CHAPTER-III

GEODESIC GRAPHOIDAL COVERING NUMBER OF BICYCLIC GRAPHS

3.1 Introduction

The concept of Geodesic Graphoodal cover was introduced by S.Arumugam and J.Suresh Suseela [4]. A geodesic graphoodal cover of a graph $G$ is a collection $\psi$ of shortest paths in $G$ such that every path in $\psi$ has at least two vertices, every vertex of $G$ is an internal vertex of at most one path in $\psi$ and every edge of $G$ is in exactly one path in $\psi$. The minimum cardinality of a geodesic graphoodal cover of $G$ is called the geodesic graphoodal covering number of $G$ and is denoted by $\eta_g(G)$ or $\eta_g(G)$. They proved the following results.

**Theorem 3.1.1** [4] Let $G$ be a unicyclic graph with unique cycle $C$ which is even. Let $n$ denote the number of pendant vertices of $G$ and let $m$ denote the number of vertices on $C$ with degree greater than 2. Then

$$
\eta_g(G) = \begin{cases} 
2 & \text{if } m = 0 \\
\left( \begin{array}{l}
\text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree 2 is a shortest path} \\
n + 1 & \text{otherwise}
\end{array} \right) & \text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree 2 is a shortest path}
\end{cases}
$$

**Theorem 3.1.2** [4] Let $G$ be a unicyclic graph with unique cycle $C$ of odd length $2k+1$, $k \geq 1$. Let $n$ denote the number of pendant vertices of $G$ and let $m$ denote the number of vertices of degree greater than 2 on $C$. Then

$$
\eta_g(G) = \begin{cases} 
3 & \text{if } m = 0 \\
n + 2 & \text{if } m = 1 \\
\left( \begin{array}{l}
\text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree 2 is a shortest path} \\
n + 1 & \text{otherwise}
\end{array} \right) & \text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree 2 is a shortest path}
\end{cases}
$$

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Corollary 3.1.3

(i) \( \eta \leq \eta_a \leq \eta_g \) and these inequalities can be strict and also for a tree \( \eta = \eta_a = \eta_g = n - 1 \) where \( n \) is the number of pendant vertices of tree

(ii) \( \eta_g = q \) if and only if \( G \) is Complete. Further for a cycle \( C_m \),

\[
\eta_g(C_m) = \begin{cases} 
2 & \text{if } m \text{ is even} \\
3 & \text{if } m \text{ is odd} 
\end{cases}
\]

(iii) For any graph \( \eta_a = \eta_g = q - p + t \)

The concept of bicyclic graphs was introduced by K.Ratan singh and P.K. Das [28]. In this Chapter, The geodesic graphoidal covering number of bicyclic graphs containing \( U(l, m), D(l, m, i), C_m(l, i) \) and for various types of graphs like Triangular snake, Double Triangular snake, Alternate double Triangular snake, Triple Triangular snake, Web graph, Gear graph, Double wheel, Triangular Cactus, Helm graph, Mobious Ladder, Mongolian Tent, \( P_m \times P_n \), Shell graph, Multiplate shell graph, Ladder graph, t-ply, Book graph, Pumpkin graph, \( P_m(Q_s) \), \( C_m(Q_s) \), Fan graph and Windmill graph were found.

3.2 Geodesic Graphoidal Covering Number of Bicyclic Graphs

Theorem 3.2.1

Let \( G \) be a bicyclic graph containing a \( U(l, m) \) and both the cycles are of even length.

Let \( n \) denote the number of pendant vertices of \( G \) and let \( k \) denote the number of vertices of degree greater than 2 on \( U(l, m) \) other than \( u_0 \). Then
\[ \eta_g(G) = \begin{cases} 
3 & \text{if } k = 0 \\
1 + \begin{cases} 
0 & \text{if } k \geq 2 \text{ and every } (v, w)\text{-section of } C \text{ in which all vertices } v \text{ and } w \text{ have degree 2 is a shortest path} \\
n + 2 & \text{if } k = 1 \text{ and } \deg u > 3 \& s = i \text{ or } j \\
n + 3 & \text{otherwise} 
\end{cases} 
\end{cases} \]

**Proof:**

Let \( V(U(l, m)) = \{u_0, u_1, u_2, \ldots, u_{t-1}, u_t, u_{t+1}, \ldots, u_{t+m-2}\} \)

\[ V(C_i) = \{u_0, u_1, u_2, \ldots, u_{t-1}, u_0\} \]

\[ V(C_m) = \{u_0, u_t, u_{t+1}, \ldots, u_{t+m-2}, u_0\} \] where \( l \) and \( m \) are even.

**Case 1:** \( k = 0 \)

**Case 1a:** Let \( G = U(l, m) \)

The geodesic graphoidal cover of \( G \) is as follows

\[ P_1 = \{u_i, u_{i-1}, \ldots, u_1, u_0, u_t, u_{t+1}, \ldots, u_t\} \quad [i = \frac{l}{2} \& t = l + \frac{m}{2} - 1] \]

\[ P_2 = \{u_t, u_{t+1}, \ldots, u_0\} \]

\[ P_3 = \{u_0, u_{t+m-2}, \ldots, u_t\} \]

\( \psi = \{P_1, P_2, P_3\} \) is a geodesic graphoidal cover of \( G \)

\( \Rightarrow \eta_g(G) \leq 3 \)

Since in any minimum geodesic graphoidal cover at least two vertices are exterior points in \( U(l, m) \) so that \( t \geq 2 \)

\[ \eta_g = q - p + t \geq 3 \]

\[ \therefore \eta_g = 3 \]

**Case 1b:**

Let \( G = U(l, m) \) with \( \deg u_0 = 5 \) and a tree attached at \( u_0 \) with \( n \) pendant vertices.
The geodesic graphoidal cover of $G$ is as follows

$$P_1 = \{u_i, u_{i-1}, \ldots, u_1, u_0, u_i, u_{i+1}, \ldots, u_t\} \quad [i = \frac{l}{2} & t = l + \frac{m}{2} - 1]$$

$$P_2 = \{u_i, u_{i+1}, \ldots, u_0\}$$

$$P_3 = \{u_0, u_{m+2}, \ldots, u_i\}$$

Let $G_1 = G - \{u_i, u_{i+1}, \ldots, u_{t-1}, u_i, u_{i+1}, \ldots, u_{t+1}\}$ is a tree with $n + 1$ pendant vertices.

Let $\psi_1$ be any minimum geodesic graphoidal cover of $G_1$ is with $u_0$ as exterior point.

By corollary 3.1.3, $\eta_g(G_1) = n$

Then $\psi = \psi_1 \cup \{P_1, P_2, P_3\}$ is a geodesic graphoidal cover of $G$

$$\Rightarrow \eta_g(G) \leq n + 3$$

Since all the pendant vertices and atleast two vertices on $U(l, m)$ are exterior vertices of any minimum geodesic graphoidal cover so that, $t \geq n + 2$

Hence $\eta_g(G) = q - p + t \geq n + 3$

$$\therefore \eta_g(G) = n + 3$$

The case is similar if $\deg u_0 > 5$

**Case 2:** $k = 1$

Let $u_s$ be the unique vertex of degree greater than 2 on $U(l, m)$ other than $u_0$

Without loss of generality assume that $u_s = u_i$ lies on $C_i$

**Sub Case 2a:**

Let $P_i = \{u_0, u_1, u_2, \ldots, u_i\}$

Let $G_i = G - \{u_1, u_2, \ldots, u_{i-1}\}$ is a unicyclic graph with $n$ pendant vertices and $k = 1$.

By Theorem 3.1.1, $\eta_g(G_i) = n + 1$
Let $\psi_1$ be a minimum geodesic graphoidal cover of $G_1$.

Clearly any path in $\psi_1$ is a shortest path in $G$ also and hence

$\psi = \psi_1 \cup P_1$ is an acyclic geodesic graphoidal cover of $G$.

$\Rightarrow \eta_g(G) \leq n + 1 + 1 = n + 2$

$\eta_g(G) \leq n + 2$

Further all the $n$ pendant vertices and at least one vertex on $U(l, m)$ are exterior points in any minimum geodesic graphoidal cover so that $t \geq n + 1$

$\eta_g(G) = q - p + t \geq n + 2$

$\therefore \eta_g(G) = n + 2$

Sub Case 2b:

Suppose $k < \frac{l}{2}$

Let $P_1 = \{u_0, u_1, u_{l+1}, \ldots, u_{f-1}, u_l\}$

Let $G_1 = G - \{u_f, u_{f+1}, \ldots, u_{f-1}\}$ is a unicyclic graph with $n + 1$ pendant vertices and $k = 1$.

By Theorem 3.1.1, $\eta_g(G_1) = n + 2$

Let $\psi_1$ be a minimum geodesic graphoidal cover of $G_1$

Clearly any path in $\psi_1$ is a shortest path in $G$ also and hence

$\psi = \psi_1 \cup P_1$ is an acyclic geodesic graphoidal cover of $G$.

$\Rightarrow \eta_g(G) \leq n + 2 + 1 = n + 3$

$\eta_g(G) \leq n + 3$

Further all the $n$ pendant vertices and at least two vertices on $U(l, m)$ are exterior points in any minimum geodesic graphoidal cover so that $t \geq n + 2$
\[ \eta_g(G) = q - p + t \geq n + 3 \]

\[ \therefore \eta_g(G) = n + 3 \]

**Case 3:** \( k \geq 2 \) and every \((v,w)\) section of each of the cycles on \( U(l,m)\) in which all the vertices except \( v \) and \( w \) have degree 2 and this \((v,w)\) section is not a shortest path.

Let the \((v,w)\) section be denoted by \( (v = u_s, u_{s+1}, \ldots, u_t = w) \) where \( 1 < s, t < \frac{l}{2} \)

& \( u_s, u_t \) lies on \( C_l \)

Let \( G_i = G - \{u_{i-1}, u_{i-2}, \ldots, u_{i-1}\} \) is a unicyclic graph with \( n + 1 \) pendant vertices and \( k = 1 \)

Let \( \psi_1 \) be a minimum geodesic graphoidal cover of \( G_i \)

Let \( P_i = \{u_0, u_{i-1}, u_{i-2}, \ldots, u_i\} \)

Clearly any path in \( \psi_1 \) is a shortest path in \( G \) also and hence

\[ \psi = \psi_1 \cup P_i \]

is an acyclic geodesic graphoidal cover of \( G \).

\[ \Rightarrow \eta_g(G) \leq n + 2 + 1 = n + 3 \]

\[ \eta_g(G) \leq n + 3 \]

Further all the \( n \) pendant vertices and at least two vertices on \( U(l,m) \) are exterior points in any minimum geodesic graphoidal cover so that \( t \geq n + 2 \)

\[ \eta_g(G) = q - p + t \geq n + 3 \]

\[ \therefore \eta_g(G) = n + 3 \]

Suppose the \((v,w)\) section be denoted by \( (v = u_s, u_{s+1}, \ldots, u_t = w) \)

where \( 1 < s < \frac{l}{2}, l < t < \frac{m}{2} \) & \( u_s \) lies on \( C_l, u_t \) lies on \( C_m \)

Then the proof is similar by considering \( G_i = G - \{u_{i-1}, u_{i-2}, \ldots, u_{i-1}\} \)
**Case 4:** $k \geq 2$ and every $(v,w)$ section of each of the cycles on $U(l,m)$ in which all the vertices except $v$ and $w$ have degree 2 and this $(v,w)$ section is a shortest path.

In this case we prove the result by induction on $n$.

When $n = 2$, then $k = 2$ and $G$ consists of $U(l,m)$ and two paths.

These two paths lie either in the same cycles or in different cycles.

Without loss of generality assume the paths lies on the different cycles such that $P_1 = \{u_t, v_t, v_{t-1}, \ldots, v_1\} \& P_2 = \{u_t, w_t, w_{t-1}, \ldots, w_1\}$ where $u_t$ on $C_l$ & $u_t$ on $C_m$.

Now $G_1 = G - \{u_t, u_2, \ldots, u_{t-1}\}$ is a unicyclic graph with 2 pendant vertices and $k = 2$

By Theorem 3.1.1, $\eta_g(G_1) = 2$

Let $\psi_1$ be a minimum geodesic graphoidal cover of $G_1$

Clearly any path in $\psi_1$ is a shortest path in $G$ also and hence $\psi = \psi_1 \cup P$

where $P = \{u_0, u_1, \ldots, u_t\}$ is a minimum geodesic graphoidal cover of $G$.

$\Rightarrow \eta_g(G) = 3 = n + 1$

We now assume that the result is true for all bicyclic graph contains a $U(l,m)$ satisfying the condition stated in case 4 with $n-1$ pendant vertices with $k \geq 2$.

Let $G$ be a bicyclic graph contains a $U(l,m)$ satisfying the condition stated in case 4 with $n$ pendant vertices where $n \geq 3$ with $k \geq 2$.

Let $w$ be any pendant vertex of $G$. Choose a vertex $v$ such that $\deg v \geq 3$ and $d(w,v)$ is maximum.Let $P$ denote this $(w,v)$ path.

Consider the graph $G_1 = G - \{P \setminus \{v\}\}$

Then $G_1$ is a bicyclic graph containing $U(l,m)$ satisfying the condition stated in case 4 with $n-1$ pendant vertices.
Using induction hypothesis, \( \eta_g(G_i) = n \)

Let \( \psi_1 \) be any minimum geodesic graphoidal cover of \( G_i \)

Then \( \psi = \psi_1 \cup P \) is a minimum geodesic graphoidal cover of \( G \) in which every nonpendant vertex is an internal vertex.

\[ \therefore \eta_g(G_i) = n + 1 \]

**Theorem 3.2.2**

Let \( G \) be a bicyclic graph containing a \( U(l,m) \) and any one of the cycle is of odd length. Let \( n \) denote the number of pendant vertices of \( G \) and let \( k \) denote the number of vertices of degree greater than 2 on \( U(l,m) \) other than \( u_0 \). Then

\[
\eta_g(G) = \begin{cases} 
4 & \text{if } k = 0 \\
 n + 1 & \text{if } k \geq 2 \text{ and every } (v, w)-\text{section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree } 2 \text{ is a shortest path} \\
n + 3 & \text{otherwise}
\end{cases}
\]

**Proof:**

Let \( V(U(l,m)) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_l, u_{l+1}, \ldots, u_{l+m-2}\} \)

\[ V(C_i) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_l\} \]

\[ V(C_m) = \{u_0, u_l, u_{l+1}, \ldots, u_{l+m-2}, u_0\} \] where \( l \) is odd and \( m \) is even.

**Case 1:** \( k = 0 \)

Then \( G = U(l,m) \)

The geodesic graphoidal cover of \( G \) is as follows

\[ P_1 = \{u_l, u_{l-1}, \ldots, u_0, u_0, u_l, u_{l+1}, \ldots, u_l\} \]

\[ P_2 = \{u_l, u_{l+1}\} \]
\[ P_3 = \{u_{i+1}, \ldots, u_0\} \]

\[ P_4 = \{u_0, u_{i+m-2}, \ldots, u_i\} \quad \text{where} \quad [i = \frac{l-1}{2} \quad \& \quad t = \frac{m}{2} - 1] \]

\[ \psi = \{P_1, P_2, P_3, P_4\} \] is a minimum geodesic graphoidal cover of \( G \)

\[ \Rightarrow \eta_g \leq 4 \]

Since in any minimum geodesic graphoidal cover at least three vertices are exterior points in \( U(l,m) \) so that \( t \geq 3 \)

\[ \eta_g = q - p + t \geq 4 \]

\[ \therefore \eta_g = 4 \]

For the remaining cases the proof is similar to that of Theorem 3.2.1. Choose the deletion vertices from the odd cycle only so that the graph \( G_1 \) always will be a unicyclic graph with even cycle.

**Theorem 3.2.3**

Let \( G \) be a bicyclic graph containing a \( U(l,m) \) and both the cycles are of odd length.

Let \( n \) denote the number of pendant vertices of \( G \) and let \( k \) denote the number of vertices of degree greater than 2 on \( U(l,m) \) other than \( u_0 \). Then

\[ \eta_g(G) = \begin{cases} 5 & \text{if } k = 0 \\ n + 4 & \text{if } k = 1 \text{ and if } k = 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree 2 is not a shortest path} \\ n + 3 & \text{otherwise} \end{cases} \]

**Proof:**

Let \( V(U(l,m)) = \{u_0, u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_{i+m-2}\} \)

\( V(C_i) = \{u_0, u_1, u_2, \ldots, u_{i-1}, u_0\} \)
\[ V(C_m) = \{u_0, u_1, u_{r+1}, \ldots, u_{r+m-2}, u_0\} \]  where \( l \) and \( m \) are odd.

**Case 1:** \( k = 0 \)

Then \( G = U(l, m) \)

The geodesic graphoidal cover of \( G \) is as follows

\[ P_1 = \{u_i, u_{i-1}, \ldots, u_0, u_1, u_2, u_3, \ldots, u_l\} \]

\[ P_2 = \{u_i, u_{i+1}\} \]

\[ P_3 = \{u_{i+1}, \ldots, u_0\} \]

\[ P_4 = \{u_{i+1}, u_i\} \]

\[ P_5 = \{u_0, u_{i+2}, \ldots, u_0\} \]  where \( i = \left\lfloor \frac{m-1}{2} \right\rfloor \) and \( t = l + \left( \frac{m-1}{2} \right) - 1 \)

\[ \psi = \{P_1, P_2, P_3, P_4, P_5\} \]  is a minimum geodesic graphoidal cover of \( G \)

\( \Rightarrow \eta_g \leq 5 \)

Since in any minimum geodesic graphoidal cover at least four vertices are exterior points in \( U(l, m) \) so that \( t \geq 4 \)

\[ \eta_g = q - p + t \geq 5 \]

\( \therefore \eta_g = 5 \)

The proof for the remaining cases is similar to that of Theorem 3.2.1.

**Similar to the Theorem 3.2.1 to Theorem 3.2.3** we have the following results for

the bicyclic graphs \( D(l,m,i), C_m(i;l) \)

**Theorem 3.2.4**

Let \( G \) be a bicyclic graph containing a long dumbbell graph \( D(l,m,i) \) if both cycles are of even length (or any one of the cycle is even). Let \( n \) denote the number of
pendant vertices of $G$ and let $k$ denote the number of vertices of degree greater than 2 on the cycles of $D(l,m,i)$ other than $u_{t-1} \& u_{t+j-1}$. Then

$$
\eta_g (G) = \begin{cases} 
3 & \text{if } k = 0 \\
 n + 2 \left( \text{if } k = 1 \& \deg u_r > 3 \text{ and } r = i \text{ or } j \right) & \text{if } k \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree } 2 \text{ is a shortest path} \\
 n + 1 & \text{otherwise}
\end{cases}
$$

Case 1: $k = 0$

Then $G = D(l,m,i)$

The geodesic graphoidal cover of $G$ is as follows

$P_1 = \{u_i,u_{i-1},\ldots,u_0,u_{l-1},u_i,u_{l+1},\ldots,u_{t+1},u_{t+j},\ldots,u_j\}$

$P_2 = \{u_i,u_{i+1},u_{l+2},\ldots,u_{i-2},u_{i-1}\}$

$P_3 = \{u_j,u_{j+1},\ldots,u_{t+j-2},u_{i+j-1}\}$

$\psi = \{P_1,P_2,P_3\}$ is a minimum geodesic graphoidal cover of $G$

$\Rightarrow \eta_g \leq 3$

Since in any minimum geodesic graphoidal cover at least two vertices are exterior points in $D(l,m,i)$ so that $t \geq 2$

$\eta_g = q - p + t \geq 4$

$\therefore \eta_g (G) = 3$

The proof for the remaining cases is similar to that of Theorem 3.2.1.

**Theorem 3.2.5**

Let $G$ be a bicyclic graph containing a long dumbbell graph $D(l,m,i)$ if both cycles are of odd length. Let $n$ denote the number of pendant vertices of $G$ and let $k$ denote
the number of vertices of degree greater than 2 on the cycles of $D(l,m,i)$ other than $u_{i-1}$ & $u_{i+1}$. Then

$$
\eta_g(G) = \begin{cases} 
5 & \text{if } k = 0 \\
4 & \text{if } k = 1 \text{ & if } k = 2 \text{ and every } (v,w) \text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree 2 is not a shortest path} \\
n + 2 & \text{if } k \geq 2 \text{ and every } (v,w) \text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree 2 is a shortest path} \\
n + 3 & \text{otherwise}
\end{cases}
$$

Proof:

Case 1: $k = 0$

Then $G = D(l,m,i)$

The geodesic graphoidal cover of $G$ is as follows

$$
P_1 = \{u_i, u_{i-1}, \ldots, u_1, u_{t-1}, u_i, u_{i+1}, \ldots, u_{t+1}, u_{i+2}, \ldots, u_j\} \quad i = \frac{l-1}{2} - 1, j = l + i + \frac{m-1}{2} - 1
$$

$$
P_2 = \{u_{i+1}, u_{i+2}, \ldots, u_{i-2}, u_{i-1}\}
$$

$$
P_3 = \{u_j, u_{i+1}\}
$$

$$
P_4 = \{u_{j+1}, u_j\}
$$

$$
P_5 = \{u_j, \ldots, u_{t+m+i-2}, u_{i+1}\}
$$

$\psi = \{P_1, P_2, P_3, P_4, P_5\}$ is a minimum geodesic graphoidal cover of $G$

$\Rightarrow \eta_g \leq 5$

Since in any minimum geodesic graphoidal cover at least four vertices are exterior points in $D(l,m,i)$ so that $t \geq 4$

$$
\eta_g = q - p + t \geq 5
$$
\[ \therefore \eta_g(G) = 5 \]

The proof for the remaining cases is similar to that of Theorem 3.2.1.

**Theorem 3.2.6**

Let \( G \) be a bicyclic graph containing a \( C_m(i;l) \) if both cycles are of even length. Let \( n \) denote the number of pendant vertices of \( G \) and let \( k \) denote the number of vertices of degree greater than 2 on the boundary of \( C_m(i;l) \) other than \( u_0 \) & \( u_l \) with \( l < i, m-i \). Then

\[
\eta_g(G) = \begin{cases} 
3 & \text{if } k = 0 \\
 n + 3 & \text{if } k = 1 \deg u_r \geq 3, r \neq \frac{i + m - i}{2}, \\
 n + 1 \left( \text{if } k \geq 2 \text{ and } (v, w)\text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree } 2 \text{ is a shortest path} \right) \\
 n + 4 \left( \text{if } k \geq 2 \text{ and } (v, w)\text{-section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree } 2 \text{ is not a shortest path} \right) 
\end{cases}
\]

**Proof:**

**Case 1:** \( k = 0 \)

Then \( G = C_m(i;l) \)

The geodesic graphoidal cover of \( G \) is as follows

\[ P_1 = \{ u_t, u_{t-1}, \ldots, u_2, u_1, u_0, u_{m-1}, u_{m-2}, \ldots, u_t \} \]

In the upper half cycle the length of \( (u_0, u_t) \) section equal to length of \( (u_t, u_l) \) section and in the lower half cycle the length of \( (u_0, u_t) \) section is equal to length of \( (u_t, u_0) \) section

\[ P_2 = \{ u_t, u_{t+1}, \ldots, u_{t+1}, u_{t+1}, \ldots, u_t \} \]
\[ P_3 = \{ u_0, u_m, u_{m+1}, \ldots, u_1 \} \]

\[ \psi = \{ P_1, P_2, P_3 \} \] is a minimum geodesic graphoidal cover of \( G \)

\[ \Rightarrow \eta_g \leq 3 \]

Since in any minimum geodesic graphoidal cover at least two vertices are exterior points in \( C_m(i, l) \) so that \( \tau \geq 3 \)

\[ \eta_g = q - p + \tau \geq 3 \]

\[ \therefore \eta_g(G) = 3 \]

The proof for the remaining cases is similar to that of Theorem 3.2.1.

**Theorem 3.2.7**

Let \( G \) be a bicyclic graph containing a \( C_m(i; l) \) if both cycles are of odd length. Let \( n \) denote the number of pendant vertices of \( G \) and let \( k \) denote the number of vertices of degree greater than 2 on the boundary of \( C_m(i; l) \) other than \( u_0 \) & \( u_i \) with \( l < i, m-i \).

Then

\[ \eta_g(G) = \begin{cases} 
3 & \text{if } k = 0 \\
3 + 2 & \text{if } k = 1 \land k = 2 \text{ every } (v, w)-\text{section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree 2 is a shortest path} \\
3 + 4 & \left\{ \begin{array}{l}
\text{if } k = 2 \text{ and every } (v, w)-\text{section of } C \text{ in which all vertices except } v \text{ and } w \text{ have degree 2 is a not shortest path} \\
3 & \text{otherwise}
\end{array} \right. 
\end{cases} \]

**Proof:**

**Case 1: \( k = 0 \)**

Then \( G = C_m(i; l) \) \( (l > i) \)

The geodesic graphoidal cover of \( G \) is as follows
\[ P_1 = \{ u_i, u_{i-1}, \ldots, u_2, u_i, u_0, u_{m-1}, u_{m-2}, \ldots, u_j \} \]

In the upper half cycle the length of \((u_0, u_i)\) section equal to length of \((u_i, u_j)\) section.

\[ P_2 = \{ u_j, u_{j+1}, \ldots, u_0, u_1, \ldots, u_i \} \]

\[ P_3 = \{ u_0, u_m, u_{m+1}, \ldots, u_i \} \]

\[ \psi = \{ P_1, P_2, P_3 \} \] is a minimum geodesic graphoidal cover of \(G\)

\[ \Rightarrow \eta_g \leq 3 \]

Since in any minimum geodesic graphoidal cover atleast two vertices are exterior points in \(C_m(i, l)\) so that \(t \geq 3\)

\[ \eta_g = q - p + t \geq 3 \]

\[ \therefore \eta_g(G) = 3 \]

The proof for the remaining cases is similar to that of Theorem 3.2.1.

### 3.3 Geodesic graphoidal covering number of triangular graphs

**Theorem 3.3.1**

Let \(G\) be triangular snake graph with \(2n-1\) vertices. Then \(\eta_g(G) = 2n - 3\)

**Proof**

Let \(V(G) = \{ v_1, v_2, v_3, \ldots, v_n, w_1, w_2, \ldots, w_{n-1} \} \)

Here \(w_i\) adjacent to \(v_i\) and \(v_{i+1}\), \(q = 3(n-1)\), \(p = 2n-1\)

The geodesic graphoidal cover of \(G\) is as follows

\[ P_i = \{ v_i, w_i \}, \quad i = 2, 3, \ldots, n-1 \]

\[ P_i = \{ w_i, v_1, v_2, v_3, \ldots, v_n, w_{n-1} \} \]

\[ R_j = \{ v_{j+1}, w_j \}, \quad j = 1, 2, 3, \ldots, n-2 \]
\[ \psi = P_1 \cup \ldots \cup P_{n-1} \cup R_l \cup \ldots \cup R_{n-2} \] is a geodesic graphoidal cover of \( G \)

\[ \Rightarrow \eta_g \leq 2n - 3 \]

Since in any minimum geodesic graphoidal cover \( t \geq n - 1 \)

\[ \eta_g = q - p + t \geq 3(n-1) - 2n + 1 + n - 1 = 2n - 3 \]

\[ \therefore \eta_g(G) = 2n - 3 \]

**Theorem 3.3.2**

Let \( G \) be double triangular snake graph with \( 3n-2 \) vertices. Then \( \eta_g(G) = 4n - 5 \)

**Proof**

Let \( V(G) = \{v_1, v_2, v_3, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}\} \)

Here \( w_i \) adjacent to \( v_i \) and \( v_{i+1} \) in upward direction and \( u_i \) adjacent to \( v_i \) and \( v_{i+1} \) in downward direction.

\[ p = 3n - 2, \quad q = 5(n-1) \]

The geodesic graphoidal cover of \( G \) is as follows

\[ P_i = \{v_{i+1}, u_i\}, \quad i = 1, 2, 3, \ldots, n - 2 \]

\[ P_{n-1} = \{u_{n-1}, v_n, w_{n-1}\} \]

\[ Q_j = \{v_j, u_j\}, \quad j = 2, 3, \ldots, n - 1 \]

\[ Q_n = \{v_1, v_2, v_3, \ldots, v_n\} \]

\[ R_k = \{v_k, w_k\}, \quad i = 2, 3, \ldots, n - 1 \]

\[ R_n = \{u_i, v_1, w_1\} \]

\[ S_r = \{v_{r+1}, w_r\}, \quad r = 1, 2, 3, \ldots, n - 2 \]
\[ \psi = \{ P_i, Q_j, R_k, S_r / 1 \leq i \leq n-1, 2 \leq j, k \leq n, 1 \leq r \leq n-2 \} \] is a geodesic graphoidal cover of \( G \)

\[ \Rightarrow \eta_g \leq 4(n-2) + 3 = 4n - 5 \]

Since in any geodesic graphoidal cover \( t \geq 2n - 2 \)

\[ \eta_g = q - p + t \geq 5(n-1) - 3n + 2 + 2n - 2 \]

\[ \eta_g \geq 4n - 5 \]

\[ \therefore \eta_g(G) = 4n - 5 \]

Theorem 3.3.3

Let \( G \) be a triple triangular snake graph with \( 4n-3 \) vertices. Then \( \eta_g(G) = 6n - 7 \)

Proof:

Let \( V(G) = \{ v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_{n-1}, u_1, u_2, \ldots, u_{n-1}, z_1, z_2, \ldots, z_{n-1} \} \)

\( w_i, u_i, z_i \) is adjacent to \( v_i \) and \( v_{i+1} \)

The geodesic graphoidal cover of \( G \) is as follows:

\[ P_i = \{ w_i, v_1, v_2, v_3, \ldots, v_n, w_{n-1} \} \]

\[ Q_r = \{ w_r, v_{r+1} \}, r = 1, 2, \ldots, n-2 \]

\[ R_j = \{ w_j, v_j \}, j = 2, 3, \ldots, n-1 \]

\[ S_i = \{ z_i, v_i \}, i = 1, 2, 3, \ldots, n-1 \]

\[ T_i = \{ z_i, v_{i+1} \}, i = 1, 2, 3, \ldots, n-1 \]

\[ X_i = \{ u_i, v_i \}, i = 1, 2, 3, \ldots, n-1 \]

\[ Y_i = \{ u_i, v_{i+1} \}, i = 1, 2, 3, \ldots, n-1 \]
\( \psi = \{P_i, Q_i, R, S, T_i, X, Y_i / 1 \leq r \leq n-2, 2 \leq j \leq n-1, 1 \leq i \leq n-1\} \) is a geodesic graphoidal cover of \( G \)

\[ \Rightarrow \eta_g \leq 6n - 7 \]

Since in any geodesic graphoidal cover for the triple triangular graph, we have

\[ t \geq 3(n-1) \]

\[ \eta_g \geq q - p + t = 7n - 7 - 4n + 3 + 3n - 3 = 6n - 7 \]

\[ \therefore \eta_g(G) = 6n - 7 \]

**Theorem 3.3.4**

Let \( G \) be alternate double triangular snake graph. Then

\[ \eta_g(G) = \begin{cases} 2n + 1 & \text{if } n \text{ is even} \\ 2n - 1 & \text{if } n \text{ is odd} \end{cases} \]

**Proof:**

Let \( V(G) = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_{\frac{n-1}{2}}, u_1, u_2, \ldots, u_{\frac{n-1}{2}}\} \) if \( n \) is odd

or \( V(G) = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_{\frac{n}{2}}, u_1, u_2, \ldots, u_{\frac{n}{2}}\} \) if \( n \) is even

\( w_i \) is adjacent to \( v_j \) and \( v_{i+1}, i = 1, 3, 5, \ldots \)

\( u_j \) is adjacent to \( v_i \) and \( v_{i+1}, i = 1, 3, 5, \ldots \)

The geodesic graphoidal cover of \( G \) is as follows

\[ P_1 = \{v_1, v_2, \ldots, v_n\} \]

\[ P_i = \{v_i, w_j\}, i = 1, 3, 5, \ldots, n-1 \text{ if } n \text{ is even, } n-2 \text{ if } n \text{ is odd} \ 	ext{ and } j = 1, 2, 3, \ldots, \frac{n-1}{2} \]

\[ Q_i = \{v_i, w_j\}, j = 1, 2, 3, \ldots, \frac{n-1}{2} \text{ if } n \text{ is odd or } j = 1, 2, 3, \ldots, \frac{n}{2} \text{ if } n \text{ is even} \ 	ext{ and } i = 2, 4, 6, \ldots \text{ if } n \text{ is even or } i = 2, 4, 6, \ldots, n-1 \text{ if } n \text{ is odd} \]
$R_i = \{v_i, u_j\}, i = 1, 3, 5, \ldots & j = 1, 2, 3, \ldots$

$S_i = \{v_i, u_j\}, i = 2, 4, 6, \ldots & j = 1, 2, 3, \ldots$

$\psi = \{P, P_i, Q, R_i, S_i\}$ is a geodesic graphoidal cover of $G$

If $n$ is odd then $\eta_g(G) = 1 + 4 \left(\frac{n-1}{2}\right) = 2n - 1$

If $n$ is even then $\eta_g(G) = 1 + 4 \left(\frac{n}{2}\right) = 2n + 1$

**Theorem 3.3.5**

Let $G$ be triangular cactus graph with $n$ triangles. Then $\eta_g(G) = 3n - 1$

**Proof**

Let $V(G) = \{v_0, v_1, v_2\}$ for $i = 1$ to $n$

$v_0$ is adjacent to $v_1$ & $v_2, i = 1, 2, \ldots, n$ and $\deg v_0 = 2n$, $p = 2n + 1$ and $q = 3n$

The geodesic graphoidal cover of $G$ is as follows

$P = \{v_{i_2}, v_0, v_{i_2}\}$

$S =$ The remaining edges

$\psi = \{P, S\}$ is a geodesic graphoidal cover of $G$

$\Rightarrow \eta_g \leq 2n - 1$

Since in any minimum geodesic graphoidal cover $t \geq n$

$\eta_g = q - p + t \geq 3n - 2n - 1 + 2n$

$\eta_g \geq 3n - 1$

$\therefore \eta_g(G) = 3n - 1$
3.4 Geodesic graphoidal covering number of some classes of graphs

Theorem 3.4.1

For $P_m \times P_n$, the geodesic graphoidal covering number is $\eta_g (P_m \times P_n) = q - p + 2$.

Proof:

Let $V(G) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$, $i = 1, 2, \ldots, m$

Here $p = mn$ and $q = m(n-1) + n(m-1)$

The geodesic graphoidal cover of $P_m \times P_n$ is as follows:

$P_i = \{v_{i+1,1}, v_{i+1,2}, \ldots, v_{i+1,m}\}$, $i = 1, 2, \ldots, m - 1$

$P_m = \{v_{1n}, v_{2n}, v_{3n}, \ldots, v_{mn}, v_{m,n-1}, v_{m,n-2}, \ldots, v_{m2}, v_{m1}\}$

$S =$ The remaining edges not covered by $P_1, P_2, P_3, \ldots, P_{m-1}, P_m$

$\psi = P_1 \cup P_2 \cup \ldots \cup P_m \cup S$ is a minimum geodesic graphoidal cover of $G$

From above we see that all the paths are shortest paths and all the vertices of $P_m \times P_n$

are internal vertices in at least one path except $v_{i_n}$ and $v_{m_1}$

$\Rightarrow \eta_g \leq q - p + 2$

Since $\eta_g = q - p + t \geq q - p + 2$

Therefore $\eta_g = q - p + 2$

Note:

For $P_m \times P_n$, $\eta_a = \eta_g = q - p + 2$

Theorem 3.4.2

Let $G$ be a gear graph with $2n+1$ vertices and $3n$ edges. Then

$\eta_g (G) = 2n - 1 = q - p + n$. 
Proof:

Let \( V(G) = \{v_0, v_1, v_2, v_3, ..., v_n, w_1, w_2, ..., w_n \} \)

where \( v_0 \) is the centre vertex of wheel and \( w_j \) adjacent to \( v_i \) & \( v_{i+1} \), \( w_n \) is adjacent to \( v_1 \) and \( v_0 \) and \( p = 2n + 1 \) and \( q = 3n \)

The geodesic graphoidal cover of \( G \) is as follows

\[
P_i = \{v_i, w_i, v_{i+1}\} \quad i = 1, 2, ..., n - 1
\]

\[
P_n = \{v_n, w_n, v_1\}
\]

\[
Q_j = \{v_0, v_j\} \quad j = 2, 3, ..., n - 1
\]

\[
Q_n = \{v_1, v_n, v_n\}
\]

\[
\psi = \{P_i, Q_j / 1 \leq i \leq n, 2 \leq j \leq n\} \quad \text{is a geodesic graphoidal cover of} \ G
\]

\[
\Rightarrow \eta_g \leq 2n - 1
\]

Since in any minimum geodesic graphoidal cover \( t \geq n \)

\[
\eta_g = q - p + t \geq 3n - 2n - 1 + n
\]

\[
\eta_g \geq 2n - 1
\]

\[\therefore \eta_g (G) = 2n - 1 = q - p + n \]

Theorem 3.4.3

Let \( G \) be a helm graph with \( 2n + 1 \) vertices and \( 3n \) edges. Then \( \eta_g = 2n - 1 = q - p + n \).

Proof:

Let \( V(G) = \{v_0, v_1, v_2, v_3, ..., v_n, t_1, t_2, ..., t_n\} \)

Where \( v_0 \) is the centre vertex of wheel and is \( v_i \) adjacent to \( t_i \) and \( v_0 \)

and \( P = 2n + 1 \) and \( q = 3n \)
The geodesic graphoidal cover of $G$ is as follows

$$P_i = \{t_i, v_i, v_{i+1}\}, i = 1,2,\ldots,n-1$$

$$P_n = \{t_n, v_n, v_1\}$$

$$Q_j = \{v_0, v_j\}, j = 2,4,5,\ldots,n$$

$$Q_3 = \{v_3, v_0, v_3\}$$

$$\psi = \{P_i, Q_j | 1 \leq i \leq n, 2 \leq j \leq n\}$$ is a geodesic graphoidal cover of $G$

$$\Rightarrow \eta_g(G) \leq 2n - 1$$

Since in any minimum geodesic graphoidal cover $t \geq n$

$$\eta_g = q - p + t \geq 3n - 2n - 1 + n$$

$$\eta_g \geq 2n - 1$$

$$\therefore \eta_g(G) = 2n - 1$$

**Theorem 3.4.4**

Let $G$ be $P_m(QS_n)$ graph. Then $\eta_g(G) = m(n+1) - 1$

**Proof**

Let $V(G) = \{v_1, v_2, \ldots, v_m, l_{i,j}, l_{i',j'}, l_{i''}, l_{i'''}, l_{i''''}, r_{i,j}, r_{i',j'}, r_{i''}, r_{i'''}, r_{i''''}, w_{i,j}, w_{i',j'}, w_{i''}, w_{i'''}, w_{i''''}\} i = 1,2,\ldots,n$

**Case 1: $m$ is even**

The geodesic graphoidal cover of $G$ is as follows

$$P_j = \{w_{i,n}, l_{i,n}, w_{i,(n-1)}, \ldots, w_{i,j}, l_{i,j}, w_{i,j+1}, l_{i,j+1}, v_i, v_{j+1}, l_{j+1}, \ldots, l_{i,n}, w_{i,n}\}, j = 1,3,\ldots,m-1$$

$$T_k = \{v_k, v_{k+1}\}, k = 2,4,\ldots,m-2$$

$$Q_i = \{v_i, r_{i,j}, w_{i,j}\}, i = 1,2,\ldots,m$$

$$R_i = \{w_{i,j}, r_{i,j}, w_{i,j}\}, i = 1,2,\ldots,m$$
\[ S_i = \{ w_{(n-i)}, r_m, w_m \}, \quad i = 1, 2, \ldots, m \]

\[ \psi = \{ P_j, Q_i, R_k, S_t, T_j / j = 1, 3, \ldots, m-1, k = 2, 4, \ldots, m-2, 1 \leq i \leq m \} \]

is a geodesic graphoidal cover of \( G \)

\[ \Rightarrow \eta_g \leq m - 1 + mn = m(n+1) - 1 \]

Since in any minimum geodesic graphoidal cover \( t \geq m \) (\( w_m \) are exterior points)

\[ \eta_g = q - p + t \geq 4mn + m - 1 - 3mn - m + m \]

\[ \eta_g \geq m(n+1) - 1 \]

\[ \therefore \eta_g (G) = m(n+1) - 1 \]

**Case 2:** \( m \) is odd

The geodesic graphoidal cover of \( G \) is as follows

\[ P_j = \{ w_{(j)}, l_j, w_{(j-1)}, \ldots, w_{(1)}, l_1, v_j, v_{j+1}, l_{j+1}, \ldots, w_{(j-1)}, l_{j-1}, w_{j+n} \}, \quad j = 1, 3, \ldots, m - 1 \]

\[ P_m = \{ v_{m-1}, v_m, l_{m-1}, w_{m-1}, \ldots, l_m, w_m \} \]

\[ T_k = \{ v_k, v_{k+1} \}, \quad k = 2, 4, \ldots, m - 3 \]

\[ Q_i = \{ v_i, r_i, w_i \}, \quad i = 1, 2, \ldots, m \]

\[ R_i = \{ w_{i}, r_i, w_{i} \}, \quad i = 1, 2, \ldots, m \]

\[ \ldots \]

\[ S_i = \{ w_{(m-i)}, r_m, w_m \}, \quad i = 1, 2, \ldots, m \]

\[ \psi = \{ P_m, P_j, T_k, Q_i, R_s, S_t / j = 1, 3, 5, \ldots, m-1, k = 2, 4, \ldots, m-2, 1 \leq i \leq m \} \]

is a geodesic graphoidal cover of \( G \)

\[ \Rightarrow \eta_g \leq m - 1 + mn = m(n+1) - 1 \]
Since in any minimum geodesic graphoidal cover $t \geq m$

$$\eta_g = q - p + t \geq 4nm + m - 3mn - m + m$$

$$\eta_g \geq m(n + 1) - 1$$

$$\therefore \eta_g(G) = m(n + 1) - 1$$

**Theorem 3.4.5**

Let $G$ be $C_m(QS_n)$ graph. Then $\eta_g(G) = m(n + 1)$

**Proof**

Let $V(G) = \{v_1, v_2, v_3, \ldots, v_m, l_{j1}, l_{j2}, \ldots, l_{jn}, r_{j1}, r_{j2}, \ldots, r_{jm}, w_{j1}, w_{j2}, \ldots, w_{jn}\}$ $i = 1, 2, \ldots, n$

**Case 1: $m$ is even**

The geodesic graphoidal cover of $G$ is as follows

$$P_j = \{w_{jn}, l_{jn}, w_{j(n-1)}, \ldots, w_{j1}, l_j, v_j, v_{j+1}, l_{j+1}, v_{j+1}, \ldots, w_{j(n-1)}, l_{j+1}, w_{j+1}\}, j = 1, 3, \ldots, m - 1$$

$$T_k = (v_k, v_{k+1}), k = 2, 4, \ldots, m - 2$$

$$w = (v_m, v_1)$$

$$Q_i = \{v_i, r_{i1}, w_{i1}\}, i = 1, 2, \ldots, m$$

$$R_i = \{w_{i1}, r_{i2}, w_{i2}\}, i = 1, 2, \ldots, m$$

$$\vdots$$

$$S_i = \{w_{i(m-1)}, r_{im}, w_{im}\}, i = 1, 2, \ldots, m$$

$$\psi = \{P_j, w, T_k, Q_i, R_j, S_i / j = 1, 3, \ldots, m - 1, k = 2, 4, \ldots, m - 2, 1 \leq i \leq m\}$$ is a geodesic graphoidal cover of $G$

$$\Rightarrow \eta_g \leq m + mn$$
Since in any minimum geodesic graphoidal cover \( t \geq m \)

\[ \eta_g = q - p + t \geq 4mn + m - 3mn - m + m \]

\[ \eta_g \geq mn + m \]

\[ \therefore \eta_g(G) = m(n+1) \]

**Case 2: \( m \) is odd**

The geodesic graphoidal cover of \( G \) is as follows

\[
P_j = \{ w_{jn}, l_{jn}, w_{j(n-1)}, \ldots, l_{j1}, v_j, v_{j+1}, l_{j+1,1}, w_{j+1,1}, \ldots, w_{j+n(n-1)}, l_{j+n,1}, w_{j+n,1} \}, \quad j = 1, 3, \ldots, m - 1
\]

\[
P_m = \{ w_{m-1,n}, u_{m-1,n}, \ldots, u_{n-1,1}, v_m \}
\]

\[ w = (v_m, v_1) \]

\[ T_k = (v_k, v_{k+1}), \quad i = 2, 4, \ldots, m - 3 \]

\[ Q_i = \{ v_i, r_i, w_i \}, \quad i = 1, 2, \ldots, m \]

\[ R_i = \{ w_{1i}, r_{i2}, w_{i2} \}, \quad i = 1, 2, \ldots, m \]

\[
S_i = \{ w_{(i-1)m}, r_{im}, w_{im} \}, \quad i = 1, 2, \ldots, m
\]

\[ \psi = \{ P_j, P_m, Q, W, T_k, R, S_i \} / j = 1, 3, \ldots, m - 1, k = 2, 4, \ldots, m - 3, 1 \leq i \leq m \}
\]

is a geodesic graphoidal cover of \( G \)

\[ \Rightarrow \eta_g \leq m + mn \]

Since in any minimum geodesic graphoidal cover \( t \geq m \)

\[ \eta_g = q - p + t \geq mn + m \]

\[ \eta_g \geq mn + m \]

\[ \therefore \eta_g(G) = m(n+1) \]
Theorem 3.4.6

Let $G$ be a web graph with $3n+1$ vertices and $5n$ edges. Then $\eta_g(G) = q - p + n$.

Proof:

Let $V(G) = \{v_0, v_1, v_{i+1}, v_{i+2}\}$ $i = 1, 2, ..., n$

Here $v_{i+2}$ is adjacent to $v_0$ and $v_{i+1}$ and $v_{i+1}$ is adjacent to $v_i$ and $v_{i+2}$

The geodesic graphoidal cover of $G$ is as follows

$P_i = \{v_{i+1}, v_{i+2}v_{i+1}\}$ $i = 1, 2, ..., n - 1$

$P_n = \{v_{i+1}, v_{i+2}\}$

$Q_i = \{v_{i+1}, v_{i+2}, v_{i+2}\}$ $i = 1, 2, ..., n - 1$

$Q_n = \{v_{i+1}, v_{i+2}, v_{i+2}\}$

$R_j = \{v_0v_{j-2}\}$ $j = 2, 3, 4, ..., n - 1$

$R_n = \{v_{i+2}, v_{i+2}, v_{i+2}\}$

$\psi = \{P_i\} \cup \{Q_i\} \cup \{R_j\}$ $1 \leq i \leq n, j = 2, 3, 4, ..., n - 1$ is a geodesic graphoidal cover of $G$

$\Rightarrow \eta_g \leq 3n - 1 = q - p + n$

Since in any minimum geodesic graphoidal cover $t \geq n$

$\eta_g = q - p + t \geq 5n - 3n - 1 + n$

$\eta_g \geq 3n - 1$

$\therefore \eta_g(G) = 3n - 1$
Theorem 3.4.7

Let $G$ be a ladder graph. Then $\eta_g(G) = q - p + 2$.

Proof:

Let $V(G) = \{u_1, u_2, u_3, ..., u_n, l_1, l_2, ..., l_n\}$

The geodesic graphoidal cover of $G$ is as follows

$P_i = \{u_i, l_i\} \text{ } i = 2, 3, ..., n - 1$

$P_2 = \{u_1, l_2, ..., l_n\}$

$P_n = \{u_1, u_2, u_3, ..., u_n, l_n\}$

$\psi = R \cup P_2 \cup \{P_i\}_{i=2}^{n-1}$ is a geodesic graphoidal cover of $G$

$\Rightarrow \eta_g \leq n = q - p + 2$

Since in any minimum geodesic graphoidal cover $t \geq 2$

$\eta_g = q - p + t \geq 3n - 2 - 2n + 2$

$\eta_g \geq n$

$\therefore \eta_g(G) = n$

Theorem 3.4.8

Let $G$ be a shell graph with $n + 1$ vertices. Then $\eta_g(G) = \frac{3n - 3}{2}$.

Proof:

Let $V(G) = \{v_0, v_1, v_2, v_3, ..., v_n\}$

$p = n + 1, q = 2n - 2$

The geodesic graphoidal cover of $G$ is as follows

$P_i = \{v_0, v_i\} \text{ } i = 2, 3, ..., n - 1$
Let $G$ be a book graph with $2n+2$ vertices. Then $\eta_g(G) = q - p + n = 2n - 1$.

**Proof:**

Let $V(G) = \{v_i, v_2, b_{i1}, b_{i2}\}$ $i = 1, 2, ..., n$

The geodesic graphoidal cover of $G$ is as follows

$P_1 = \{v_1, v_2\}$

$P_2 = \{b_{11}, b_{12}, v_2, b_{22}\}$

$P_{n+1} = \{b_{n1}, v_1, b_{21}, b_{22}\}$

$P_i = \{b_{i1}, b_{i2}, v_2\} \quad i = 3, 4, ..., n$

$Q_i = \{v_i, b_{i1}\} \quad i = 3, 4, ..., n$

$\psi = \{P_i\} \cup \{Q_i\} / 3 \leq i \leq n$ is a geodesic graphoidal cover of $G$
\[ \Rightarrow \eta_g \leq 2n - 1 = q - p + n \]

Since in any minimum geodesic graphoidal cover \( t \geq n \)
\[ \eta_g = q - p + t \geq 3n + 1 - 2n - 2 + n \]
\[ \eta_g \geq 2n - 1 \]
\[ \therefore \eta_g(G) = 2n - 1 \]

**Theorem 3.4.10**

Let \( G \) be mongolian tent graph. Then \( \eta_g(M_{m,n}) = q - p + 2 \)

**Proof:**

Let \( G = M_{m,n} \)

Let \( V(G) = \{v_0, v_1, v_2, \ldots, v_m\} \)
\( i = 1, 2, \ldots, m \)

The geodesic graphoidal cover of \( G \) is as follows
\[ P_i = \{v_{i+1}, v_{i+2}, \ldots, v_m\} \quad i = 1, 2, \ldots, m-1 \]
\[ P_n = \{v_{1n}, v_{2n}, v_{3n}, \ldots, v_{mn}, v_{mn-1}, v_{m,n-2}, \ldots, v_{m2}, v_{m1}\} \]
\[ P_{n+1} = \{v_{1n}, v_0, v_{1n}\} \]

\( S = \) The remaining edges

From above paths all the vertices are exterior points except \( v_{1n} \) and \( v_{m1} \)
\[ \psi = P_1 \cup P_2 \cup \ldots \cup P_n \cup P_{n+1} \cup S \] is a geodesic graphoidal cover of \( G \)

\[ \Rightarrow \eta_g \leq q - p + 2 \]

Since in any minimum geodesic graphoidal cover \( t \geq 2 \)
\[ \eta_g = q - p + t \geq q - p + 2 \]
\[ \therefore \eta_g = q - p + 2 \]
Theorem 3.4.11

Let $G$ be double wheel graph with $3n+1$ vertices. Then $\eta_g(G) = \begin{cases} 2n+6 & \text{if } n \text{ is odd} \\ 2n+4 & \text{if } n \text{ is even} \end{cases}$.

**Proof:**

Let $V(G) = \{v_0, c_{11}, c_{12}, \ldots, c_{1n}, c_{21}, c_{22}, \ldots, c_{2n}\}$

Clearly all the $2n$ edges which are incident to $v_0$ are the shortest paths.

Therefore the remaining edges are nothing but the two cycles.

We know that $\eta_g(G) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$

$\Rightarrow \eta_g(G) = \begin{cases} 2n+6 & \text{if } n \text{ is odd} \\ 2n+4 & \text{if } n \text{ is even} \end{cases}$

Theorem 3.4.12

Let $G$ be a $t$-ply graph. Then $\eta_g(G) = t$

**Proof:** Let $V(G) = \{u, v_{i1}, v_{i2}, \ldots, v_{in}, v\} \ i = 1, 2, \ldots, t$

The geodesic graphoidal cover of $G$ is as follows:

$P_i = \{u, v_{i1}, v_{i2}, \ldots, v_{in}, v\} \ i = 1, 2, 3, 4, \ldots, t$

Here $P_i$ is a geodesic graphoidal cover of $G$,

$\eta_g \leq t$

Since any cycle at least two vertices are exterior points, hence we have $t \geq 2$

$\eta_g(G) = q - p + t \geq nt + t - nt - 2 + 2 = t$

$\eta_g(G) = t$
Theorem 3.4.13

Let $G$ be a mobius graph $M_{2,5}$. Then $\eta_g(G)=7$

Proof: Let $V(G)=\{v_1, v_2, \ldots, v_5, w_1, w_2, \ldots, w_3\}$

The geodesic graphoidal cover of $G$ is as follows:

$P_1 = \{w_1, w_2, w_3, w_4\}$

$P_2 = \{w_1, v_1, w_5, w_4\}$

$P_3 = \{w_1, v_5, v_4, w_4\}$

$P_4 = \{v_1, v_2, v_3, v_4\}$

$P_5 = \{w_2, v_2\}$

$P_6 = \{w_3, v_3\}$

$P_7 = \{w_5, v_5\}$

$\psi = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$ is a geodesic graphoidal path covering of $G$

$\Rightarrow \eta_g \leq 7$

$\therefore \eta_g(G) \geq q - p + t = 15 - 10 + 2 = 7$

$\therefore \eta_g = 7$

Theorem 3.4.14

Let $G$ be a pumpkin graph $P_4^5$. Then $\eta_g(P_4^5) = 13 = q - p + t$

Proof:

Let $V(G) = \{y_1, y_2, y_3, y_4, x_{i,1} (i = 1to5), x_{2,i,1}, x_{2,i,2} (i = 1to5), x_{3,i,1}, x_{3,i,2}, x_{3,i,3} (i = 1to5)\}$
Let \( G \) be pumpkin graph \( P_4^5 \)

The geodesic graphoidal cover of \( G \) is as follows:

\[
P_i = \{y_1, x_{i,1}, y_2, x_{i,2}, y_3, x_{i,3}, x_{i,1,2}, x_{i,1,3}, y_4\}
\]

\[
P_i = \{y_1, x_{i,1}, y_2\} \quad i = 2, 3, 4, 5
\]

\[
Q_i = \{y_2, x_{i,2}, x_{i,1,2}, y_3\} \quad i = 2, 3, 4, 5
\]

\[
Q_i = \{y_3, x_{i,3}, x_{i,1,3}, y_4\} \quad i = 2, 3, 4, 5
\]

\[
\psi = \{P_i, P_j, Q_i, R_i \mid 2 \leq i \leq 5\} \text{ be a geodesic graphoidal cover of } G
\]

\[
\eta_g \leq 13
\]

Since \( \eta_g \geq q - p + t = 13 \)

\[
\therefore \eta_g(G) = 13 = q - p + t
\]

**Theorem 3.4.15**

Let \( G \) be a fan graph with \( n \ (n \geq 3) \) vertices. Then \( \eta_g(G) = \begin{cases} \frac{3n - 2}{2} & \text{if } n \text{ is even} \\ \frac{3n - 3}{2} & \text{if } n \text{ is odd} \end{cases} \)

**Proof:** Let \( V(G) = \{v_0, v_1, v_2, \ldots, v_n\} \)

The geodesic graphoidal cover of \( G \) is as follows:

\[
P = \{v_1, v_0, v_3\}
\]

\[
Q_i = \{v_i, v_{i+1}, v_{i+2}\}, i = 1, 3, 5, \ldots, n - 2 \text{ if } n \text{ is odd} \& i = 1, 3, 5, \ldots, n - 3 \text{ if } n \text{ is even}
\]

\[
S = \text{The remaining edges}
\]

\[
\psi = \{Q_i\} \cup \{P, S\} \text{ be a geodesic graphoidal cover of } G
\]
\[ \eta_g(G) = \begin{cases} 
\frac{3n-2}{2} & \text{if } n \text{ is even} \\
\frac{3n-3}{2} & \text{if } n \text{ is odd} 
\end{cases} \]

Since \( \eta_g(G) = q - p + t \geq \begin{cases} 
\frac{3n-3}{2}, & n \text{ is odd} \\
\frac{3n-2}{2}, & n \text{ is even} 
\end{cases} \)

\[ \therefore \eta_g(G) = \begin{cases} 
\frac{3n-2}{2} & \text{if } n \text{ is even} \\
\frac{3n-3}{2} & \text{if } n \text{ is odd} 
\end{cases} \]

**Theorem 3.4.16**

Let \( G \) be windmill graph \( (k_4^{(n)}) \) graph with \( mn+1 \) vertices. Then \( \eta_g(G) = 3(2n-1) \)

**Proof:**

Let \( V(G) = \{v_{i1}, v_{i2}, v_{i3}, v_i \} \), \( i = 1, 2, ..., n \)

The geodesic graphoidal cover of \( G \) is as follows

\[ P_1 = \{v_{i1}, v_i, v_{i2} \} \]

\( S = \) The remaining edges

\[ \psi = \{P_1, S\} \] be a minimum geodesic graphoidal path cover of \( G \)

\[ \eta_g = n \left( \frac{4(4-1)}{2} \right) - 2 = 6n - 2 \]
3.5 Conclusion

In this Chapter, the geodesic graphoidal covering number \( \eta_g \) of bicyclic graphs containing \( U(l,m), D(l,m,i), C_m(l,i) \) and for various types of graphs like Triangular snake, Double Triangular snake, Alternate double Triangular snake, Triple Triangular snake, Web graph, Gear graph, Double wheel, Triangular Cactus, Helm graph, Mobious Ladder, Mongolian Tent, \( P_m \times P_n \), Shell graph, Multipler shell graph, Ladder graph, t-ply, Book graph, Pumpkin graph, \( P_m(Q_n) \), \( C_m(Q_n) \), Fan graph and Windmill graph were found.

We observe that for the following graphs The geodesic graphoidal covering number and graphoidal covering number are same.

(i) Book graph  (ii) Helm graph (iii) Web graph  (iv) Ladder graph  
(v) Mongolian Tent graph  (v) Triangular cactus graph  (vi) Triangular snake graph

In a similar way we can characterize for which classes of graphs have \( \eta_a = \eta_g \).