, a_3, a_4, a_5, b_1, b_2 \} \) where \( a_1 = m_1 \times R_2 \times F_1, a_2 = R_1 \times m_2 \times F_1, a_3 = R_1 \times R_2 \times (0), a_4 = (0) \times R_2 \times F_1, a_5 = R_1 \times (0) \times F_1, \)
\( b_1 = R_1 \times (0) \times (0), b_2 = (0) \times R_2 \times (0). \) Then \( \langle \Omega' \rangle \) is a subgraph of \( \mathcal{I}(R) \), \( \langle \Omega' \rangle \) contains a subdivision of \( K_{5} \) (see Fig. 4.2(b)) and hence \( \mathcal{I}(R) \) is nonplanar.
Subcase 4. $m = 1$ and $n = 2$. Then $R = R_1 \times F_1 \times F_2$. Suppose $I$ is any nonzero proper ideal in $R_1$ and $I \subset m_1$. Let $\Omega'' = \{d_1, d_2, d_3, e_1, e_2, e_3\}$ where $d_1 = m_1 \times F_1 \times F_2$, $d_2 = I \times F_1 \times F_2$, $d_3 = (0) \times F_1 \times F_2$, $e_1 = R_1 \times (0) \times (0)$, $e_2 = R_1 \times (0) \times F_2$, $e_3 = R_1 \times F_1 \times (0)$. Then $\langle \Omega'' \rangle$ is a subgraph of $\mathcal{I}(R)$, $\langle \Omega'' \rangle$ contains a $K_{3,3}$ as a subgraph and so $\mathcal{I}(R)$ is nonplanar. Hence $m_1$ is the only nonzero proper ideal in $R_1$. Let $V(\mathcal{I}(R)) = \{v_1, \ldots, v_9\}$ where $v_1 = (0) \times F_1 \times F_2$, $v_2 = m_1 \times F_1 \times F_2$, $v_3 = R_1 \times (0) \times F_2$, $v_4 = R_1 \times F_1 \times (0)$, $v_5 = R_1 \times (0) \times (0)$, $v_6 = (0) \times F_1 \times (0)$, $v_7 = m_1 \times F_1 \times (0)$, $v_8 = (0) \times (0) \times F_2$, $v_9 = m_1 \times (0) \times F_2$. Since $\mathcal{I}(R)$ is planar and by Fig. 4.2(c), $R \cong R_1 \times F_1 \times F_2$ and $m_1$ is the only nonzero proper ideal in $R_1$.

Case 2. $n + m = 2$. Then by Theorem 6.2.7, $\mathcal{I}(R)$ is a complete bipartite graph. Since $\mathcal{I}(R)$ is planar, $R \cong R_1 \times F_1$ or $F_1 \times F_2$.

If $R \cong R_1 \times R_2$, then by Theorem 6.2.7, $\mathcal{I}(R) \cong K_{t,k}$ where $t$ and $k$ are number of nonzero ideals in $R_1$ and $R_2$ respectively. Since $\mathcal{I}(R)$ is
planar, either $t \leq 3$ and $k \leq 2$ or $t \leq 2$ or $k \leq 3$.

Converse is obvious.  \qed