Chapter 4

Zero-divisor graphs with respect to nilpotent elements

In this Chapter, we construct a graph called zero-divisor graph of a commutative ring $R$ with respect to nilpotent elements as a simple undirected graph $\Gamma_N^*(R)$ with vertex set $\mathcal{Z}_N(R)^*$, and two vertices $x$ and $y$ are adjacent if and only if $xy$ is nilpotent and $xy \neq 0$, where $\mathcal{Z}_N(R) = \{ x \in R : xy \text{ is nilpotent, for some } y \in R^* \}$. We investigate the interplay between the graph theoretic properties of $\Gamma_N^*(R)$ and the ring theoretic properties of $R$. Also we characterize the class of rings for which $\Gamma_N^*(R)$ is planar. Finally, we determine all isomorphism classes of finite commutative rings with identity whose $\Gamma_N^*(R)$ has genus one. The contents of this chapter have been published in Transactions on Combinatorics.
4.1 Basic Properties of $\Gamma^*_N(R)$

Definition 4.1.1. Let $R$ be a commutative ring with identity. The zero-divisor graph of a commutative ring $R$ with respect to nilpotent elements is a simple undirected graph $\Gamma^*_N(R)$ with vertex set $\mathcal{Z}_N(R)^*$, and two vertices $x$ and $y$ are adjacent if and only if $xy$ is nilpotent and $xy \neq 0$, where $\mathcal{Z}_N(R) = \{x \in R : xy$ is nilpotent, for some $y \in R^*\}$.

Remark 4.1.2. Let $R$ be a reduced ring. Then $\mathcal{Z}_N(R)^* = Z(R)^*$, $\Gamma^*_N(R) \cong \Gamma(R)$ and by definition, $\Gamma^*_N(R)$ is an empty graph.

Remark 4.1.3. Let $R$ be a local ring, but not a field. Then $\Gamma^*_N(R)$ is connected and $\text{diam}(\Gamma^*_N(R)) = 2$.

Theorem 4.1.4. Let $R$ be a finite commutative ring with identity, where each $R_i$ is a local ring and $n \geq 2$. Then $\Gamma^*_N(R)$ is connected if and only if $R_i$ is not a field for every $i$. Further $\text{diam}(\Gamma^*_N(R)) = 2$.

Proof. By Theorem 2.1.28, $R \cong R_1 \times \cdots \times R_n$, where each $R_i$ is a local ring.

Suppose $R_i$ is not field for every $i$. Then $Z(R_i) \neq \{0\}$ for every $i$. Let $x, y \in \mathcal{Z}_N(R)^*$ and $x \neq y$. Note that $\mathcal{J}(R) = \mathfrak{M}(R)$.

If $\mathfrak{M}(R)^* = \emptyset$, then $\mathfrak{M}(R) = 0$ which is a contradiction as $Z(R_i) \neq \{0\}$ for every $i$. Therefore $\mathfrak{M}(R)^* \neq \emptyset$. 

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Case 1. \( x, y \in R^\times \)

Then there exists \( a \in \mathfrak{N}(R)^* \) such that \( xa, ya \in \mathfrak{N}(R)^* \). Thus \( x - a - y \) is a path in \( \Gamma_N^*(R) \).

Case 2. \( x \in R^\times \) and \( y \in \mathfrak{N}(R)^* \)

Then \( xy \in \mathfrak{N}(R)^* \) and so \( x \) and \( y \) are adjacent in \( \Gamma_N^*(R) \).

Case 3. \( x \in R^\times \) and \( y \in Z(R) - \mathfrak{N}(R) \)

Then there exists \( b \in \mathfrak{N}(R)^* \) such that \( xb, yb \in \mathfrak{N}(R)^* \). Thus \( x - b - y \) is a path in \( \Gamma_N^*(R) \).

Case 4. \( x, y \in Z(R) - \mathfrak{N}(R) \)

Then there exists \( c \in \mathfrak{N}(R)^* \) such that \( xc, yc \in \mathfrak{N}(R)^* \). Thus \( x - c - y \) is a path in \( \Gamma_N^*(R) \).

Case 5. \( x, y \in \mathfrak{N}(R)^* \)

Then there exists \( u \in R^\times \) such that \( xu, yu \in \mathfrak{N}(R)^* \) and so \( x - u - y \) is a path in \( \Gamma_N^*(R) \).

Case 6. \( x \in Z(R) - \mathfrak{N}(R) \) and \( y \in \mathfrak{N}(R)^* \).

If \( xy \neq \{0\} \), then \( xy \in \mathfrak{N}(R)^* \) and so \( x \) and \( y \) are adjacent in \( \Gamma_N^*(R) \).

If \( xy = 0 \), then there exists \( z \in \mathfrak{N}(R)^* \) such that \( xz, yz \in \mathfrak{N}(R)^* \). Thus \( x - z - y \) is a path in \( \Gamma_N^*(R) \). Hence \( \Gamma_N^*(R) \) is connected.

Conversely, let \( \Gamma_N^*(R) \) be a connected graph. Suppose \( R_i \) is a field for some \( i \). Then there exists an element \( x = (0, \ldots, 0, 1, 0 \ldots, 0) \in Z_N(R)^* \), with 1 in the \( i^{th} \) place of \( x \), such that \( x \) is not adjacent to any other vertex of \( \Gamma_N^*(R) \), a contradiction. \( \square \)
Theorem 4.1.5. Let $R$ be a finite commutative ring with identity such that $\Gamma_N^*(R)$ is connected. Then $\Gamma(R)^c$ is a subgraph of $\Gamma_N^*(R)$ if and only if $R$ is a local ring.

Proof. By Theorems 2.1.28 and 4.1.4, $R \cong R_1 \times \cdots \times R_n$, where each $(R_i, m_i)$ is a local ring but not a field. Suppose $R$ is a local ring. Then by definition of $Z_N(R)^*$, $Z_N(R)^* = R^*$ and so $Z(R)^* \subset Z_N(R)^*$. Also $R \cong R_1$ and so $x$ is nilpotent for all $x \in m_1^*$. Let $x, y \in m_1^*$ with $x \neq y$. If $x$ and $y$ are adjacent in $\Gamma(R)$, then $xy = 0$, and so $x$ and $y$ are non-adjacent in $\Gamma_N^*(R)$. If $x$ and $y$ are non-adjacent in $\Gamma(R)$, then $xy \neq 0$, $xy \in \mathfrak{m}(R)^*$ and so $x$ and $y$ are adjacent in $\Gamma_N^*(R)$. Hence $\Gamma(R)^c$ is a subgraph of $\Gamma_N^*(R)$.

Conversely, let $\Gamma(R)^c$ be a subgraph of $\Gamma_N^*(R)$. Suppose $R$ is a non-local ring and $n \geq 2$. Let $a = (1, 0, \ldots, 0)$, $b = (u, 0, \ldots, 0) \in Z(R)^* \subset Z_N(R)$, where $u$ is a unit in $R_1$. Then $ab \neq 0$ in $R$ and so $a$ and $b$ are non-adjacent in $\Gamma(R)$. Clearly $ab \notin \mathfrak{m}(R)$ and so $a$ and $b$ are non-adjacent in $\Gamma_N^*(R)$, a contradiction. \hfill \Box

Theorem 4.1.6. Let $R$ be a finite commutative ring with identity and $\Gamma_N^*(R)$ be a connected graph. Then $\Gamma_N^*(R)$ is a complete bipartite graph if and only if $\Gamma(R)$ is a complete graph.

Proof. Suppose $\Gamma(R)$ is complete. Then by Theorems 1.3.2 and 4.1.4, $R$ is a local ring with unique maximal ideal $m$ and $xy = 0$ for all $x, y \in m^*$. 59
Note that $\mathcal{Z}_N(R)^* = R^* = m^* \cup (R \setminus m)$. By the definition of $\Gamma_N^*(R)$, $m^*$ and $R \setminus m$ are independent sets of $\Gamma_N^*(R)$. Also each edge in $\Gamma_N^*(R)$ has one end in $m^*$ and other end in $R \setminus m$. Hence $\Gamma_N^*(R)$ is a complete bipartite graph with bipartition $(m^*, R \setminus m)$.

Conversely, suppose $\Gamma_N^*(R)$ is complete bipartite graph. Note that $R \cong R_1 \times \cdots \times R_n$, where each $(R_i, m_i)$ is a local ring. If $R$ is a non-local ring, then $n \geq 2$. Let $a \in m_1^*$, $b \in m_2^*$. Then $(a, b, \ldots, 0) - (1, 0, \ldots, 0) - (a, 1, 0, \ldots, 0) - (a, b, 0\ldots, 0)$ is a cycle of length 3 in $\Gamma_N^*(R)$, a contradiction. Hence $R \cong R_1$ is a local ring. Suppose $\Gamma(R)$ is not complete. Then there exists elements $x_1, y_1 \in \mathcal{Z}(R)^*$ such that $x_1y_1 \neq 0$ and $x_1y_1 \in \mathfrak{M}(R)^*$. Thus $x_1 - u - y_1 - x_1$ is a cycle of length 3 in $\Gamma_N^*(R)$, where $u \in R^*$, a contradiction. Hence $\Gamma(R)$ is complete. \qed

In view of Theorems 1.3.2 and 4.1.6, if $\Gamma_N^*(R)$ is connected then we have $\Gamma_N^*(R)$ is complete bipartite if and only if $R$ is a local ring with $\text{char}(R) = p$ or $p^2$.

**Theorem 4.1.7.** Let $R$ be a finite commutative ring with identity and $\Gamma_N^*(R)$ be a connected graph. Then the following are equivalent:

(i) $\Gamma_N^*(R)$ is a star

(ii) $\Gamma_N^*(R)$ is a tree

(iii) $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

**Proof.** (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (iii) Suppose $\Gamma^*_N(R)$ is tree. Then $\Gamma^*_N(R)$ contains no cycle. Since $R$ is connected, by Remark 4.1.3 and Theorem 4.1.4, $R \cong R_1 \times \cdots \times R_n$ where each $(R_i, m_i)$ is a local ring, but not a field. If $n \geq 2$, then $(a, b, \ldots, 0) - (1, 0, \ldots, 0) - (a, 1, 0, \ldots, 0) - (a, b, 0, \ldots, 0)$ is a cycle in $\Gamma^*_N(R)$, where $a \in m_1^*, b \in m_2^*$, a contradiction. Thus $n = 1$ and so $R$ is local. Suppose $|m_1^*| \geq 2$. Then $|R^x| \geq 3$. Let $x, y \in m_1^*$ with $xy = 0$ and $u, v \in R^x$. Then $x - u - y - v - x$ is a cycle in $\Gamma^*_N(R)$, a contradiction. Hence $|m_1^*| = 1$ and so $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

(iii) $\Rightarrow$ (i) If $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, then $\Gamma^*_N(R) \cong K_{1,2}$.

\[\blacksquare\]

**Theorem 4.1.8.** Let $R$ be a finite commutative ring with identity and $\Gamma^*_N(R)$ be a connected graph.

(i) $gr(\Gamma^*_N(R)) = \infty$ if and only if $R \cong \mathbb{Z}_4$ and $\mathbb{Z}_2[x]/(x^2)$.

(ii) $gr(\Gamma^*_N(R)) = 4$ if and only if $R$ is local with $Z(R)^2 = 0$ and $|Z(R)^*| \geq 2$.

(iii) $gr(\Gamma^*_N(R)) = 3$ if and only if $R \not\cong \mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$ and $R$ is not a local ring with $Z(R)^2 = 0$ and $|Z(R)^*| \geq 2$.

**Proof.** (i) follows from Theorem 4.1.7.

(ii) Suppose $R$ is local with $Char(R) = p^2$ and $|Z(R)^*| \geq 2$. Then by Theorems 1.3.3 and 4.1.6, $\Gamma^*_N(R)$ is a complete bipartite graph and so $gr(\Gamma^*_N(R)) = 4$.

Conversely, let $gr(\Gamma^*_N(R)) = 4$. Then $\Gamma^*_N(R)$ does not contain a cycle...
of length 3 and \(|Z(R)^*| \geq 4\). Since \(R\) is finite, \(R \cong R_1 \times \cdots \times R_n\), where each \((R_i, m_i)\) is a local ring. Since \(\Gamma_N^*(R)\) is connected, \(m_i \neq \{0\}\) for all \(i\). Suppose \(n \geq 2\). Let \(x_1 \in m_1^*\) and \(x_2 \in m_2^*\). Then \((x_1, 0, \ldots, 0) - (0, x_2, 0, \ldots, 0) - (1, 0, \ldots, 0) - (x_1, 0, \ldots, 0)\) is a cycle in \(\Gamma_N^*(R)\), a contradiction. Thus \(R\) is local. If \(Z(R)^2 \neq 0\), then there exists \(x, y \in Z(R)^*\) such that \(xy \neq 0\) and so \(x - u - y - x\) is a cycle in \(\Gamma_N^*(R)\), a contradiction.

(iii) follows from (i) and (ii).

\[\text{Theorem 4.1.9.} \quad \text{Let} \ R \ \text{be a finite commutative ring with identity and} \ \Gamma_N^*(R) \ \text{be a connected graph. Then} \ \Gamma_N^*(R) \ \text{is a split graph if and only if} \ R \cong \mathbb{Z}_4 \ \text{or} \ \frac{\mathbb{Z}_2[x]}{(x^2)}. \]

\[\text{Proof.} \quad \text{Suppose} \ \Gamma_N^*(R) \ \text{is a split graph. Since} \ R \ \text{is finite,} \ R \cong R_1 \times \cdots \times R_n, \ \text{where each} \ (R_i, m_i) \ \text{is a local ring. If} \ n \geq 2, \ \text{then} \ (1, \ldots, 1) - (a_1, 0, \ldots, 0) - (u_1, u_2, \ldots, u_n) - (0, a_2, 0, \ldots, 0) \ \text{is a cycle of length} \ 4 \ \text{in} \ \Gamma_N^*(R), \ \text{where} \ a_1 \in m_1, \ a_2 \in m_2, \ u_i \in R_i^x. \ \text{By Theorem 2.2.40,} \ \Gamma_N^*(R) \ \text{is not split, a contradiction. Hence} \ R \ \text{is local. If} \ |m_1^*| \geq 2, \ \text{then} \ C_4 \ \text{is a subgraph of} \ \Gamma_N^*(R), \ \text{a contradiction. Hence} \ |m_1| = 2 \ \text{and so} \ R \cong \mathbb{Z}_4 \ \text{or} \ \frac{\mathbb{Z}_2[x]}{(x^2)}. \]

Conversely, if \(R \cong \mathbb{Z}_4 \ \text{or} \ \frac{\mathbb{Z}_2[x]}{(x^2)}\), then by Theorem 4.1.7(i), \(\Gamma_N^*(R)\) is star and so \(\Gamma_N^*(R)\) is a split graph. \qed

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4.2 Eulerian and Hamiltonian Properties of $\Gamma^*_N(R)$

In this section, we investigate the Eulerian and Hamiltonian properties of $\Gamma^*_N(R)$.

**Theorem 4.2.1.** Let $(R, \mathfrak{m})$ be a finite local ring but not a field, $|R| = p^n$, where $p$ is prime and $n > 1$. If $\Gamma^*_N(R)$ is a connected graph with $|\mathfrak{m}^*| \geq 2$, then $\Gamma^*_N(R)$ is Eulerian if and only if $|R|$ is odd and $x^2 = 0$ for all $x \in \mathfrak{m}^*$.

**Proof.** Note that $|\mathfrak{m}| = p^k$ for some $k < n$ and $|R^\times| = p^k(p^{n-k} - 1)$.

Suppose $\Gamma^*_N(R)$ is Eulerian. Then $deg_{\Gamma^*_N(R)}(v)$ is even for all $v \in \mathcal{Z}_N(R)^\times$.

Suppose $|R|$ is even. Then $|\mathfrak{m}^*|$ is odd and so $deg_{\Gamma^*_N(R)}(u) = p^k - 1$ is odd for all $u \in R^\times$, a contradiction. Hence $|R|$ is odd and so $|\mathfrak{m}^*|$, $|R^\times|$ are even. If $x^2 \neq 0$ for some $x \in \mathfrak{m}^*$, then $|ann(x)| = p^\ell$ for some $\ell < n$ and so $deg_{\Gamma^*_N(R)}(x) = |R^\times| + |\mathfrak{m}| - p^\ell - 1$ is odd, a contradiction. Hence $x^2 = 0$ for all $x \in \mathfrak{m}^*$.

Conversely, let $|R|$ be an odd integer such that $x^2 = 0$ for all $x \in \mathfrak{m}^*$. Then $deg_{\Gamma^*_N(R)}(u) = p^k - 1$ is even for all $u \in R^\times$ and $deg_{\Gamma^*_N(R)}(z) = |R^\times|$ is even for all $z \in \mathfrak{m}^*$. Hence $\Gamma^*_N(R)$ is Eulerian. \[\square\]

**Theorem 4.2.2.** Let $R$ be a finite commutative nonlocal ring with identity and $\Gamma^*_N(R)$ be a connected graph. Then $\Gamma^*_N(R)$ is Eulerian if and only if $|R|$ is odd and $x^2 = 0$ for all $x \in \mathfrak{J}(R)^\times$.  

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Proof. Suppose $\Gamma_N^*(R)$ is Eulerian. Then $deg_{\Gamma_N^*(R)}(x)$ is even for all $x \in \mathcal{Z}_N(R)^*$. Since $R$ is finite, $R \cong R_1 \times \cdots \times R_n$ where each $(R_i, m_i)$ is a local ring and $n \geq 2$. If $|R|$ is even, then $|R_i|$ is even for some $i$, $|m_i|$ is even and so $deg_{\Gamma_N^*(R)}(u) = \left(\prod_{i=1}^{n} |m_i|\right) - 1$ is odd for all $u \in R^\times$, a contradiction. Hence $|R|$ is odd and so $|m_i|$ is odd.

If $x^2 \neq 0$ for some $x \in \mathfrak{J}(R)^*$, then $deg_{\Gamma_N^*(R)}(x) = |R| - (|ann(x)| + 1)$ is odd, a contradiction. Hence $x^2 = 0$ for all $x \in \mathfrak{J}(R)^*$.

Conversely, let $|R|$ be an odd integer and $x^2 = 0$ for all $x \in \mathfrak{J}(R)^*$. Then $|R_i|$ is odd for all $i$ and so $|m_i|$ is odd. Let $y = (y_1, \ldots, y_n) \in \mathcal{Z}_N(R)^*$. Then

$$deg_{\Gamma_N^*(R)}(y) = \begin{cases} |R| - |ann(y)| & \text{if } y \in \mathfrak{J}(R)^* \\ \left(\prod_{i=1}^{n} |m_i|\right) - 1 & \text{if } y \in R^\times \end{cases}$$

In equation 4.1, $deg_{\Gamma_N^*(R)}(y)$ is even. If $y \in Z(R) \setminus \mathfrak{J}(R)$, then $deg_{\Gamma_N^*(R)}(y) = |Z(R)| - |ann(y)| - 1$ is even. Hence $deg_{\Gamma_N^*(R)}(a)$ is even for all $a \in \mathcal{Z}_N(R)^*$ and so $\Gamma_N^*(R)$ is Eulerian. \qed

Theorem 4.2.3. Let $(R, m)$ be a finite local ring but not a field, $|R| = p^n \geq 4$, where $p$ is prime and $n > 1$. Then $\Gamma_N^*(R)$ has a Hamiltonian path if and only if $R/m \cong \mathbb{Z}_2$. Hence $|R|$ is even.

Proof. Note that $|m| = p^k$ for some $k < n$ and $|R^\times| = p^k(p^{n-k} - 1) = t$. Clearly $K_{p^k - 1, t}$ is a subgraph of $\Gamma_N^*(R)$ and $R^\times$ is an independent subset
of $\Gamma_N^*(R)$. Suppose $R/m \cong \mathbb{Z}_2$. Then $|m| = |R^\times|$ and $|\mathcal{Z}_N(R)| = |m^*| + |R^\times|$. Note that $K_{p^{k-1},p^k}$ is a subgraph of $\Gamma_N^*(R)$. From this, we get $u_1 - z_1 - u_2 - z_2 - u_3 - \cdots - u_{p^{k-1}} - z_{p^{k-1}} - u_{p^k}$ is a Hamiltonian path in $\Gamma_N^*(R)$, where $u_i \in R^\times$ and $z_j \in m^*$.

Conversely, suppose $\Gamma_N^*(R)$ has a Hamiltonian path. Suppose $|R/m| > 2$. Then $|m| < |R^\times|$ and so $|R - m| > 2$. Note that the closure of $\Gamma_N^*(R)$ is $K_{p^{k-1}} + \overline{K}_{p^k(p^{n-k}-1)}$. Clearly $K_{p^{k-1}} + \overline{K}_{p^k(p^{n-k}-1)}$ is not Hamiltonian and so $\Gamma_N^*(R)$ is not Hamiltonian. Hence $|R/m| = 2$ and so $R/m \cong \mathbb{Z}_2$.

\[\square\]

**Corollary 4.2.4.** Let $(R, m)$ be a finite local ring but not a field. Then $\Gamma_N^*(R)$ is nonhamiltonian.
4.3 Genus of $\Gamma_N^*(R)$

In this section, we characterize the class of rings for which $\Gamma_N^*(R)$ is planar. Also we determine all isomorphism classes of finite commutative rings with identity whose $\Gamma_N^*(R)$ has genus one.

**Theorem 4.3.1.** Let $R$ be a finite commutative ring with identity and $\Gamma_N^*(R)$ be a connected graph. Then $\Gamma_N^*(R)$ is planar if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_9$, or $\mathbb{Z}_3[x]/(x^2)$.

**Proof.** Suppose $\Gamma_N^*(R)$ is planar. Clearly if $R$ is a non-local ring, then $K_{3,3}$ is a subgraph of $\Gamma_N^*(R)$. Hence $(R, m)$ is a local ring with $|R| = p^n$ where $p$ is a prime and $n > 1$. Note that $|m| = p^k$ for some $k < n$ and $|R^x| = |R \setminus m| = p^k(p^{n-k} - 1)$. If $|m^*| \geq 3$, then $|R^x| \geq 4$ and so $K_{3,4}$ is a subgraph of $\Gamma_N^*(R)$, a contradiction. Hence $|m^*| \leq 2$ and so $R \cong \mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_9$, or $\mathbb{Z}_3[x]/(x^2)$. Converse is obvious. □

**Theorem 4.3.2.** Let $R$ be a finite commutative ring with identity and $\Gamma_N^*(R)$ be a connected graph. Then $g(\Gamma_N^*(R)) = 1$ if and only if $R$ is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_2[x,y]/((x,y)^2)$ or $\mathbb{Z}_4[x]/(2x,x^2-2)$.

**Proof.** We only need to prove the necessary part. Suppose $g(\Gamma_N^*(R)) = 1$. Then $\Gamma_N^*(R)$ is non-planar. Since $R$ is finite, $R \cong R_1 \times \cdots \times R_n$,
Therefore, then by Example 1.3.2, subgraph of \( u \) where each \( u_i \) for every \( i \). Suppose \( n \geq 2 \). Let \( a_1 \in m_1, a_2 \in m_2 \) with \( a_1^2 = 0 \) and \( a_2^2 = 0 \).

Consider \( x_1 = (0, a_2, 0, \ldots, 0), x_2 = (a_1, 0, \ldots, 0), x_3 = (1, 1, 0, \ldots, 0), x_4 = (1, u_2, 0, \ldots, 0), x_5 = (u_1, 1, 0, \ldots, 0, 0), x_6 = (a_1, a_2, 0, \ldots, 0), x_7 = (u_1, a_2, 0, \ldots, 0), x_8 = (1, a_2, 0, 0, \ldots, 0), x_9 = (0, u_2, 0, \ldots, 0), x_{10} = (a_1, 1, 0, 0, \ldots, 0) \) and \( x_{11} = (a_1, u_2, 0, \ldots, 0) \) \( \in \mathbb{Z}(R)^* \), where \( u_1 \in R_1^\times, u_2 \in R_2^\times \). Let \( \Omega = \{x_1, \ldots, x_{11} \} \). Then \( G \) is a subgraph of \( \langle \Omega \rangle \) in \( \Gamma_N^*(R) \) (see, Fig. 2.9). Note that \( g(G) = 2[52, \text{ C. Wickham}] \). Therefore \( g(\Gamma_N^*(R)) \geq 2 \), a contradiction. Hence \( (R, m) \) is local.

Since \( \Gamma_N^*(R) \) is non-planar, by Theorem 4.3.1, \( |m^*| \geq 3 \). If \( |m^*| = 4 \), then by Example 1.3.2, \( R \cong \mathbb{Z}_{25} \) or \( \mathbb{Z}_5[x] \frac{(x^2)}{(x^2)} \), \( |R^\times| = 20 \) and so \( K_{4,20} \) is a subgraph of \( \Gamma_N^*(R) \). By Lemma 2.3.7, \( g(\Gamma_N^*(R)) > 1 \), a contradiction.

If \( |m^*| \geq 5 \), then \( |R^\times| \geq 6 \) and so \( K_{5,6} \) is a subgraph of \( \Gamma_N^*(R) \), now \( g(\Gamma_N^*(R)) > 1 \), a contradiction. Thus \( |m^*| = 3 \) and by Example 1.3.2, \( R \) is isomorphic to one of the following rings: \( \mathbb{Z}_8, \mathbb{Z}_2[x] \frac{(x)}{(x)}, \mathbb{Z}_4[x] \frac{(x^2)}{(x^2-2)}, \mathbb{Z}_2[x,y] \frac{(x,y)}{(x,y)^2} \), \( \frac{\mathbb{Z}_4[x]}{(2,x)}, \frac{\mathbb{F}_4[x]}{(x^2)}, \frac{\mathbb{Z}_4[x]}{(x^2+x+1)} \).

Suppose \( R \cong \frac{\mathbb{F}_4[x]}{(x^2)} \) or \( \frac{\mathbb{Z}_4[x]}{(x^2+x+1)} \). Then by Example 1.3.2, \( |R| = 16, |R^\times| = 12 \) and so \( K_{3,12} \) is a subgraph of \( \Gamma_N(R) \). By Lemma 2.3.7, \( g(\Gamma_N^*(R)) \geq 2 \), a contradiction.
Fig. 3.1(a): Embedding of $\Gamma^*_N(\mathbb{Z}_8)$ in $S_1$

Fig. 3.1(b): Embedding of $\Gamma^*_N(R)$ in $S_1$

Since $\Gamma^*_N(\mathbb{Z}_8) \cong \Gamma^*_N\left(\mathbb{Z}_2\left[\frac{x}{x^2+2}\right]\right) \cong \Gamma^*_N\left(\mathbb{Z}_4\left[\frac{x}{2x^2-2}\right]\right) \cong K_{3,12}$ and Fig. 3.1(b), hence $R$ is isomorphic to one of the following rings:

$\mathbb{Z}_8$, $\mathbb{Z}_2\left[\frac{x}{x^2+2}\right]$, $\mathbb{Z}_4\left[\frac{x}{(x,y)^2}\right]$ or $\mathbb{Z}_4\left[\frac{x}{(2,x)^2}\right]$.

**Theorem 4.3.3.** There exists no finite local ring $R$ with $g(\Gamma^*_N(R)) = 2$.

**Proof.** Let $(R, \mathfrak{m})$ be a finite local ring with $|R| = p^n$, where $p$ is a prime and $n > 1$. Then $|\mathfrak{m}| = p^k$ for some $k < n$ and $|R^\times| = p^k(p^{n-k} - 1) > |\mathfrak{m}^*| = p^k - 1$. By Theorems 4.3.1 and 4.3.2, $|\mathfrak{m}| \geq 3$.

If $|\mathfrak{m}| = 3$, then by Example 1.3.2 and Theorem 4.3.2, $R \cong \mathbb{F}_4[x] / (x^2)$ or $\mathbb{Z}_4[x] / (x^2+2)$. In this case, $\Gamma^*_N(R) \cong K_{3,12}$ and by Lemma 2.3.7, $g(\Gamma^*_N(R)) = 5$.

If $|\mathfrak{m}^*| \geq 4$, then $H = K_{p^{n-k}-1, p^{n-k-1}}$ is a subgraph of $\Gamma^*_N(R)$ and so $g(\Gamma^*_N(R)) \geq g(H) > 2$.

Hence there exists no finite local ring $R$ with $g(\Gamma^*_N(R)) = 2$. □