CHAPTER-II

ACYCLIC GRAPHOIDAL COVERING NUMBER OF PRODUCT GRAPHS

2.1 Introduction

The concept of graphoidal cover was introduced by B.D. Acharaya and E. Sampathkumar [1]. A graphoidal cover of a graph \( G \) is a collection \( \psi \) of (not necessarily open) paths in \( G \) satisfying the following conditions:

(i) Every path in \( \psi \) has at least two vertices.

(ii) Every vertex of \( G \) is an internal vertex of at most one path in \( \psi \).

(iii) Every edge of \( G \) is in exactly one path in \( \psi \)

The minimum cardinality of a graphoidal cover of \( G \) is called the graphoidal covering number of \( G \) and is denoted by \( \eta(G) \).

B.D. Acharaya and E. Sampathkumar observe that, in the city road network with \( G \) as the underlying graph, the ideal limit for any reasonable measure of “mobility” of traffic flow is \( \eta(G) = \min \{|\psi|: \psi \in G'\} \) where \( G' \) denotes the set of all distinct graphoidal covers of \( G \). A Study of this parameter was initiated by B.D. Acharya and E. Sampath Kumar. Further results on graphoidal covering number were obtained by C. Packkiam and S. Arumugam [26]. Various types of graphoidal covers and corresponding parameters have been studied by several authors by imposing conditions on the members of the decomposition.

The concept of acyclic graphoidal covers was introduced by S. Arumugam and J. Suresh Suseela [3]. In the graphoidal cover every path is open then it is called acyclic graphoidal cover. The minimum cardinality of an acyclic graphoidal cover of
\( G \) is called the acyclic graphoidal covering number of \( G \) and is denoted by \( \eta_a(G) \) or \( \eta_a \).[8]

A graphoidal cover \( \psi \) of a graph \( G \) is called an induced acyclic graphoidal cover if every member of \( \psi \) is an induced path. The minimum cardinality of an induced acyclic graphoidal cover of \( G \) is called the induced acyclic graphoidal covering number of \( G \) and is denoted by \( \eta_{ia}(G) \) or \( \eta_{ia} \) [29].

Let \( \psi \) be a collection of internally edge disjoint paths in \( G \). A vertex of \( G \) is said to be an internal vertex of \( \psi \) if it is an internal vertex of some path in \( \psi \), otherwise it is called an external vertex of \( \psi \).

For any graphoidal cover \( \psi \) of \( G \), let \( t_\psi \) denote the number of exterior vertices of \( \psi \) [6,7].

Let \( t = \min t_\psi \) where the minimum is taken over all graphoidal covers of \( G \). Then

\[ \eta = q - p + t. \]

For any graph \( G \), \( \eta \geq q - p \). Moreover the following are equivalent [6,7]

(i) \( \eta = q - p \)

(ii) There exists a graphoidal cover without exterior vertices.

(iii) There exists a set of internally disjoint and edge disjoint paths without exterior vertices.

For any graph \( G \), \( \delta \geq 3 \), \( \eta = q - p \).

In this chapter we find minimum acyclic graphoidal covering number of Cartesian product and weak product of graphs and also determine \( \eta_a \) for some classes of graphs.
2.2 Graphoidal covering number of product graphs

**Theorem 2.2.1**

The acyclic graphoidal covering number of the rectangular grid \( P_m \times P_n \) is

\[
\eta_a(P_m \times P_n) = q - p.
\]

**Proof:**

Let \( p = mn \) and \( q = m(n-1) + n(m-1) \)

Let \( V(G) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}, i = 1, 2, \ldots, m \)

The acyclic graphoidal cover of \( P_m \times P_n \) is as follows:

\[
P_1 = \{v_{i_2}, v_{i_1}, v_{2i_1}, v_{3i_1}, \ldots, v_{mi_1}, v_{m_1} \}
\]

\[
P_2 = \{v_{i_3}, v_{i_2}, v_{2i_2}, v_{3i_2}, \ldots, v_{mi_2}, v_{m_2} \}
\]

\[
P_3 = \{v_{i_4}, v_{i_3}, v_{2i_3}, v_{3i_3}, \ldots, v_{mi_3}, v_{m_3} \}
\]

\[\vdots\]

\[
P_{n-1} = \{v_{2i_n}, v_{i_n}, v_{1i_n}, v_{2i_{n-1}}, v_{3i_{n-1}}, \ldots, v_{mi_{n-1}}, v_{m_{n-1}} \}
\]

\[
P_n = \{v_{2i_{n-1}}, v_{2i_n}, v_{3i_n}, v_{4i_n}, \ldots, v_{mi_{n-1}}, v_{m_{n-1}} \}
\]

\[P_{n+1} = \text{The remaining edges}\]

\[\psi = \{P_1, P_2, \ldots, P_n, P_{n+1}\} \text{ is an acyclic graphoidal cover of } G.\]

From above we see that all the vertices of \( P_m \times P_n \) are internal vertices.

\[
\eta_a(P_m \times P_n) \leq m(n-1) - n = q - p
\]

\[\therefore \eta \geq q - p\]

Therefore \( \eta_a(P_m \times P_n) = m(n-1) - n = q - p \)
Theorem 2.2.2

For the graph $P_m \circ P_n$, the acyclic graphoidal covering number is $\eta_a(P_m \circ P_n) = q - p + 6$

Proof:

Case (i): $m$ is even ($m \geq 6$)

Let $V(G) = \{v_{i1}, v_{i2}, \ldots, v_{im}\}, i = 1, 2, \ldots, m$

$P_1 = \{v_{11}, v_{22}, v_{31}, v_{42}, \ldots, v_{m-1,1}, v_{m2}, v_{m1,3}\}$

$P_2 = \{v_{12}, v_{23}, v_{32}, v_{43}, \ldots, v_{m-1,2}, v_{m3}, v_{m1,4}\}$

$P_3 = \{v_{13}, v_{24}, v_{33}, v_{44}, \ldots, v_{m-1,3}, v_{m4}, v_{m1,5}\}$

$\vdots$

$P_{n-1} = \{v_{2,n-2}, v_{1,n-1}, v_{2n}, v_{3n-1}, v_{4n}, \ldots, v_{m-1,n-1}, v_{mn}\}$

$LP_1 = \{v_{32}, v_{41}, v_{52}\}$

$LP_2 = \{v_{52}, v_{61}, v_{72}\}$

$\vdots$

$LP_{\lfloor \frac{m}{2} \rfloor - 1} = \{v_{m-3,2}, v_{m-2,1}, v_{m-1,2}\}$

$RP_1 = \{v_{2,n-1}, v_{3,n}, v_{4,n-1}\}$

$RP_2 = \{v_{4n-1}, v_{5n}, v_{6n-1}\}$

$\vdots$

$RP_{\lfloor \frac{m}{2} \rfloor - 1} = \{v_{m-2,n-1}, v_{m-3,n}, v_{m-4,n-1}\}$

$S = \text{The remaining edges}$

$\psi = \{P_i\} \cup \{LP_j\} \cup \{RP_k\} \cup S / 1 \leq i \leq n-1, 1 \leq j, k \leq \frac{(m-3)-1}{2}$ is an acyclic graphoidal cover of $G$. 
From above we see that all the vertices of $P_m \circ P_n$ are internal vertices except the vertices $v_{11}, v_{21}, v_{mn}, v_{1n}, v_{m-1n}, v_{m1}$.

Therefore $\eta_a (P_m \circ P_n) = q - p + 6$

**Case (ii):** $m$ is odd ($m \geq 7$).

$P_1 = \{v_{11}, v_{22}, v_{31}, v_{42}, \ldots, v_{m-12}, v_{m1}\}$

$P_2 = \{v_{21}, v_{12}, v_{23}, v_{32}, v_{43}, \ldots, v_{m-1,3}, v_{m2}, v_{m-1,1}\}$

$P_3 = \{v_{22}, v_{13}, v_{24}, v_{33}, v_{44}, v_{53}, \ldots, v_{m-1,4}, v_{m3}, v_{m-1,2}\}$

$\vdots$

$P_{n-1} = \{v_{2n-2}, v_{1n-1}, v_{2n}, v_{3n-1}, v_{4n}, \ldots, v_{m-1,n-1}, v_{m,n-1}, v_{m-1,n-2}\}$

$L_P_i = \{v_{i2}, v_{i1}, v_{i52}\}$

$L_P_2 = \{v_{i52}, v_{i61}, v_{i72}\}$

$\vdots$

$L_P_{(m-4)/2} = \{v_{m-2,2}, v_{m-3,1}, v_{m-4,2}\}$

$RP_i = \{v_{i2,n-1}, v_{i3n}, v_{i4n-1}\}$

$RP_2 = \{v_{i4n-1}, v_{i5n}, v_{i6n-1}\}$

$\vdots$

$RP_{(m-4)/2} = \{v_{m-1,n-1}, v_{m-2,n}, v_{m-3,n-1}\}$

$S$ = The remaining edges that are not covered by the above paths.

$\psi = \{P_1\} \cup \{L_P_i\} \cup \{RP_i\} \cup S, 1 \leq i \leq n-1, 1 \leq j \leq \frac{(m-4)+1}{2}$ is an acyclic graphoidal cover of $G$. 
\[ \eta_a(P_m \circ P_n) \leq q - p + 6 \]

From above we see that all the vertices of \( P_m \circ P_n \) are internal vertices except the vertices \( v_{11}, v_{21}, v_{m-1,1}, v_{m1}, v_{1n}, v_{mn} \) so that \( t \geq 6 \).

Hence \[ \eta_a(P_m \circ P_n) \geq q - p + 6 \]

Therefore \[ \eta_a(P_m \circ P_n) = q - p + 6 \]

**Theorem 2.2.3**

For the graph \( P_m \otimes P_n \), the acyclic graphoidal covering number is

\[ \eta_a(P_m \otimes P_n) = 2q - 2p + 6 \]

**Proof:**

The proof is similar to that of Theorem 2.2.1

**Theorem 2.2.4**

For \( C_m \times P_n \), the acyclic graphoidal covering number is \( \eta_a(C_m \times P_n) = q - p \)

**Proof:**

Let \( V(G) = \{v_{1i}, v_{2i}, \ldots, v_{ni}\}, i = 1, 2, \ldots, m \)

**Case (i):** \( m \) is even and \( n \) is odd.

\[
P_1 = \left\{ v_{m1}, v_{m-1,1}, \ldots, v_{11}, v_{12}, \ldots, v_{m2}, v_{m3}, v_{m-1,3}, \ldots \right\}
\]

\[
P_2 = \{v_{11}, v_{m1}, v_{m2}\}
\]

\[
P_3 = \{v_{mn}, v_{ln}, v_{ln-1}\}
\]

\[ P_4 = \text{The remaining edges that are covered by } \{P_1, P_2, P_3\} \]

\[ \psi = \{P_1, P_2, P_3, P_4\} \text{ is an acyclic graphoidal cover of } G. \]
From the above paths all the vertices are internal vertices.

$$\eta_a(C_m \times P_n) \leq q - p$$

Since $$\eta_a \geq q - p$$

Therefore $$\eta_a(C_m \times P_n) = q - p$$

**Case (ii):** $m$ and $n$ is even.

$$P_1 = \left\{ v_{m1}, v_{m-1,1}, \ldots, v_{11}, v_{12}, \ldots, v_{m2}, v_{m3}, \ldots, v_{m-1,3}, \ldots, v_{13}, v_{14}, \ldots, v_{mn} \right\}$$

$$P_2 = \{v_{11}, v_{m1}, v_{m2}\}$$

$$P_3 = \{v_{ln}, v_{mn}, v_{m,n-1}\}$$

$$P_4 =$$ The remaining edges that are not covered by $$\{P_1, P_2, P_3\}$$

$$\psi = \{P_1, P_2, P_3, P_4\}$$ is an acyclic graphoidal cover of $G$.

From the above paths all the vertices are internal vertices.

$$\eta_a(C_m \times P_n) \leq q - p$$

Since $$\eta_a \geq q - p$$

Therefore $$\eta_a(C_m \times P_n) = q - p$$

**Case (iii):** $m$ is odd and $n$ is even.

$$P_1 = \left\{ v_{m1}, v_{m-1,1}, \ldots, v_{11}, v_{12}, \ldots, v_{m2}, v_{m3}, \ldots, v_{m-1,3}, \ldots \right\}$$

$$P_2 = \{v_{11}, v_{m1}, v_{m2}\}$$

$$P_3 = \{v_{ln}, v_{mn}, v_{m,n-1}\}$$

$$P_4 =$$ The remaining edges that are not covered by $$\{P_1, P_2, P_3\}$$

$$\psi = \{P_1, P_2, P_3, P_4\}$$ is an acyclic graphoidal cover of $G$. 

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From the above paths all the vertices are internal vertices.

\[ \eta_a(C_m \times P_n) \leq q - p \]

Since \( \eta_a \geq q - p \)

Therefore \( \eta_a(C_m \times P_n) = q - p \)

**Case (iv):** \( m \) and \( n \) is odd.

\[
P_1 = \left\{ v_{m1}, v_{m-1l}, \ldots, v_{l1}, v_{l2}, \ldots, v_{m2}, v_{m3}, \ldots, v_{m-1,3}, \ldots, v_{l3}, v_{l4}, v_{l24}, \ldots, v_{m4}, \ldots, v_{m,n-1}, v_{mn}, v_{m-1,n}, \ldots, v_{ln} \right\}
\]

\[
P_2 = \{v_{11}, v_{m1}, v_{m2}\}
\]

\[
P_3 = \{v_{mn}, v_{ln}, v_{l,n-1}\}
\]

\[
P_4 = \text{The remaining edges that are not covered by} \ \{P_1, P_2, P_3\}
\]

\[\psi = \{P_1, P_2, P_3, P_4\} \] is an acyclic graphoidal cover of \( G \).

From the above paths all the vertices are internal vertices.

\[ \eta_a(C_m \times P_n) \leq q - p \]

Since \( \eta_a \geq q - p \)

Therefore \( \eta_a(C_m \times P_n) = q - p \)

**Theorem 2.2.5**

For the rectangular grid \( P_m \times P_n \), the induced acyclic graphoidal covering number is

\[ \eta_a(P_m \times P_n) = q - p + 2 \].

**Proof:**

Let \( p = mn \) and \( q = m(n-1) + n(m-1) \)

Let \( V(G) = \{v_{i1}, v_{i2}, \ldots, v_{im}\}, i = 1, 2, \ldots, m \)
The induced acyclic graphoidal cover of $P_m \times P_n$ is as follows:

$$P_1 = \{v_{12}, v_{11}, v_{21}, v_{31}, \ldots, v_{m1}\}$$

$$P_2 = \{v_{13}, v_{12}, v_{22}, v_{32}, \ldots, v_{m2}\}$$

$$P_3 = \{v_{14}, v_{13}, v_{23}, v_{33}, \ldots, v_{m3}\}$$

$$\vdots$$

$$P_{n-1} = \{v_{1n}, v_{1n-1}, v_{2n-1}, v_{3n-1}, \ldots, v_{nn-1}\}$$

$$P_n = \{v_{m1}, v_{m2}, v_{m3}, v_{m4}, \ldots, v_{mn}, v_{m1, n}, v_{m2, n}, \ldots, v_{1n}\}$$

$S =$ The remaining edges not covered by $P_1, P_2, P_3, \ldots, P_{n-1}, P_n$

$\psi = \{P_1, P_2, \ldots, P_n\} \cup S$ is a minimum acyclic graphoidal cover of $G$.

From above we see that all the vertices of $P_m \times P_n$ are internal vertices except $v_{m1}$ and $v_{1n} (t = 2)$

Therefore $\eta_{ac}(P_m \times P_n) = q - p + 2$

### 2.3 Graphoidal covering number of some classes of graphs

**Theorem 2.3.1**

Let $G$ be triangular snake graph with $2n-1$ vertices. Then $\eta_{ac}(G) = n$

**Proof**

Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n, w_1, w_2, \ldots, w_{n-1}\}$

Here $w_i$ adjacent to $v_i$ and $v_{i+1}$

The acyclic graphoidal cover of $G$ is as follows

$$P_i = \{v_i, v_{i+1}\}, i = 1, 2, \ldots, n-1$$

$$P_n = \{v_1, w_1, v_2, w_2, v_3, \ldots, w_{n-1}, v_n\}$$
\[ \psi = P_1 \cup \ldots \cup P_n \text{ is an acyclic graphoidal cover of } G \]

\[ \Rightarrow \eta_\psi \leq n \]

Since \( \eta_\psi = q - p + t \geq 3n - 3 - 2n + 1 + 2 = n \)

\[ \therefore \eta_\psi(G) = n \]

**Theorem 2.3.2**

Let \( G \) be double triangular snake graph with \( 3n - 2 \). Then \( \eta_\psi(G) = 2n - 1 \)

**Proof**

Let \( V(G) = \{v_1, v_2, v_3, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{n-1}\} \)

Here \( w_j \) adjacent to \( v_j \) and \( v_{i+1} \) in upward direction and \( u_j \) adjacent to \( v_j \) and \( v_{i+1} \) in downward direction.

\[ p = 3n - 2, \; q = 5(n - 1) \]

The acyclic graphoidal cover of \( G \) is as follows

\[ P_1 = \{v_1, v_2, v_3, \ldots, v_n\} \]

\[ Q_i = \{v_i, u_i, v_{i+1}\}, i = 1, 2, \ldots, n - 1 \]

\[ R_j = \{v_j, w_j, v_{j+1}\}, j = 1, 2, \ldots, n - 1 \]

\[ \psi = \{P_1\} \cup \{Q_i\} \cup \{R_j\} / 1 \leq i, j \leq n - 1 \text{ is an acyclic graphoidal cover of } G \]

\[ \Rightarrow \eta_\psi \leq 2n - 1 \]

Since in any minimum acyclic graphoidal cover \( \eta_\psi \geq 2 \)

\[ \eta_\psi = q - p + t \geq 5(n - 1) - 3n + 2 + 2 = 2n - 1 \]

\[ \therefore \eta_\psi(G) = 2n - 1 \]
Theorem 2.3.3

Let $G$ be triple triangular snake graph with $4n-3$ vertices. Then $\eta_a(G) = 3n - 2$

Proof

Let $V(G) = \{v_1, v_2, v_3, ..., v_n, u_1, u_2, ..., u_{n-1}, w_1, w_2, ..., w_{n-1}, z_1, z_2, ..., z_{n-1}\}$

Here $w_i$ adjacent to $v_i$ and $v_{i+1}$ in upward direction, $z_i$ adjacent to $v_i$ and $v_{i+1}$ in upward direction and $u_i$ adjacent to $v_i$ and $v_{i+1}$ in downward direction.

$p = 4n - 3, \ q = 7(n - 1)$

The acyclic graphoidal cover of $G$ is as follows

$P_i = \{v_i, v_2, v_3, ..., v_n\}$

$Q_i = \{v_i, u_i, v_{i+1}\} i = 1, 2, ..., n - 1$

$R_j = \{v_j, w_j, v_{j+1}\} j = 1, 2, ..., n - 1$

$w_k = \{v_k, z_k, v_{k+1}\} k = 1, 2, ..., n - 1$

$\psi = P_i \cup Q_i \cup R_j \cup w_k / 1 \leq i, j, k \leq n - 1 \ \text{is an acyclic graphoidal cover of } G$

$\Rightarrow \eta_a(G) \leq 3n - 2$

Since in any minimum acyclic graphoidal cover $t \geq 2$

$\eta_a = q - p + t \geq 7(n - 1) - 4n + 3 + 2 = 3n - 2$

$\therefore \eta_a(G) = 3n - 2$

Theorem 2.3.4

Let $G$ be a gear graph with $2n+1$ vertices and $3n$ edges. Then $\eta_a(G) = q - p$.

Proof:
Let $V(G) = \{v_0, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$ where $v_0$ is the centre vertex of wheel and $w_i$ adjacent to $v_i$ & $v_{i+1}$ and $w_n$ is adjacent to $v_l$ & $v_0$ and $P = 2n+1$ and $q = 3n$

The acyclic graphoidal cover of $G$ is as follows

$P_1 = \{v_1, w_1, v_2, v_3, \ldots, v_n, v_0\}$

$P_2 = \{v_n, w_n, v_l, v_0, v_2\}$

$S = \text{The remaining edges}$

$\psi = \{P_1, P_2, S\}$ is an acyclic graphoidal cover of $G$

From above all the vertices are internal vertices in exactly one path

$\Rightarrow \eta_o(G) \leq q - p$

Since $\eta_o(G) \geq q - p$

$\therefore \eta_o(G) = q - p$

**Theorem 2.3.5**

Let $G$ be a helm graph $2n+1$ vertices and $3n$ edges. Then $\eta_o(G) = q - p + n = 2n - 1.$

**Proof:**

Let $V(G) = \{v_0, v_1, v_2, \ldots, v_n, t_1, t_2, \ldots, t_n\}$

Where $v_0$ is the centre vertex of wheel and $v_i$ is adjacent to $t_i$ and $v_0$

and $P = 2n+1$ and $q = 3n$

The acyclic graphoidal cover of $G$ is as follows

$P_1 = \{t_1, v_1, v_0, v_2, v_3, \ldots, v_n, t_n\}$

$P_2 = \{v_1, v_2\}$

$P_3 = \{v_1, v_n\}$
\( Q_i = \{ v_i, t_i \}, i = 2, 3, ..., n - 1 \)

\( R_j = \{ v_0, v_j \}, j = 3, 4, ..., n \)

\( \psi = P_1 \cup P_2 \cup P_3 \cup \{ R_j \} \cup \{ Q_i \}/2 \leq i \leq n - 1, 3 \leq j \leq n \) is an acyclic graphoidal cover of \( G \)

Here all the vertices except the pendant vertices are internal vertices.

\[ \Rightarrow \eta_a(G) \leq 2n - 1 = q - p + n \]

Since all pendant vertices are exterior points \( t \geq n \)

\[ \Rightarrow \eta_a(G) \geq q - p + n \]

\[ \therefore \eta_a(G) = q - p + n \]

**Theorem 2.3.6**

Let \( G \) be a shell graph with \( n \) vertices. Then \( \eta_a(G) = q - p + 1 \).

**Proof:**

Let \( V(G) = \{ v_1, v_2, v_3, ..., v_n \} \)

The acyclic graphoidal cover of \( G \) is as follows

\( P_i = \{ v_i, v_j \}, i = 3, 4, ..., n - 2 \)

\( P_1 = \{ v_0, v_n \} \)

\( P_2 = \{ v_1, v_2, ..., v_{n-1}, v_n \} \)

\( \psi = \{ P_i \}_{i=1}^{n-2} \) is a minimum acyclic graphoidal cover of \( G \)

Here all the vertices (except \( v_n \)) are internal vertices.

\[ \Rightarrow \eta_a(G) = n - 2 = q - p + 1 \]
Theorem 2.3.7

Let $G$ be a multiple shell graph. Then $\eta_a(G) = q - p + m$

Proof:

Let $V(G) = \{v_0, v_{i1}, v_{i2}, \ldots, v_{im}\}$ $i = 1, 2, \ldots, m$

The acyclic graphoidal cover of $G$ is as follows:

$P_i = \{v_0, v_{i1}, v_{i2}, \ldots, v_{im}\}$ $i = 1, 2, \ldots, m$

$P = \{v_{i2}, v_0, v_{im-1}\}$

$S$ = The remaining edges

$\psi = \{P, P_i, S\}$ $1 \leq i \leq m$ is an acyclic graphoidal cover of $G$

$\Rightarrow \eta_a \leq q - p + t = q - p + m$

Since in any graphoidal cover the number of exterior vertices $t \geq m$

Hence $\eta_a \geq q - p + t = q - p + m$

$\therefore \eta_a(G) = q - p + m$

2.4 Conclusion

In this Chapter we found the parameter $\eta_a$ for some product graphs such as $P_m \times P_n$, $P_m \circ P_n$, $P_m \boxtimes P_n$, $C_m \times P_n$, and for some classes of graphs such as Triangular snake, Double triangular snake graph, Triple triangular snake graph, Helm graph, Gear graph, shell and multiple shell. Also we found the induced acyclic graphoidal covering number of $P_m \times P_n$ has been founded and we observe that for thr graphs antiprism and $D_n^m$ the acyclic graphoidal covering number is $q - p$. Similarly we can find the simple path covering number $\pi_s$ for the above classes of graphs.