Chapter 5

Cozero-divisor graph of commutative rings

In this Chapter, we characterize all finite commutative non-local rings for which the cozero-divisor graph $\Gamma'(R)$ has genus one. Further, we characterize all finite commutative non-local rings $R$ for which the reduced cozero-divisor graph $\Gamma_r(R)$ is planar.

5.1 Introduction

In [5], Khashyarmanesh defined the cozero-divisor graph of commutative rings. The cozero-divisor graph of $R$, denoted by $\Gamma'(R)$, is a graph with vertex-set $W^*(R)$, which is the set of all non-zero non-unit elements of $R$, and two distinct vertices $x$ and $y$ in $W^*(R)$ are adjacent if and only if $x \notin Ry$ and $y \notin Rx$, where for $z \in R$, $Rz$ is the ideal generated
by $z$. In [5, 6], some basic results on the structure of this graph were obtained. Also, in [7], the situation where $\Gamma'(R)$ is planar and outer planar is investigated. In [12], A. Wilkens et. al defined the reduced cozero-divisor graph of commutative rings. The reduced cozero-divisor graph of $R$, denoted by $\Gamma_r(R)$, is a graph with vertex-set $\Omega(R)$, which is the set of all non-zero nontrivial principal ideals of $R$, and two distinct vertices $Rx$ and $Ry$ in $\Omega(R)$ are adjacent if and only if $Rx \not\subseteq Ry$ and $Ry \not\subseteq Rx$.

## 5.2 Definitions and Examples

In this section, we observe basic properties of the cozero-divisor graph of a finite commutative ring.

**Definition 5.2.1.** Let $R$ be a commutative ring with identity. The cozero-divisor graph of $R$, denoted by $\Gamma'(R)$, is a graph with vertex-set $W^*(R)$, which is the set of all non-zero non-unit elements of $R$, and two distinct vertices $x$ and $y$ in $W^*(R)$ are adjacent if and only if $x \not\in Ry$ and $y \not\in Rx$, where for $z \in R$, $Rz$ is the ideal generated by $z$.

By above definition, if $R$ is a finite commutative ring with identity, then $W^*(R) = Z(R)^*$.

**Remark 5.2.2.** Let $R$ be a field. Then $W^*(R) = \emptyset$ and hence $\Gamma'(R)$ is
an empty graph.

**Remark 5.2.3.** Let $R$ be a finite commutative ring. Then $\Gamma'(R)$ is a complete graph if and only if $Rx = \{0, x\}$ for all $x \in W^*(R)$.

**Remark 5.2.4.** Let $R = F_1 \times F_2$ be a finite commutative ring, where $F_1$ and $F_2$ are fields. Then $\Gamma'(R)$ is a complete bipartite graph and hence $\Gamma_N(R) \cong K_{|F_1^*|, |F_2^*|}$.

**Example 5.2.5.** Let $R = \mathbb{Z}_{p^k}$, where $p$ is prime and $k > 1$. Then $W^*(R) = \{\lambda p : \lambda \in \mathbb{Z}_{p^k}\}$ and $\Gamma'(R) \cong \overline{K_t}$, where $t = p^k - p^{k-1}(p-1) - 1$.

**Example 5.2.6.** Let $R = \mathbb{Z}_{12}$. Then $W^*(R) = \{2, 3, 4, 6, 8, 9, 10\}$ and $\langle 2 \rangle = \langle 10 \rangle = \{0, 2, 4, 6, 8, 10\}$, $\langle 3 \rangle = \langle 9 \rangle = \{0, 3, 6, 9\}$, $\langle 4 \rangle = \langle 8 \rangle = \{0, 4, 8\}$.
5.3 Genus of cozero-divisor graphs

In this paper, we determine all isomorphism classes of finite commutative non-local rings with identity whose $\Gamma(R)$ has genus one.

**Theorem 5.3.1.** Let $R = F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where each $F_j$ is a field and $n \geq 2$. Then $g(\Gamma(R)) = 1$ if and only if $R$ is isomorphic to one of the following rings: $F_4 \times F_4, F_4 \times Z_5, Z_5 \times Z_5, F_4 \times F_7$ or $Z_3 \times Z_2 \times Z_2$.

**Proof.** Assume that $g(\Gamma(R)) = 1$. Suppose $n \geq 4$.

Let $A = \{x_1, x_2, \ldots, x_{11}\}$, where $x_1 = (0, 0, 0, 1, 0, \ldots, 0)$, $x_2 = (0, 0, 1, 0, \ldots, 0)$, $x_3 = (0, 0, 1, 1, 0, \ldots, 0)$, $x_4 = (0, 1, 0, \ldots, 0)$, $x_5 = (1, 1, 0, 0 \ldots, 0)$, $x_6 = (1, 0, 0, \ldots, 0)$, $x_7 = (0, 1, 1, 0, \ldots, 0)$, $x_8 = (0, 1, 1, 0, \ldots, 0)$, $x_9 = (1, 0, 0, 1, 0, \ldots, 0)$, $x_{10} = (1, 0, 1, 0, \ldots, 0)$, $x_{11} = (1, 0, 1, 1, 0 \ldots, 0) \in W^*(R)$. Then the subgraph induced by $A$ in $\Gamma(R)$ contains $G$(see Fig. 2.9) as a subgraph and so $g(\Gamma(R)) \geq 2$. Hence $n \leq 3$.

**Case 1.** $n = 2$.

Note that $\Gamma(R) \cong K_{\lvert F_1 \rvert - 1, \lvert F_2 \rvert - 1}$. As $g(\Gamma(R)) = 1$, by Lemma 2.3.7, $R$ is isomorphic to one of the following rings $F_4 \times F_4, F_4 \times Z_5, Z_5 \times Z_5$ or $F_4 \times F_7$.

**Case 2.** $n = 3$.

Suppose at least two of $F_i$’s are such that $\lvert F_i \rvert \geq 3$. Without loss of
generality let $|F_1| \geq 3$ and $|F_2| \geq 3$. Let $B = \{z_1, \ldots, z_{11}\}$, where $z_1 = (1,0,0)$, $z_2 = (1,1,0)$, $z_3 = (1,v_1,0)$, $z_4 = (0,1,1)$, $z_5 = (0,v_1,1)$, $z_6 = (0,0,1)$, $z_7 = (0,1,0)$, $z_8 = (0,v_1,0)$, $z_9 = (1,0,1)$, $z_{10} = (u_1,0,0)$, $z_{11} = (u_1,0,1)$, $1 \neq u_1 \in F_1^*$, $1 \neq v_1 \in F_2^*$. Then the subgraph induced by $B$ in $\Gamma'(R)$ contains $G$ (see Fig. 2.9) as a subgraph and so $g(\Gamma'(R)) \geq 2$, a contradiction. Hence $|F_1| = |F_2| = 2$ for some $i \neq j$.

We assume that $F_2 = F_3 = \mathbb{Z}_2$.

Suppose $|F_1| \geq 4$. Let $B' = \{b_1, \ldots, b_{11}\}$, where $b_1 = (u_1,0,1)$, $b_2 = (1,0,1)$, $b_3 = (u_2,0,1)$, $b_4 = (u_1,1,0)$, $b_5 = (0,1,0)$, $b_6 = (u_2,1,0)$, $b_7 = (0,0,1)$, $b_8 = (0,1,1)$, $b_9 = (u_2,0,0)$, $b_{10} = (1,0,0)$, $b_{11} = (u_1,0,1)$ $\in W^*(R)$ and $1 \neq u_i \in F_1^*$. Then the subgraph induced by $B'$ in $\Gamma'(R)$ contains $G$ (see Fig. 2.9) as a subgraph and so $g(\Gamma'(R)) \geq 2$, a contradiction. Hence $|F_1| \leq 3$. Since $\Gamma'(R)$ is non-planar, by Theorem 1.3.19, $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and hence $F_1 = \mathbb{Z}_3$.

Converse is obvious. \[\square\]
Remark 5.3.2. Note that if \( R = R_1 \times \cdots \times R_n \) is a commutative ring with identity, where each \((R_i, m_i)\) is a local ring with \( m_i \neq \{0\} \) and \( n \geq 2 \), then \( \Gamma'(\mathbb{Z}_4 \times \mathbb{Z}_4) \) is a subgraph of \( \Gamma'(R) \). Since \( g(\Gamma'(\mathbb{Z}_4 \times \mathbb{Z}_4)) \geq 2 \), we have \( g(\Gamma'(R)) \geq 2 \).

Note that \( \Gamma'(\mathbb{Z}_8 \times \mathbb{Z}_2) \cong \Gamma'(R) \), where \( R \) is isomorphic to one of the following rings: 
\[
\begin{align*}
\mathbb{Z}_2[x]/(x^4) \times \mathbb{Z}_2, & \quad \mathbb{Z}_4[x]/(2x^2-2) \times \mathbb{Z}_2, \quad \mathbb{Z}_4[x]/(2x^2) \times \mathbb{Z}_2 \\
\text{or } \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_7, & \quad \mathbb{Z}_9 \times \mathbb{Z}_2, \quad \mathbb{Z}_3[x]/(x^2) \times \mathbb{Z}_2, \quad \mathbb{Z}_8 \times \mathbb{Z}_2, \quad \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2, \quad \mathbb{Z}_4[x]/(2x^2-2) \times \mathbb{Z}_2,
\end{align*}
\]

Theorem 5.3.3. Let \( R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m \) be a commutative ring with identity, where each \((R_i, m_i)\) is a local ring with \( m_i \neq \{0\} \), \( F_j \) is a field and \( n, m \geq 1 \). Then \( g(\Gamma'(R)) = 1 \) if and only if \( R \) is isomorphic to one of the following rings: 
\[
\begin{align*}
\mathbb{Z}_4 \times \mathbb{F}_4, & \quad \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4, \quad \mathbb{Z}_4 \times \mathbb{F}_5, \quad \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_7 \\
\text{or } \mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_7, & \quad \mathbb{Z}_9 \times \mathbb{Z}_2, \quad \mathbb{Z}_3[x]/(x^2) \times \mathbb{Z}_2, \quad \mathbb{Z}_8 \times \mathbb{Z}_2, \quad \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2, \quad \mathbb{Z}_4[x]/(2x^2-2) \times \mathbb{Z}_2,
\end{align*}
\]

Proof. Assume that \( g(\Gamma'(R)) = 1 \). Suppose \( n + m \geq 3 \). Let \( \Omega = \{x_1, \ldots, x_{11}\} \), where 
\[
x_1 = (1,0,0,\ldots,0), \quad x_2 = (u,0,0,\ldots,0), \quad x_3 = (a,0,0,\ldots,0), \quad x_4 = (0,0,1,0,\ldots,0), \quad x_5 = (0,1,0,\ldots,0),
\]
\[
x_6 = (0,1,1,0,\ldots,0), \quad x_7 = (1,1,0,\ldots,0), \quad x_8 = (u,1,0,\ldots,0), \quad x_9 = (1,0,1,0,\ldots,0), \quad x_{10} = (u,0,1,0,\ldots,0), \quad x_{11} = (a,0,1,0,\ldots,0) \in W^*(R), \quad 1 \neq u \in R_i^* \quad \text{and} \quad a \in m_i^*.
\]
Then the subgraph induced by \( \Omega \) in \( \Gamma'(R) \) contains \( G \) (see Fig. 2.9) as a subgraph and so \( g(\Gamma'(R)) \geq 2 \), a contradiction. Hence \( R = R_1 \times F_1 \).
Suppose \(|F_1| \geq 3.\) If \(|m_1^*| \geq 2,\) then \(|R_1^x| \geq 4.\)

Let \(\Omega_1 = \{(a, 0), (b, 0), (u_1, 0), (u_2, 0), (u_3, 0), (u_4, 0), (0, 1), (0, v_1), (a, 1), (a, v_1), (b, 1), (b, v_1) : a, b \in m_1^*, u_i \in R_1^x, \text{ and } 1 \neq v_1 \in F_1^*\} \subseteq W^*(R).\) Then the subgraph induced by \(\Omega_1\) contains \(K_{6,6}\) which turns to be a subgraph of \(\Gamma'(R)\) so that \(g(\Gamma'(R)) \geq g(K_{6,6}) \geq 2,\) a contradiction. Hence \(|m_1^*| = 1\) and so \(R_1 \cong \mathbb{Z}_4\) or \(\mathbb{Z}_2[\langle x \rangle^*].\) By Theorem 1.3.19, \(F_1 \neq \mathbb{Z}_3\) and \(|F_1| \geq 4.\)

If \(|F_1| \geq 8,\) then \(K_{3,7}\) is a subgraph of \(\Gamma'(R)\) and by Lemma 2.3.7, \(g(\Gamma'(R)) \geq 2,\) a contradiction. Hence \(|F_1| \leq 7\) and so \(F_1\) is isomorphic to one of the following fields: \(\mathbb{F}_4, \mathbb{F}_5\) or \(\mathbb{F}_7.\)

Suppose \(|m_1^*| \geq 2.\) Then by above argument, we have \(F_1 = \mathbb{Z}_2.\) If \(|m_1^*| \geq 4,\) then \(|R_1^x| \geq 5\) and so \(K_{5,5}\) is a subgraph of \(\Gamma'(R).\) By Lemma 2.3.7, \(g(\Gamma'(R)) \geq 2,\) a contradiction. Hence \(|m_1^*| = 2\) or 3 and so \(R_1\) is isomorphic to one of the following rings:

\[
\mathbb{Z}_9, \mathbb{Z}_3[z], \mathbb{Z}_8, \mathbb{Z}_2[x], \mathbb{Z}_4[x], \mathbb{Z}_4[x], \mathbb{Z}_2[\langle x \rangle^*], \mathbb{F}_4[x], \mathbb{Z}_4[\langle x \rangle^*].
\]
Fig. 5.4: Embedding of $\Gamma' \left( \mathbb{Z}_8 \times \mathbb{Z}_2 \right)$

Fig. 5.5: Embedding of $\Gamma(\mathbb{Z}_4 \times F_4) \cong \Gamma' \left( \mathbb{Z}_2[x] / (x^2) \times F_4 \right)$

Fig. 5.6: Embedding of $\Gamma(\mathbb{Z}_4 \times F_5) \cong \Gamma' \left( \mathbb{Z}_2[x] / (x^2) \times F_5 \right)$
Fig. 5.7: Embedding of $\Gamma'(\mathbb{Z}_4 \times \mathbb{F}_7) \cong \Gamma'\left(\frac{\mathbb{Z}_4[x]}{(x^2)} \times \mathbb{F}_7\right)$

If $R_1 \cong \frac{\mathbb{F}_4[x]}{(x^2)}$ or $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$, then $|R_1^\times| = 13$ and so $K_{4,13}$ is a subgraph of $\Gamma'(R)$. By Lemma 2.3.7, $g(\Gamma'(R)) \geq 2$, a contradiction. Hence $R_1 \not\cong \frac{\mathbb{F}_4[x]}{(x^2)}$ and $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$. \(\square\)
5.4 Basics of reduced cozero-divisor graphs

In this section, we observe basic properties of the reduced cozero-divisor graph of a finite commutative ring.

Definition 5.4.1. Let $R$ be a commutative ring with identity. The reduced cozero-divisor graph of $R$, denoted by $\Gamma_r(R)$, is a graph with vertex-set $\Omega(R)$, which is the set of all non-zero nontrivial principal ideals of $R$, and two distinct vertices $Rx$ and $Ry$ in $\Omega(R)$ are adjacent if and only if $Rx \not\subseteq Ry$ and $Ry \not\subseteq Rx$.

Example 5.4.2. If $R = \mathbb{Z}_4 = \{0, 1, 2, 3\}$, then $\Omega(R) = \{\langle 2 \rangle\}$ and hence $\Gamma_r(R) \cong K_1$.

Remark 5.4.3. Let $R = F_1 \times F_2$ be a ring, where $F_1$ and $F_2$ are fields. Then $\Omega(R) = \{\langle (1, 0) \rangle, \langle (0, 1) \rangle\}$ and hence $\Gamma_r(R) \cong K_2$.

Example 5.4.4. Let $R = \mathbb{Z}_{12}$. Then $\Omega(R) = \{\langle 2 \rangle, \langle 4 \rangle, \langle 3 \rangle, \langle 6 \rangle\}$. Clearly $\langle 4 \rangle \subset \langle 2 \rangle$, $\langle 6 \rangle \subset \langle 2 \rangle$ and $\langle 6 \rangle \subset \langle 3 \rangle$.

\[\langle 2 \rangle \quad \langle 3 \rangle \quad \langle 4 \rangle \quad \langle 6 \rangle\]

Fig. 5.8: $\Gamma_r(\mathbb{Z}_{12})$
Example 5.4.5. Let $R = \mathbb{Z}_{p^k}$, where $p$ is prime and $k > 1$. Then\n$\Omega(R) = \{\langle p^i \rangle : 1 \leq i \leq k - 1 \}$ and $\langle p^{k-1} \rangle \subset \langle p^{k-2} \rangle \subset \cdots \subset \langle p \rangle$ hence $\Gamma_r(R) \cong K_{\tau(p^k)-2}$, where $\tau(n)$ is the number of positive divisors of $n$.

5.5 Planarity of reduced cozero-divisor graphs

In this paper, we determine all isomorphism classes of finite commutative non-local rings $R$ with identity whose reduced cozero-divisor graph $\Gamma_r(R)$ is planar.

Theorem 5.5.1. Let $R = F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where each $F_j$ is a field and $n \geq 2$. Then $\Gamma_r(R)$ is planar if and only if $n \leq 3$.

Proof. Suppose $R \cong F_1 \times F_2 \times F_3, F_1 \times F_2$. Then it is easy to see that $\Gamma_r(R)$ is planar (see, Fig. 5.9).

![Fig. 5.9](image-url)
Assume $\Gamma_r(R)$ is planar. Suppose $n \geq 4$.

Let $A = \{x_1, x_2, x_3, x_4, x_5, x_6\} \subseteq V(\Gamma_r(R))$, where $x_1 = F_1 \times (0) \times (0) \times \cdots \times (0), x_2 = F_1 \times (0) \times (0) \times F_4 \times (0) \times \cdots \times (0), x_3 = F_1 \times F_2 \times (0) \times \cdots \times (0), x_4 = 0 \times (0) \times F_3 \times F_4 \times (0) \times \cdots \times (0), x_5 = (0) \times F_2 \times F_3 \times (0) \times \cdots \times (0), x_6 = (0) \times F_2 \times F_3 \times (0) \times (0) \times \cdots \times (0)$.

Then the subgraph induced by $A$ in $\Gamma_r(R)$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence $n \leq 3$ and so $R \cong F_1 \times F_2 \times F_3$ or $F_1 \times F_2$. □

**Theorem 5.5.2.** Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity 1, where each $(R_i, m_i)$ is a local ring with $m_i \neq \{0\}$ and $n \geq 2$. Then $\Gamma_r(R)$ is planar if and only if $R \cong R_1 \times R_2$ such that $m_i$ is the only non-zero principal ideal in $R_i$.

**Proof.** Suppose $R = R_1 \times R_2$ such that $m_i$ is the only non-zero principal ideal in $R_i$ for $i = 1, 2$. Then it is easy to see that the graph $\Gamma_r(R)$ is planar (see Fig. 5.10).

Assume that $\Gamma_r(R)$ is planar. Suppose $n \geq 3$.

Let $A = \{x_1, x_2, x_3, x_4, x_5, x_6\} \subseteq V(\Gamma_r(R))$, where $x_1 = (0) \times R_2 \times (0) \times \cdots \times (0), x_2 = (0) \times R_2 \times I \times (0) \times \cdots \times (0), x_3 = (0) \times R_2 \times R_3 \times (0) \times \cdots \times (0), x_4 = R_1 \times (0) \times \cdots \times (0), x_5 = R_1 \times (0) \times I \times (0) \times \cdots \times (0), x_6 = R_1 \times (0) \times R_3 \times (0) \times \cdots \times \cdots \times (0)$. Here $I$ is non-zero principal ideal in $R_3$. Then the subgraph induced by $A$ in $\Gamma_r(R)$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence $n = 2$. 80
Suppose $R_i$ contains at least two non-zero principal ideals for some $i$. Without loss of generality, we assume that $i = 2$. Let $I_1 = \langle x \rangle$, $I_2 = \langle y \rangle$ be two distinct principal ideals in $R_2$.

![Fig. 5.10: $\Gamma_r(R_1 \times R_2)$](image)

**Case 1.** $I_1 \not\subset I_2$ and $I_2 \not\subset I_1$

Let $B = \{x_1, x_2, \ldots, x_6\} \subseteq V(\Gamma_r(R))$, where $x_1 = (0) \times I_1$, $x_2 = R_1 \times I_1$, $x_3 = J_1 \times R_2$, $x_4 = (0) \times I_2$, $x_5 = R_1 \times I_2$, $x_6 = J_1 \times I_2$ and $J_1$ is a principal ideal in $R_1$. Then the subgraph induced by $B$ in $\Gamma_r(R)$ contains $K_{3,3}$ as a subgraph, a contradiction.

**Case 2.** $I_1 \subset I_2$ or $I_2 \subset I_1$. Without loss of generality, we assume that $I_1 \subset I_2$. Let $C = \{y_1, y_2, \ldots, y_7, y_8\} \subseteq V(\Gamma_r(R))$, where $y_1 = R_1 \times (0)$, $y_2 = R_1 \times I_2$, $y_3 = (0) \times R_2$, $y_4 = J_1 \times I_2$, $y_5 = (0) \times I_2$, $y_6 = J_1 \times R_2$, $y_7 = J_1 \times (0)$, $y_8 = R_1 \times I_2$ and $J_1$ is a principal ideal in $R_1$. Then the subgraph induced by $C$ in $\Gamma_r(R)$ contains a subdivision of $K_{3,3}$, a contradiction. Hence $R_i$ contains only one principal ideal. We claim that this principal ideal is maximal. It is enough to prove that $m_1$ is a principal ideal in $R_1$.

Suppose $m_1$ is not a principal ideal in $R_1$. Then there exists two distinct
elements \( x, y \in m_1 \) such that \( \langle x \rangle \) and \( \langle y \rangle \) are distinct ideals in \( R_1 \). Thus \( R_1 \) contains two non-zero principal ideals and by Cases 1 and 2, \( \Gamma_r(R) \) contains \( K_{3,3} \) as a subgraph, a contradiction. Hence \( m_i \) is principal for all \( i = 1, 2 \).

Converse follows from Fig. 5.10.

\[ \square \]

**Theorem 5.5.3.** Let \( R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m \) be a finite commutative ring with identity, where each \( (R_i, m_i) \) is a local ring and each \( F_j \) is a field, \( m, n \geq 1 \). Then \( \Gamma_r(R) \) is planar if and only if \( R \) satisfies the following conditions:

(i) \( n + m = 2 \)

(ii) There exists only two non-zero principal ideals \( \langle a_1 \rangle, \langle a_2 \rangle \) in \( R_1 \) such that \( \langle a_1 \rangle \not\subseteq \langle a_2 \rangle \) and \( \langle a_2 \rangle \not\subseteq \langle a_1 \rangle \).

(iii) \( m_1 = \langle a \rangle \) is a principal ideal with nilpotency at most \( k = 4 \) and

if \( k = 2 \), then \( \langle a \rangle \) is the only ideal in \( R_1 \)

if \( k = 3 \), then \( \langle a \rangle, \langle a^2 \rangle \) are the only ideals in \( R_1 \)

if \( k = 4 \), then \( \langle a \rangle, \langle a^2 \rangle, \langle a^3 \rangle \) are the only ideals in \( R_1 \).

**Proof.** Suppose \( R \) is isomorphic to \( R_1 \times F_1 \) where \( R_1 \) has at most three proper ideals. It is easy to see that \( \Gamma_r(R) \) is planar (see Fig. 5.11 and 5.12).

Conversely, assume that \( \Gamma_r(R) \) is planar. Suppose \( n \geq 2 \). Let \( A = \{ x_1, x_2, x_3, y_1, y_2, y_3 \} \subseteq V(\Gamma_r(R)) \) where \( x_1 = \langle x \rangle \times \langle y \rangle \times \cdots \times R_n \times F_1 \times (0) \times \cdots \times (0) \),
\[ x_2 = (0) \times R_2 \times \cdots \times R_n \times F_1 \times (0) \times \cdots \times (0), \quad x_3 = \langle 0 \rangle \times \langle y \rangle \times R_3 \times \cdots \times R_n \times F_1 \times (0) \times \cdots \times (0), \quad y_1 = R_1 \times (0) \times \cdots \times (0),
\]
\[ y_2 = R_1 \times R_2 \times (0) \times \cdots \times (0), \quad y_3 = R_1 \times (0) \times \cdots \times (0) \times F_1 \times (0) \times \cdots \times (0). \]

Then the subgraph induced by \( A \) in \( \Gamma_r(R) \) contains \( K_{3,3} \) as a subgraph, a contradiction. Hence \( n = 1 \).

Suppose \( m \geq 2 \). Then \( n + m \geq 3 \). Let \( B = \{ z_1, z_2, z_3, u_1, u_2, u_3 \} \subseteq V(\Gamma_r(R)) \), where \( z_1 = \langle x \rangle \times F_1 \times F_2 \times (0) \times \cdots \times (0), \)

\[ z_2 = (0) \times F_1 \times F_2 \times (0) \times \cdots \times (0), \quad z_3 = (0) \times (0) \times F_2 \times (0) \times \cdots \times (0), \]
\[ u_1 = R_1 \times F_1 \times (0) \times \cdots \times (0), \quad u_2 = R_1 \times F_1 \times (0) \times \cdots \times (0), \]
\[ u_3 = R_1 \times (0) \times \cdots \times (0). \]

Then the subgraph induced by \( B \) in \( \Gamma_r(R) \) contains \( K_{3,3} \) as a subgraph, a contradiction. Hence \( m = 1 \) and so \( R = R_1 \times F_1 \).

Since \( R \) is finite, \( R_1 \) is an Artinian ring and so every ideal in \( R_1 \) is finitely generated. Let \( \Omega = \{ a_1, \ldots, a_t : a_i \in R_1, \ a_i \neq a_j, \text{ for } i \neq j \} \) be a minimal generating set for \( m_1 \) in \( R_1 \). Then \( t \geq 1 \) and \( \langle a_i \rangle \nsubseteq \langle a_j \rangle \) for all \( i \neq j \).

Suppose \( t \geq 3 \). Let \( B = \{ x_1, x_2, x_3, x_4, x_5, x_6 \} \) where \( x_1 = \langle a_1 \rangle \times (0), \)
\[ x_2 = \langle a_2 \rangle \times (0), \quad x_3 = R_1 \times (0), \quad x_4 = \langle a_3 \rangle \times (0), \quad x_5 = \langle a_3 \rangle \times F_1, \]
\[ x_6 = (0) \times F_1. \]

Then the subgraph induced by \( B \) in \( \Gamma_R(R) \) contains \( K_{3,3} \) as a subgraph, a contradiction. Hence \( t \leq 2 \).
Assume that \( t = 2 \). Suppose \( \langle b_1 \rangle \) is a non-zero ideal in \( R_1 \) such that \( \langle b_1 \rangle \subset \langle a_1 \rangle \). Then the subgraph induced by \( \{ \langle a_1 \rangle \times (0), \langle b_1 \rangle \times (0), R_1 \times (0), \langle a_2 \rangle \times (0), \langle a_2 \rangle \times F_1, (0) \times F_1 \} \) in \( \Gamma_r(R) \) contains \( K_{3,3} \) as a subgraph, a contradiction.

Hence \( \langle a_1 \rangle, \langle a_2 \rangle \) are the only non-zero principal ideals in \( R_1 \)

Assume that \( t = 1 \). Then \( m_1 \) is a principal ideal generated by \( a_1 \).

Since \( R_1 \) is Artinian, \( m_1 \) is nil-ideal with nilpotency \( k \) and so \( m_1^k = (0) \), \( m_1^{k-1} \neq (0) \). From this, we get \( (0) \subset \langle a_1^{k-1} \rangle \subset \langle a_1^{k-2} \rangle \subset \cdots \subset \langle a_1 \rangle \) are the only ideals in \( R_1 \). Suppose \( k \geq 5 \). Then the subgraph induced by \( \{ \langle a_1 \rangle \times (0), (0) \times F_1, R_1 \times (0), \langle a_2^2 \rangle \times F_1, \langle a_1^3 \rangle \times F_1, \langle a_4^3 \rangle \times F_1, \langle a_2^3 \rangle \times F_1 \} \)
\((0), \langle a_i^3 \rangle \times (0) \rangle \) in \(\Gamma_r(R)\) contains a subdivision of \(K_{3,3}\), a contradiction. Hence \(k \leq 4\) and so \(R_1\) contains at most three non-zero principal ideals.

\(\square\)