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Editorial office
e-mail: imf@m-hikari.com

Postal address: HIKARI Ltd, P.O. Box 85
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Path Double Covering Number

of a Bicyclic Graphs

T. Gayathri
Department of Mathematics
Sri Manakula Vinayagar Engineering College
Puducherry-605 107, India

S. Meena
Department of Mathematics
Government Arts and Science College
Chidamambaram-608 102, India

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Abstract

A path double cover of a graph $G$ is a collection $\mathcal{P}$ of paths in $G$ such that every edge of $G$ belongs to exactly two paths in $\mathcal{P}$. The minimum cardinality of a path double cover is called the path double covering number of $G$ and is denoted by $\eta_{pd}(G)$. In this paper we determine the exact value of this parameter for several classes of graphs.

Keywords: Graphoidal covers, path double covers, path double covering number of a graph, bicyclic graphs
1 Introduction

A graph is a pair \(G = (V, E)\), where \(V\) is the set of vertices and \(E\) is the set of edges. Here we consider only nontrivial, finite, connected undirected graph without loops or multiple edges. The order and size of \(G\) are denoted by \(p\) and \(q\) respectively. For graph theoretic terminology we refer to Harary [6]. The concept of graphoidal cover was introduced by B.D Acharya and E. Sampathkumar [1] and the concept of acyclic graphoidal cover was introduced by Arumugam and Suresh Suseela [4]. The reader may refer [6] and [2] for the terms not defined here.

Let \(P = (v_1, v_2, v_3, \ldots, v_r)\) be a path or a cycle in a graph \(G = (V, E)\). Then vertices \((v_2, v_3, \ldots, v_{r-1})\) are called internal vertices of \(P\) and \(v_1\) and \(v_r\) are called external vertices of \(P\). Two paths \(P\) and \(Q\) of a graph \(G\) are said to be internally disjoint if no vertex of \(G\) is an internal vertex of both \(P\) and \(Q\). If \(P = (v_0, v_1, v_2, \ldots, v_r)\) and \(Q = (v_r, v_0, v_1, w_2, \ldots, w_r)\) are two paths in \(G\) then the walk obtained by concatenating \(P\) and \(Q\) at \(v_r\) is denoted by \(P \circ Q\) and the path \((v_r, v_{r-1}, \ldots, v_0)\) is denoted by \(P^{-1}\) [3]. Bondy [5] introduced the concept of path double cover of a graph. This was further studied by Hao Li [7].

**Definition 1.1[3]:** A path double cover (PDC) of a graph \(G\) is a collection \(\mathcal{P}\) of paths in \(G\) such that every edge of \(G\) belongs to exactly two paths in \(\mathcal{P}\).

The collection \(\mathcal{P}\) may not necessarily consist of distinct paths in \(G\) and hence it cannot be treated as a set in the standard sense. For any graph \(G = (V, E)\), let \(\mathcal{P}\) denote the collection of all paths of length one each path appearing twice in the
collection. Clearly $\mathcal{P}$ is a path double of $G$ and hence the set of all path double covers of $G$ is non-empty.

Arumugam and Meena [3] introduced the concept of path double covering number of a graph $G$.

**Definition 1.2 [3]:** The minimum cardinality of a path double cover of a graph $G$ is called path double covering number of $G$ and is denoted by $\eta_{PD}(G)$.

In [9] it has been observed that for any graph $G$ $\eta_{PD}(G) \leq 2q$ and equality holds if and only if $G$ is isomorphic to $qK_2$ and the following results have been proved:

**Theorem 1.3 [3]:** Let $\mathcal{P}$ be any path double cover of a graph $G$. Then $|\mathcal{P}| = 2q - ip$ where $ip = \sum_{p \in \mathcal{P}} i(p)$ where $i(p)$ is the number of internal vertices of $\mathcal{P}$.

**Theorem 1.4 [3]:** $\eta_{PD} = 2q - i$ where $i = \max \mathcal{P}$ the maximum being taken over all path double covers $\mathcal{P}$ of $G$.

**Theorem 1.5 [3]:** Let $G$ be a graph with $\delta = 1$, if there exists a path double cover $\mathcal{P}$ such that every non-pendant vertex of $G$ is an internal vertex of $d(v)$ paths in $\mathcal{P}$ then $\mathcal{P}$ is minimum path double cover and $\eta_{PD} = |\mathcal{P}|$.

**Theorem 1.6 [3]:** For any tree $T$, $\eta_{PD}(T) = n$ where $n$ is the number of pendant vertices of $T$.

**Theorem 1.7 [3]:** For any graph $G$, $\eta_{PD}(T) \geq \Delta$. Further for any tree $T$, $\eta_{PD}(T) = \Delta$ if and only if $T$ is homeomorphic to a star.

**Definition 1.8 [8]:** A connected $(p, p+1)$ graph $G$ is called a bicyclic graph.
Definition 1.9 [8]: A one-point union of two cycles is a simple graph obtained from two cycles, say \( C_l \) and \( C_m \) where \( l,m \geq 3 \), by identifying one and the same vertex from both cycles. Without loss of generality, we may assume the \( l \)-cycle to be \( u_0u_1u_2u_3 \) and the \( m \)-cycle to be \( u_0u_1u_2u_3 \). We denote this graph by \( U(l;m) \).

Definition 1.10 [8]: A long dumbbell graph is a simple graph obtained by joining two cycles \( C_l \) and \( C_m \) where \( l,m \geq 3 \), with a path of length \( i \), \( i \geq 1 \). Without loss of generality, we may assume \( C_l = u_0u_1u_2u_3u_0 \), \( P_i = u_{i+1}u_{i+2}u_{i+3} \ldots u_{i+2l} \) and \( C_m = u_{i+1}u_{i+2}u_{i+3} \ldots u_{i+2m}u_{i+2l} \). We denote this graph by \( D(l,m,i) \).

Definition 1.11 [8]: A cycle with a long chord is a simple graph obtained from an \( m \)-cycle, \( m \geq 4 \), by adding a chord of length \( l \) where \( l \geq 1 \). Let the \( m \)-cycle be \( u_0u_1u_2u_3 \ldots u_{m-1}u_0 \). Without loss of generality, we may assume the chord joins \( u_0 \) with \( u_i \), where \( 2 \leq i \leq m-2 \). That is, \( u_0u_1u_2u_3 \ldots u_{m-2}u_i \) is the chord. We denote this graph by \( C_m(i;l) \).

2 Main Results

In this paper we determine the value of \( \eta_{PD} \) for \( U(l;m) \), \( D(l,m,i) \) and \( C_m(i;l) \) and for some classes of graphs.

Theorem 2.1

Let \( G \) be a bicyclic graph with \( n \) pendant vertices and \( G \) containing a \( U(l,m) \) and \( j \) be the number of vertices greater than 2 on \( U(l,m) \) other than \( u_0 \). Let \( j_1 \) and \( j_2 \) are the number of vertices of degree greater than 2 on \( C_l \) and \( C_m \) respectively.

Then the path double covering number of \( G \) is
Path double covering number

\[ \eta_{PD} = \begin{cases} 
4 & \text{if } G = U(l,m) \\
4+3 & \text{if } j_1 = 0 \text{ or } j_2 = 0 \text{ and } j = 1 \& n \geq j \\
j_1 = 0 \text{ or } j_2 = 0 \text{ and } j \geq 2 \& n > j \\
j_1 \geq 1 \text{ or } j_2 \geq 1 \text{ and } j = 2, 3 \& n = j \\
j_1 \geq 1 \text{ or } j_2 \geq 1 \text{ and } j = 2 \& n > j \\
n + 1 & \text{if } j = 4 \text{ and } n = j \\
n & \text{otherwise}
\end{cases} \]

**Proof:** Let \( V(U(l,m)) = \{u_0, u_1, u_2, u_3, \ldots, u_{l-1}, u_l, u_{l+1}, \ldots, u_{l+m-2}\} \)

And let \( j = j_1 + j_2 \) Where \( j_1 \) and \( j_2 \) are the number of vertices of degree greater than 2 on \( C_l \) and \( C_m \) respectively.

**Case 1:** \( U(l,m) \) The path double covering of \( G \) is as follows

\[ P_1 = \{u_1, u_2, u_3, \ldots, u_{l-1}, u_l, u_{l+1}, u_{l+2}, \ldots, u_{l+m-2}\} \]
\[ P_2 = \{u_{l-1}, u_{l-2}, \ldots, u_2, u_1, u_0, u_{l-1}, u_{l-2}, \ldots, u_{l+m-2}\} \]
\[ P_3 = \{u_1, u_0, u_{l+m-2}\} \]
\[ P_4 = \{u_{l-1}, u_l, u_l\} \]
\[ \eta_{PD}(G) = 4 = \Delta \]

**Case 2:** Either \( j_1 = 0 \) or \( j_2 = 0 \) and \( j = 1 \)

We prove this by induction on \( n \).

When \( j = 1 \) and \( n = 1 \)

\( G \) is isomorphic to the graph consisting of \( U(l,m) \) together with a path \( P = \{w_1, w_2, \ldots, w_\ell(= u_l)\} \) where \( u_l \) on \( C_l \)

The path double cover is as follows

\[ P_1 = \{w_1, w_2, w_3, \ldots, w_\ell = u_l, u_{l+1}, u_{l+2}, \ldots, u_{l-1}, u_{l-2}, \ldots, u_{l+m-2}\} \]
\[ P_2 = \{u_1, u_{l+1}, \ldots, u_{l+m-2}, u_0, u_1, u_2, \ldots, u_l = w_\ell, w_{\ell-1}, w_{\ell-2}, \ldots, w_1\} \]
\[ P_3 = \{u_{l-1}, u_0, u_l\} \]
\[ P_4 = \{u_{l+m-2}, u_0, u_1, \ldots, u_{l-1}\} \]
\[ \therefore \eta_{PD} \leq 4 \]
Since $\eta_{PD} \geq \Delta = 4$

$\therefore \eta_{PD} = 4 = n + 3$

Now assume that the result is true for all bicyclic graphs containing a $U(l,m)$ with $j = 1$ and having less than $n$ pendant vertices and $n \geq 2$.

Let $G$ be any bicyclic graph containing a $U(l,m)$ with $n$ pendant vertices and $j = 1$.

Let $w$ be any pendant vertex of $G$.

Choose a vertex $v$ such that $\deg v \geq 5$ and $d(w,v)$ is maximum.

Let $Q$ denote the $(w,v)$ path, Since $\deg v \geq 5$ there exist pendant vertices $w_1, w_2$ such that $(w,w_1)$ path $Q_1$ and $(w,w_2)$ path $Q_2$ both contain $Q$.

Now let $P_1$ and $P_2$ denote the $(w_1,v)$ section of $Q_1$ and $(w_2,v)$ section of $Q_2$, respectively.

Let $P = P_1 \circ P_2^1$.

Clearly $v$ is the only vertex of degree greater than two on $P$.

Let $P = (w_1 = v_0, v_1, v_2, \ldots, v_r = v_1, \ldots, v_k = w_2)$

Consider the graph $G_i = G \setminus P$.

Using induction hypothesis $\eta_{PD}(G_i) = (n-2) + 3 = n + 1$

Let $P$ be a minimum path double covering of $G_i$.

Let $P = \emptyset \cup (P, P)$ be a minimum path double cover of $G$.

$\eta_{PD}(G) = (n+1) + 2 = n + 3$

**Case 3:**

**Sub case 3a:** Either $j_1 = 0$ or $j_2 = 0$ and $n > j$ and $j \geq 2$.

We prove by induction on $n$.

Let $j = 2$, $n = 3$ (since $j \geq 2$ and $n > j$).
Without loss of generality assume that $u_i$ on $C_i$ is incident with the the path has a pendant vertex $w_1$ and $u_i$ on $C_i$ incident with the path has pendant vertices $w_2$ and $w_3$ respectively. ($i < k$)

The path double cover is as follows

$$P_1 = \{ (w_5 =) w_{21}, w_{22}, w_{23}, \ldots , w_{2n} (\neq u_i), u_{i-1}, u_{i-2}, \ldots , u_0, u_{j+m-2}, \ldots , u_i \}$$

$$P_2 = \{ (w_5 =) w_{31}, w_{32}, w_{33}, \ldots , w_{3n} (\neq u_i), u_{i+1}, \ldots , u_k, u_{i-1}, u_0, u_{j}, \ldots , u_{i+m-2} \}$$

$$P_3 = \{ (w_1 =) w_{i1}, w_{i2}, w_{i3}, \ldots , w_{in} (\neq u_k), u_{k+1}, \ldots , u_{i-1}, u_0, u_j \}$$

$$P_4 = \{ (w_1 =) w_{i1}, w_{i2}, w_{i3}, \ldots , w_{in} (\neq u_k), u_{k-1}, \ldots , u_1, u_0, u_{j+k-2}, \ldots , u_i \}$$

$$P_5 = \{ (w_2 =) w_{21}, w_{22}, w_{23}, \ldots , w_{2n} (\neq u_j), (w_5 =) w_{31}, w_{32}, w_{33}, \ldots , w_{3n} (\neq u_i) \}$$

$\therefore n_{PD} = n + 2$

Let us assume that the result is true for all bicyclic graph containing a $U(l,m)$ with $n-1$ pendant vertices, $j_1 = 0$ or $j_2 = 0$ and $n > j$ and $j \geq 2$

Now let $G$ be a bicyclic graph containing a $U(l,m)$ with $n$ pendant vertices $(w_1, w_2, \ldots , w_n)$ and $j_1 = 0$ or $j_2 = 0$ and $n > j$ and $j \geq 2$

As before the mentioned paths $P_1$ to $P_5$ covers the bicyclic as well as three pendant vertices $w_1, w_2, w_3$

Let $G_i = G - \{ P_1, P_2, P_3, P_4, P_5 \}$ is a tree with $n-3$ pendant vertices.

$\Rightarrow n_{PD} \left( G_i \right) = n-3$

$\therefore n_{PD} \left( G \right) = n-3 + 5 = n + 2$

**Sub case 3b:** Let $j_1 \geq 1, j_2 \geq 1, j = 2, 3$ and $j = n$

Let $j = 2$ and $n = j$

Since $j = 2$, clearly $j_1 = 1$ and $j_2 = 1$

Let $u_i$ be the vertex on $C_i$ and $u_k$ be the vertex on $C_m$ are of degree greater than 2.

$u_i$ and $u_k$ are incident with the paths has pendant vertices $w_1$ and $w_2$ respectively.
The path double cover is as follows

\[ P_1 = \{u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m, w_1, \ldots, w_{2m} \} \]

\[ P_2 = \{(w_1 =)w_1, w_2, w_3, \ldots, w_m (= u_1), u_2, \ldots, u_{k-1}, u_{k+1}, \ldots, u_m, w_{2n} (= u_k), w_{2(n-1)} = (w_{2k} =)w_{2k+1} \} \]

\[ P_3 = \{(w_1 =)w_1, w_2, w_3, \ldots, w_m (= u_1), u_2, \ldots, u_{k-1}, u_{k+1}, \ldots, u_m, w_{2n} (= u_k), w_{2(n-1)} = (w_{2k} =)w_{2k+1} \} \]

\[ P_4 = \{u_1, u_0, \ldots, u_{k-2}, w_1, \ldots, w_{2m} \} \]

\[ \Rightarrow \eta_{PD}(G) \leq 4 = n + 2 = \Delta \]

We know that \( \eta_{PD}(G) \geq \Delta \)

\[ \therefore \eta_{PD}(G) = 4 = n + 2 = \Delta \]

**Sub case 3c:** \( j = 2 \) and \( n > j \)

Let \( j = 2 \)

Without loss of generality assume that \( n = j + 2 = 4 \)

\( u_i \) is incident with the pendant vertices \( w_1 \) and \( w_2 \) on \( C_i \)

\( u_k \) is incident with the pendant vertices \( w_3 \) and \( w_4 \) on \( C_m \)

The path double cover is as follows

\[ P_1 = \{w_1, u_1, u_{i+1}, u_{i-1}, u_i, u_{i+1}, \ldots, u_k, w_1, w_2 \} \]

\[ P_2 = \{w_2, u_k, u_{k+1}, \ldots, u_i, u_1, u_0, u_{i-1}, \ldots, u_{k-1}, w_2, w_3 \} \]

\[ P_3 = \{w_3, u_1, u_2, u_3, \ldots, u_{i+1}, u_{i+2}, \ldots, u_{k-1}, u_k, w_3 \} \]

\[ P_4 = \{w_4, u_1, u_2, u_3, \ldots, u_{i+1}, u_{i+2}, \ldots, u_{k-1}, u_k, w_4 \} \]

\[ \therefore \eta_{PD}(G) = 4 = \Delta \]

Similarly we can find the path double covering for the case \( j = 2, n > j \) and for the case \( j = 3 \)

So that \( \eta_{PD}(G) = n + 2 = \Delta \)

**Case 4:** Let \( j = 4, n = j = 4 \)

Let \( u_i, u_k \) are the two vertices on \( C_i \) are of degree greater than two incident with the pendant vertices \( w_1 \) and \( w_2 \) respectively.
Let \( u_m, u_n \) are the two vertices on \( C_m \) are of degree greater than two incident with the pendant vertices \( w_3 \) and \( w_4 \) respectively.

The path double cover is as follows

\[
P_1 = \{w_1, u_1, u_{i+1}, u_{j-1}, u_l, u_t, \ldots, u_m, w_3\}
\]

\[
P_2 = \{w_2, u_2, \ldots, u_j, u_{i+1}, u_{j-1}, u_l, \ldots, u_m, w_4\}
\]

\[
P_3 = \{w_l, u_l, \ldots, u_k, w_2\}
\]

\[
P_4 = \{u_k, u_{k-1}, \ldots, u_0, u_{i+2}, \ldots, u_n, w_k\}
\]

\[
P_5 = \{u_k, \ldots, u_1, u_0, u_{i+2}, \ldots, u_m, w_3\}
\]

\[
\therefore \eta_{PD}(G) = n + 1
\]

Similarly we can find the path double covering for the case \( j = 4, n > j \)
So that \( \eta_{PD}(G) = n + 1 \)

**Case 5:** Let \( j > 5, n > j \)

Here the number of pendant vertices is greater than 5, we can find the path double covering such that all the vertices are internal vertices except the pendant vertices by using above proof technique.

\[
\therefore \eta_{PD}(G) = n
\]

**Theorem 2.2**

Let \( G \) be a bicyclic graph with \( n \) pendant vertices and \( G \) containing a \( D(l, m, i) \) is the unique bicycle in \( G \) and let \( j \) be the number of vertices of degree greater than 2 on \( D(l, m, i) \) except \( u_{j-1} & u_{i+1} \). Then the path double covering of \( G \) is
\[
\eta_{PD} = \begin{cases} 
4 & \text{if } G = D(l,m,i) \\
4 + 4 & \text{if } \begin{cases} j_1 = j_2 = 1 \text{ and } j_3 = 0 \\
j_1 = j_2 = 0 \text{ and } j_3 \geq 1 \\
j_1 = 1, j_2 = j_3 = 0 \\
j_1 = 0 \text{ or } j_2 = 1, j_3 = 0 \\
j_1 = j_3 = 0 \\
j_1 = j_2 = 2, j_3 = 0 \text{ or } 2 \\
j_1 = j_2 = 1, j_3 = 0 \\
j_1 \geq 3, j_2 = 0, 1, j_3 = 0 \\
j_1 = j_2 = 2, j_3 = 0 \\
j_1 \geq 3, j_2 = 2, j_3 = 0 \\
\end{cases} \\
4 + 3 & \text{if } j_1 = j_2 = j_3 = 1 \\
n + 2 & \text{if } j_1 = j_2 = j_3 = 0 \\
n + 1 & \text{if } j_1 \geq 3, j_2 = 2, j_3 = 0 \\
n & \text{otherwise}
\end{cases}
\]

Where \( j_1, j_2 \) and \( j_3 \) are the number of vertices greater than two on \( C_l \), \( C_m \) and on the path respectively.

**Proof**

Let \( V(D(l,m,i)) = \{u_0, u_1, u_2, u_3, \ldots, u_{i-1}, u_i, u_{i+1}, u_{i+2}, \ldots, u_{l+m+i-2}\} \)

\( j = j_1 + j_2 + j_3 \) Where \( j_1 \) and \( j_2 \) are the number of vertices of degree greater than 2 on \( C_l \) and \( C_m \) and \( j_3 \) is the number of vertices of degree greater than 2 on the path respectively.

**Case 1:** \( G = D(l,m,i) \)

The path double covering of \( G \) is as follows

\[ P_1 = \{u_0, u_1, u_2, \ldots, u_{i-1}, u_i, \ldots, u_{l+m+i-2}\} \]

\[ P_2 = \{u_{i+1}, u_{i+2}, \ldots, u_{l+m+i-2}, u_{l+i-1}, u_l, u_{l-1}, u_0, u_1, u_2, \ldots, u_{l-2}\} \]

\[ P_3 = \{u_0, u_{l-3}, u_{l-2}\} \]

\[ P_4 = \{u_{l+i}, u_{l+1}, u_{l+m+i-2}\} \]

\[ \therefore \eta_{PD}(G) = 4 \]

**Case 2:**

**Sub Case 2a:** Let \( j_1 = j_2 = 1 \) and \( j_3 = 0 \)

\( j = j_1 + j_2 = 2 \)
We prove by induction on $n$

Let $n = 2$ and let $v_0, w_0$ are pendant vertices.

The path double covering of $G$ is as follows

Let $v_0, v_1, v_2, ..., v_r$ be a path attached with $u_i$ in $C_i$ and $w_0, w_1, w_2, ..., w_r$ be a path attached with $u_k$ in $C_m$

\[ P_1 = \{ v_0, v_1, v_2, ..., v_r = u_i, u_{i+1}, u_i, u_0, u_{i-1}, u_i, u_{i+1}, ..., u_{i+2}, u_i, ..., w_r = u_r, w_{r-1}, ..., w_0 \} \]

\[ P_2 = \{ v_0, v_1, v_2, ..., v_r = u_i, u_{i-1}, u_i, u_{j-2}, u_i, u_{j-1}, u_i, u_{j+1}, u_i, u_{j+2}, w_r = u_r, w_{r-1}, ..., w_0 \} \]

\[ P_3 = \{ v_r = u_i, u_i, ..., u_i \} \]

\[ P_4 = \{ v_r = u_i, u_i, ..., u_i \} \]

\[ P_5 = \{ w_r = u_r, u_{r-1}, ..., u_i \} \]

\[ P_6 = \{ w_r = u_r, u_{r+1}, ..., u_i \} \]

Here the two pendant vertices are exterior vertices and exactly two vertices on both $C_i$ and $C_m$ are exterior vertices.

\[ : \eta_{PD}(G) = n + 4 \]

Assume that the result is true for all bicyclic graph with $n - 1$ pendant vertices containing a $D(l, m, i)$

Let $G$ be a graph bicyclic graph with $n$ pendant vertices containing a $D(l, m, i)$

Let $f = 2$

The proof is similar to case 2 in Theorem 2.1

**Sub Case 2b:** Let $j_1 = j_2 = 0$ and $j_3 \geq 1$

Let $f = n = 1$

$G$ is isomorphic to the graph consisting of $D(l, m, i)$ together with a path

\[ P = \{ p_0, p_1, ..., p_j = u_i \} \]

attached with the vertex $u_i$

The path double cover is as follows

\[ P_1 = \{ u_{l-2}, u_2, u_1, u_0, u_{j-1}, u_j, ..., u_{j+1}, u_{j+2}, ..., u_{l+m+i-1} \} \]
\[ P_2 = \{p_0, p_1, \ldots, p_i(u_l), u_{l+1}, \ldots, u_k, u_{k+1}, \ldots, u_m, u_{m+1}, \ldots, u_{2g}\} \]
\[ P_3 = \{p_0, p_1, \ldots, p_i(u_l), u_{l+1}, u_{l+2}, \ldots, u_k\} \]
\[ P_4 = \{u_0, u_{j-1}, u_{j-2}\} \]
\[ P_5 = \{u_{i+1}, u_{i+2}, u_{i+3}\} \]

Here one pendant vertex and two vertices in each cycles \( C_i \) and \( C_m \) are exterior vertices.

\[ \therefore \eta_{PD} = n + 4 \]

If \( j = 1 \) and \( n > 1 \) we prove this by using the technique in Theorem 2.1 case 2.

**Case 3:**

**Sub case 3a:**

(i) \( j_1 = 1, 2; j_2 = j_3 = 0; j = n \) or \( j_1 = 1, 2; j_1 = j_3 = 0, j = n \)

(ii) \( j_1 = j_2 = j_3 = 1; j = n \)

(iii) \( j_1 = 1, j_2 = j_3 = 0; j < n \)

In the above cases we can find the path double covering such that the pendant vertices and three vertices on both \( C_i \) and \( C_m \) are exterior points.

\[ \therefore \eta_{PD} = n + 3 \]

In the remaining cases we can find the path double cover such that all the vertices are internal vertices except the pendant vertices.

\[ \therefore \eta_{PD} = n \].

**Theorem 2.3**

Let \( G \) be a unicyclic graph with \( n \) pendant vertices and \( G \) containing a \( C_m(i,l) \) is the unique cycle in \( G \) and let \( j \) be the number of vertices greater than 3 on \( C_m(i,l) \). Then the path double covering of \( G \) is

\[ \eta_{PD} = \begin{cases} 3 & \text{if } G = C_m(i,l) \\ n + 3 & \text{if } j = 1 \text{ and } \text{deg} u_i \geq 6 \text{ for some } i \\ n + 2 & \text{if } j_1 \geq 1, j_2 = 0 \text{ or } j_2 \geq 1, j_3 = 0 \\ n & \text{otherwise} \end{cases} \]
where \( j_1 \) and \( j_2 \) are the number of vertices of degree greater than 2 on the upper half and lower half on \( C_m(i,i) \) respectively and \( j = j_1 + j_2 \).

Assume that if \( \text{deg} u_i \) or \( \text{deg} u_o \geq 4 \) this will increase the number of \( j_1 \).

**Proof**

Let \( V(C_m(i,i)) = \{u_0, u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_{m-1}, u_m, u_{m+1}, \ldots, u_{i+m-2}\} \)

\( j = j_1 + j_2 \)

**Case 1:** \( G = C_m(i,i) \)

The path double covering of \( G \) is as follows

\[
P_1 = \{u_{i+m-2}, \ldots, u_{m+1}, u_m, u_0, u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_{m-1}\} \\
P_2 = \{u_{i+m-2}, \ldots, u_{m+1}, u_m, u_0, u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_{m-1}\} \\
P_3 = \{u_m, u_0, u_{m-1}\} \\
P_4 = \{u_{i+m-2}, u_i, u_{i+1}\} \\
\therefore \eta_{\text{PD}}(G) = 4
\]

**Case 2:** If \( j=1 \) and \( \text{deg} u_m \geq 6 \) for some \( m \)

Without loss of generality assume that \( \text{deg} u_i = 7 \)

The path double covering of \( G \) is as follows

\[
P_1 = \{u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+m-2}, u_{m+1}, u_m, u_0, u_1, \ldots, u_{i+1}\} \\
P_2 = \{w_1, w_2, \ldots, u_i, u_{i+1}, \ldots, u_0, u_1, u_2, \ldots, u_m\} \\
P_3 = \{w_1, w_2, \ldots, u_i, u_{i+m-2}, \ldots, u_m, u_0, u_1\} \\
P_4 = \{w_1, w_2, \ldots, u_i, u_{i+1}\} \\
P_5 = \{w_{41}, w_{42}, \ldots, u_i, u_{i+1}\} \\
P_6 = \{w_1, w_2, \ldots, u_m = u_i = w_{n+1}, w_{n+1}\} \\
P_7 = \{w_1, w_2, \ldots, w_{3n} = u_i = w_{4n}, \ldots, w_{41}\} \\
\therefore \eta_{\text{PD}}(G) = 7 = n + 3 = \Delta
\]

For the different types of paths the proof is similar to case 2 of Theorem 2.1.
If $\deg u_i \geq 7$, then the above path double cover covers four pendant vertices.

Clearly $G_i = G - \{P_1, P_2, \ldots, P_j\}$ is tree with $n - 4$ pendant vertices.

$$\eta_{PD}(G_i) = n - 4$$

$$\therefore \eta_{PD}(G) = n - 4 + 7 = n + 3$$

**Case 3:** Let $j_1 \geq 1$, $j_2 = 0$ or $j_2 \geq 1$, $j_1 = 0$

We prove by induction on $j_1$

Let $j_1 = 1$ and the number of pendant vertex = 1

Without loss of generality assume that the path $(w_{i_1}, w_{i_2}, \ldots, w_{i_n} = u_i)$ attached with $u_i$.

The path double covering of $G$ is as follows

$$P_1 = \{w_{i_1}, w_{i_2}, \ldots, u_i, u_0, u_m, u_{m+1}, \ldots, u_{i_{rs-2}}, u_i, u_{i+1}, \ldots, u_{i_{rs-1}}\}$$

$$P_2 = \{w_{i_1}, w_{i_2}, \ldots, u_i, u_0, u_{m-1}, u_m, u_{m+1}, \ldots, u_{i_{rs-2}}, u_i, u_{i+1}, \ldots, u_{i_{rs-1}}\}$$

$$P_3 = \{u_{m-1}, u_0, u_1, u_2, \ldots, u_i, u_{i+1}\}$$

$$\therefore \eta_{PD}(G) = 3 = \Delta$$

( Note that if $j_1 = 1$ and pendant vertices = n

The proof is similar to the case 2 in Theorem 2.1)

Assume that the result is true for $j_1 \leq n - 1$

Let $j_1 = n$, Since $j_2 = 0$ at least two vertices among $u_{i+1}, u_{i+2}, \ldots, u_{m-1}$ are exterior vertices and the $n$ pendant vertices are exterior points

$$\therefore \eta_{PD}(G) = n + 2$$

**Case 4:** For the remaining cases we find a path double covering such that all the vertices are internal vertices except the pendant vertices.

$$\therefore \eta_{PD}(G) = n$$
References


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GEODESIC GRAPHOIDAL COVERING NUMBER OF GRAPHS

T. Gayathri*1 and S. Meena2

1Department of Mathematics,
Sri Manakula Vinayagar Engineering College, Puducherry-605 107, India.

2Department of Mathematics,
Government Arts and Science College, Chidambaram-608 102, India.

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ABSTRACT

A geodesic graphoidal cover of a graph G is a collection \( \psi \) of shortest paths in G such that every path in \( \psi \) has at least two vertices, every vertex of G is an internal vertex of at most one path in \( \psi \) and every edge of G is in exactly one path in \( \psi \). The minimum cardinality of a geodesic graphoidal cover of G is called the geodesic graphoidal covering number of G and is denoted by \( \eta_g(G) \). In this paper we determine \( \eta_g(G) \) for bicyclic graphs.

Key words: Graphoidal covers, Acyclic graphoidal cover, Geodesic Graphoidal cover

1. INTRODUCTION

A graph is a pair \( G = (V, E) \), where \( V \) is the set of vertices and \( E \) is the set of edges. Here we consider only non-trivial, finite, connected, undirected graph without loops or multiple edges. The order and size of G are denoted by \( p \) and \( q \) respectively. For graph theoretic terminology we refer to Harary [4]. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1] and the concept of acyclic graphoidal cover was introduced by Arumugam and Suresh Suseela [4]. The reader may refer [5], [2] and [7] for the terms not defined here.

Let \( p = (v_1, v_2, v_3, \ldots, v_r) \) be a path or a cycle in a graph \( G = (V, E) \). Then vertices \( (v_2, v_3, \ldots, v_{r-1}) \) are called internal vertices of \( P \) and \( v_1 \) and \( v_r \) are called external vertices of \( P \). Two paths \( P \) and \( Q \) of a graph G are said to be internally disjoint if no vertex of \( G \) is an internal vertex of both \( P \) and \( Q \).

Definition 1.1 [1]: A graphoidal cover of a graph G is called a collection \( \psi \) of (not necessarily open) paths in G satisfying the following conditions:
(i) Every path in \( \psi \) has at least two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in \( \psi \).
(iii) Every edge of G is in exactly one path in \( \psi \).

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by \( \eta(G) \).

Definition 1.2 [3]: A graphoidal cover \( \psi \) of a graph G is called an acyclic graphoidal cover if every member of \( \psi \) is an open path. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of G and is denoted by \( \eta_a(G) \).
Definition 1.3 [4]: A geodesic graphoidal cover of a graph $G$ is a collection $\Gamma$ of shortest paths in $G$ such that every path in $\Gamma$ has at least two vertices, every vertex of $G$ is an internal vertex of at most one path in $\Gamma$ and every edge of $G$ is an exactly one path in $\Gamma$. The minimum cardinality of a geodesic graphoidal cover of $G$ is called the geodesic graphoidal covering number of $G$ and is denoted by $\eta_g$.

Definition 1.4 [1]: Let $\Gamma$ be a collection of internally disjoint paths in $G$. A vertex of $G$ is said to be in the interior of $\Gamma$ if it is an internal vertex of some path in $\Gamma$. Any vertex which is not in the interior of $\Gamma$ is said to be an exterior vertex of $\Gamma$.

Theorem 1.5 [8]: For any graphoidal cover $\Gamma$ of $G$, let $t_\Gamma$ denote the number of exterior vertices of $\Gamma$. Let $q_\Gamma = \min_t t_\Gamma$ where the minimum is taken over all graphoidal covers of $G$. Then $\eta = q - p + t$.

Corollary 1.6: For any graph $G$, $\eta \geq q - p$. Moreover the following are equivalent.

(i) $\eta = q - p$
(ii) There exists a graphoidal cover without exterior vertices.
(iii) There exists a set of internally disjoint and edge disjoint paths without exterior vertices.

In [4] given that $\eta \leq \eta_a \leq \eta_g$ and these inequalities can be strict and also for a tree $\eta = \eta_a = \eta_g = n - 1$ and Theorem 1.5 and corollary 1.6 are true for geodesic graphoidal covers.

They observe that $\eta_g = q$ if and only if $G$ is Complete. Further for a cycle $C_m$, $\eta_g = \begin{cases} 2 & \text{if } m \text{ is even} \\ 3 & \text{if } m \text{ is odd} \end{cases}$

Theorem 1.7 [4]: Let $G$ be a unicyclic graph with unique cycle $C$ which is even. Let $n$ denote the number of pendant vertices of $G$ and let $m$ denote the number of vertices on $C$ with degree greater than 2. Then

$$\eta_g = \begin{cases} 2 & \text{if } m = 0 \\ n & \text{if } m \geq 2 \text{ and every } (v, w)-\text{section of } C \text{ in which all vertices} \\ & \text{except } v \text{ and } w \text{ have degree 2 is a shortest path} \\ n + 1 & \text{otherwise} \end{cases}$$

Theorem 1.8 [4]: Let $G$ be a unicyclic graph with unique cycle $C$ of odd length $2k+1, k \geq 1$. Let $n$ denote the number of pendant vertices of $G$ and let $m$ denote the number of vertices of degree greater than 2 on $C$ with. Then

$$\eta_g = \begin{cases} 3 & \text{if } m = 0 \\ n + 2 & \text{if } m = 1 \\ n & \text{if } m \geq 2 \text{ and every } (v, w)-\text{section of } C \text{ in which all vertices} \\ & \text{except } v \text{ and } w \text{ have degree 2 is a shortest path} \\ n + 1 & \text{otherwise} \end{cases}$$

Definition 1.9: For two graphs $G$ and $H$, their Cartesian product $G \times H$ has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)$ is joined $(g_2, h_2)$ iff $g_1 = g_2$ and $h_1 h_2 \in E(H)$ or $h_1 = h_2$ and $g_1 g_2 \in E(G)$.

Definition 1.10: A triangular cactus is a connected graph all of whose blocks are triangles. A triangular snake is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path $v_1, v_2, ..., v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $i = 1, 2, ..., n-1$).

Definition 1.11: A double triangular snake consists of two triangular snakes that have a common path. That is a double triangular snake is obtained from a path $v_1, v_2, ..., v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $i = 1, 2, ..., n-1$ and to a new vertex $u_i$ for $i = 1, 2, ..., n-1$. 

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Definition 1.12: A triple triangular snake consists of three triangular snakes that have a common path. That is, a triple triangular snake is obtained from a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i \) to a new vertex \( v_i \) for \( i = 1, 2, \ldots, n-1 \) and to a new vertex \( w \) for \( i = 1, 2, \ldots, n-1 \) for and also to a new vertex \( z_i \) for \( i = 1, 2, \ldots, n-1 \).

Definition 1.13: Mongolian tent as a graph obtained from \( P_m \times P_n \), \( n \) odd, by adding one extra vertex above the grid and joining every other vertex of the top row of \( P_m \times P_n \) to the new vertex.

Definition 1.14: The book \( B_m \) is the graph \( S_m \times P_2 \) where \( S_m \) is the star with \( m + 1 \) vertices.

Definition 1.15: A gear graph, denoted \( G_n \), is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph \( W_n \). Thus, \( G_n \) has 2n+1 vertices and 3n edges. Gear graphs are examples of square graphs, and play a key role in the forbidden graph characterization of square graphs. Gear graphs are also known as cogwheels and bipartite wheels.

Definition 1.16: A helm graph, denoted \( H_n \), is a graph obtained by attaching a single edge and node to each node of the outer circuit of a wheel graph \( W_n \).

Definition 1.17: A graph \( G \) is called the flower graph with \( n \) petals if it has 3n+1 vertices which form an n-cycle.

Definition 1.18: A shell \( S_n \) is the graph obtained by taking \( n-3 \) concurrent chords in a cycle \( C_n \) on \( n \) vertices. The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called fan \( F_{n-1} \).

i.e., \( S_n = F_n - 1 = P_n - 1 + K_1 \).

Definition 1.19: The cartesian product of two paths is known as grid graph which is denoted by \( P_m \times P_n \). In particular the graph \( L_n = P_n \times P_2 \) is known as ladder graph.

Definition 1.20: A web graph is the graph obtained by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.

Definition 1.21: A double-wheel graph \( DW_N \) of size \( N \) can be composed of \( 2C + K_1 \), i.e. it consists of two cycles of size \( N \), where the vertices of the two cycles are all connected to a common hub.

Definition 1.22 [9]: For \( n \geq 3, m \geq 1, A^m_n \) the plane graph of a convex polytope which is obtained as a combination of antiprism.

Definition 1.23 [9]: The m-prism \( D^m_n \), \( n \geq 3, m \geq 1 \) is a trivalent graph of a convex polytope which can be defined as cartesian product of a path on \( m+1 \) vertices with a cycle on \( n \) vertices \( (p_{m+1} \times c_n) \) embedded in the plane.

2. MAIN RESULTS

Theorem 2.1: Let \( G \) be Triangular cactus graph with \( n \) number of triangles then \( \eta_g = 2n - 1 = q - p + n \)

Proof:
Let \( V(G) = \{v_0, v_1, v_2\} \) \( i = 1 \) to \( n \)
\( P = 2n+1 \) and \( q = 3n \)
The Geodesic graphoidal path covering of \( G \) is as follows
\( P = \{v_{i1}, v_{i2}, v_0, v_{i2}, v_{i1}\} \)
\( P_i = \{v_{i1}, v_{i2}, v_0\} \) \( i = 2, 3, \ldots, n-1 \)
\( Q_i = \{v_{i1}, v_0\} \) \( i = 1, 2, \ldots, n \)
\( \psi = P \cup \{P_i\} \cup \{Q_i\} \) is minimum geodesic graphoidal covering of \( G \)
\( \Rightarrow \eta_g = 1 + n - 2 + n = 2n - 1 \)
Theorem 2.2: Let G be Triangular snake graph with \( n-1 \) number of triangles then \( \eta_g = 2n - 3 = q - p + n - 1 \)

Proof:
Let \( V(G) = \{v_1, v_2, v_3, ... , v_n, w_1, w_2, ... , w_{n-1} \} \)

Here \( w_i \) adjacent to \( v_j \) and \( v_{i+1} \)

The Geodesic graphoidal path covering of G is as follows

\[
P_i = \{v_i w_i \}, \quad i = 1, 2, ..., n - 2
\]
\[
P_{n-1} = \{w_i v_i, v_2, v_3, ..., v_n, w_{n-1} \}
\]
\[
R_i = \{v_i w_i \}, \quad i = 1, 2, 3, ..., n - 1
\]
\[
\psi = P_1 \cup ... \cup P_{n-1} \cup R_2 \cup ... \cup R_{n-1} \text{ is minimum geodesic graphoidal covering of G}
\]
\[
\Rightarrow \eta_g = n - 1 + n - 2 = 2n - 3 = q - p + n - 1
\]

Also for triangular graph \( \eta_a = q - p + n - 1 \)

Theorem 2.3: Let G be Double Triangular snake graph with \( 2n-2 \) number of triangles then \( \eta_g = 4n - 5 = q - p + 2n - 2 \)

Proof:
Let \( V(G) = \{v_1, v_2, v_3, ... , v_n, u_1, u_2, ... , u_{n-1}, w_1, w_2, ... , w_{n-1} \} \)

Here \( u_i \) adjacent to \( v_j \) and \( v_{i+1} \) in upward direction and \( w_j \) adjacent to \( v_j \) and \( v_{i+1} \) in downward direction.

\( p = 3n - 2, \quad q = 5(n-1) \)

The Geodesic graphoidal path covering of G is as follows

\[
P_i = \{v_i u_i \}, \quad i = 1, 2, 3, ..., n - 2
\]
\[
P_n = \{u_{n-1} v_n, w_{n-1} \}
\]
\[
Q_i = \{v_i u_i \}, \quad i = 2, 3, ..., n - 1
\]
\[
Q_n = \{v_1, v_2, v_3, ... , v_n \}
\]
\[
R_i = \{v_i w_i \}, \quad i = 2, 3, ..., n - 1
\]
\[
R_n = \{u_1 v_1 w_1 \}
\]
\[
S_i = \{v_{i+1} u_i \}, \quad i = 1, 2, 3, ..., n - 2
\]
\[
\psi = P_1 \cup Q_1 \cup R_2 \cup ... \cup R_{n-1} \text{ is minimum geodesic graphoidal covering of G}
\]
\[
\Rightarrow \eta_g = 4(n - 2) + 3 = 4n - 5 = q - p + 2n - 2
\]

Note:
Let G be triple Triangular snake graph with \( 3n-3 \) number of triangles then \( \eta_g = q - p + 3(n-1) \)

Theorem 2.4: For \( p_m \times p_n \), the geodesic graphoidal covering number is \( \eta_g = q - p + 2 \).

Proof: Let \( V(G) = \{v_{ij}, v_{i2}, ... , v_{im} \} \quad i = 1, 2, ..., m \)

Here \( p = mn \) and \( q = m(n-1)+n(m-1) \)
The geodesic graphoidal cover of $P_m \times P_n$ is as follows:

$$P_i = \{v_{i+1}, v_1, v_2, \ldots, v_m \} \quad i = 1, 2, \ldots, m-1$$

$$P_n = \{v_{m+1}, v_2, v_3, \ldots, v_{m+n}, v_{m+n-1}, v_{m+n-2}, \ldots, v_{m+2}, v_1 \}$$

$S = \text{The remaining edges not covered by } P_1, P_2, P_3, \ldots, P_{n-1}, P_n$

$\psi = P_1 \cup P_2 \cup \ldots \cup P_n \cup S$ is minimum geodesic graphoidal covering of $G$

From above we see that all the paths are shortest paths and all the vertices of $P_m \times P_n$ are internal vertices in at least one path except except $v_{1n}$ and $v_{m1}$

Therefore $\eta_g = q - p + 2$

Note:
(i) For $P_m \times P_n$, $\eta_a = \eta_g = q - p + 2$
(ii) Let $G$ be a Ladder graph then $\eta_g = q - p + 2$. Since Ladder is a particular case of $P_m \times P_n$

**Theorem 2.5:** Let $G$ be a gear graph with $2n+1$ vertices and $3n$ edges then $\eta_g = q - p + n$.

**Proof:** Let $V(G) = \{v_0, v_1, v_2, v_3, \ldots, v_n, w_1, w_2, \ldots, w_n \}$

where $v_0$ is the centre vertex of wheel and $w_i$ adjacent to $v_i$ and $v_{i+1}$ and $w_n$ is adjacent to $v_1$ and $v_n$ and $P = 2n+1$ and $q = 3n$

The Geodesic graphoidal path covering of $G$ is as follows

$$P_i = \{v_i, w_i, v_{i+1} \} \quad i = 1, 2, \ldots, n-1$$

$$P_n = \{v_n, w_n, v_1 \}$$

$$Q_i = \{v_0, v_i \} \quad i = 2, 3, \ldots, n-1$$

$$Q_n = \{v_0, v_1, v_n \}$$

$\psi = \{P_1 \cup \{Q_i \}$ is minimum geodesic graphoidal covering of $G$

$\Rightarrow \eta_g = 2n-1 = q - p + n$

**Theorem 2.6:** Let $G$ be a Helm graph $2n+1$ vertices and $3n$ edges then $\eta_g = q - p + n$.

**Proof:** Let $V(G) = \{v_0, v_1, v_2, v_3, \ldots, v_n, w_1, w_2, \ldots, w_n \}$

where $v_0$ is the centre vertex of wheel and is $v_i$ adjacent to $w_i$ and $v_0$ and $P = 2n+1$ and $q = 3n$

The Geodesic graphoidal path covering of $G$ is as follows

$$P_i = \{w_i, v_i, v_{i+1} \} \quad i = 1, 2, \ldots, n-1$$

$$P_n = \{w_n, v_n, v_1 \}$$

$$Q_i = \{v_0, v_i \} \quad i = 2, 4, 5, \ldots, n$$

$$Q_n = \{v_3, v_0, v_1 \}$$

$\psi = \{P_1 \cup \{Q_i \}$ is minimum geodesic graphoidal covering of $G$

$\Rightarrow \eta_g = 2n-1 = q - p + n$

For the Helm graph the graphoidal cover is also the same.

(i.e.) $\eta_a = q - p + n$
Theorem 2.7: Let G be $P_m\left(QS_n\right)$ graph then $\eta_g = n(m+1) - 1$

Proof:
Let $V(G) = \{v_1, v_2, v_3, ..., v_m, l_{i1}, l_{i2}, ..., l_{in}, r_{i1}, r_{i2}, ..., r_{in}, w_{i1}, w_{i2}, ..., w_{in}\}$ $i = 1, 2, ..., n$

The Geodesic graphoidal path covering of G is as follows

$P_i = \{w_{in}, l_{in}, w_{in-1}, ..., w_{i2}, l_{i2}, w_{i1}, v_{i1}, l_{i1}, w_{i1}, v_{i1}, ..., l_{i1}, w_{in}, v_{in}, w_{in}\}$ $i = 1, 2, ..., n-1$

$Q_i = \{v_i, r_{i1}, w_{i1}\}$, $i = 1, 2, ..., n$

$R_i = \{w_{i1}, r_{i2}, w_{i2}\}$, $i = 1, 2, ..., n$

$S_i = \{w_{i(m-1)}, r_{in}, w_{in}\}$, $i = 1, 2, ..., n$

$\psi = P \cup \{P_i\} \cup \{Q_i\} \cup \{R_i\} \cup \{S_i\}$ is minimum geodesic graphoidal covering of G

$\Rightarrow \eta_g = n-1 + mn = n(m+1)-1$

Theorem 2.8: Let G be $C_m\left(QS_n\right)$ graph then $\eta_g = n + mn$

Proof:
Let $V(G) = \{v_1, v_2, v_3, ..., v_m, l_{i1}, l_{i2}, ..., l_{in}, r_{i1}, r_{i2}, ..., r_{in}, w_{i1}, w_{i2}, ..., w_{in}\}$ $i = 1, 2, ..., n$

The Geodesic graphoidal path covering of G is as follows

$P_i = \{w_{in}, l_{in}, w_{in-1}, ..., w_{i2}, l_{i2}, w_{i1}, v_{i1}, l_{i1}, w_{i1}, v_{i1}, ..., l_{i1}, w_{in}, v_{in}, w_{in}\}$ $i = 1, 2, ..., n-1$

$P_n = \{v_n, v_1\}$

$Q_i = \{v_i, r_{i1}, w_{i1}\}$, $i = 1, 2, ..., n$

$R_i = \{w_{i1}, r_{i2}, w_{i2}\}$, $i = 1, 2, ..., n$

$S_i = \{w_{i(m-1)}, r_{in}, w_{in}\}$, $i = 1, 2, ..., n$

$\psi = \{P_i\} \cup \{Q_i\} \cup \{R_i\} \cup \{S_i\}$ is minimum geodesic graphoidal covering of G

$\Rightarrow \eta_g = n + mn$

Theorem 2.9: Let G be a web graph with $3n+1$ vertices and $5n$ edges then $\eta_g = q - p + n$.

Proof: Let $V(G) = \{v_0, v_i, v_{i1}, v_{i2}\}$ $i = 1, 2, ..., n$

Here $v_{i2}$ is adjacent to $v_0$ and $v_{i1}$ and $v_{i1}$ is adjacent to $v_i$ and $v_{i2}$

The Geodesic graphoidal path covering of G is as follows

$P_i = \{v_{i1}, v_{i1}, v_{i1}, v_{i1}\}$ $i = 1, 2, ..., n-1$

$P_n = \{v_{i1}, v_{i1}, v_{i1}\}$

$Q_i = \{v_{i1}, v_{i2}, v_{i1,2}\}$ $i = 1, 2, ..., n-1$

$Q_n = \{v_{i1}, v_{i2}, v_{i1,2}\}$

$R_i = \{v_0, v_{i2}\}$ $i = 2, 4, 5, ..., n$

$R_n = \{v_{i2}, v_0, v_{i2}\}$
\( \psi = \{ P_i \cup Q \} \cup R_i \) is minimum geodesic graphoidal covering of G

\[ \Rightarrow \eta_n = 3n - 1 = q - p + n \]

For the Web graph the graphoidal cover is also the same.
\((i.e.) \eta_n = q - p + n\)

**Theorem 2.10:** Let G be a shell graph with \( n+1 \) vertices then \( \eta_n = q - p + \frac{3n - 6}{2} \).

**Proof:** Let \( V(G) = \{ v_1, v_2, v_3, \ldots, v_n \} \)

\( p = n, q = 2n + 1 \)

The Geodesic graphoidal path covering of G is as follows

\( P_i = \{ v_i, v_i \} \ i = 3, 4, \ldots, n - 1 \)

\( Q = \{ v_i, v_i \} \)

\( R_i = \{ v_i, v_{i-1}, v_{i+1} \} \ i = 2, 4, 6, \ldots, n - 2 \)

\[ \psi = \{ P_i \cup Q \cup R_i \} \text{ is minimum geodesic graphoidal covering of G} \]

\[ \Rightarrow \eta_n = n - 3 + 1 + \frac{n}{2} - 1 = q - p + \frac{3n - 6}{2} \]

**Theorem 2.11:** Let G be a book graph with \( 2n+2 \) vertices then \( \eta_n = q - p + n \).

**Proof:** Let \( V(G) = \{ v_1, v_2, b_{i1}, b_{i2} \} \ i = 1, 2, \ldots, n \)

The Geodesic graphoidal path covering of G is as follows

\( P_i = \{ v_i, v_i \} \)

\( P_2 = \{ b_{i1}, b_{i2}, v_{i1}, v_{i2} \} \)

\( P_{n+1} = \{ b_{i1}, v_i, b_{i+1}, b_{i2} \} \)

\( P_i = \{ b_{i1}, b_{i2}, v_i \} \ i = 3, 4, \ldots, n \)

[ If this path not exists for some \( i \) then for that particular \( i, P_i = \{ b_{i2}, b_{i1}, v_i \} \]}

\( Q_i = \{ v_i, b_{i1} \} \ o \ \{ v_i, b_{i2} \} \ i = 3, 4, \ldots, n \)

\[ \psi = \{ P_i \cup Q_i \} \text{ is minimum geodesic graphoidal covering of G} \]

\[ \Rightarrow \eta_n = 3 + n - 2 + n - 2 = 2n - 1 = q - p + n \]

For book graph the graphoidal coving number is also same as geodesic graphoidal covering number
\((i.e.) \eta_n = q - p + n\)

**Theorem 2.12:** Let G be Mongolian tent graph then \( \eta_n (M_{m,n}) = q - p + 2 \)

**Proof:** Let \( G = M_{m,n} \)

Let \( V(G) = \{ v_0, v_1, v_2, \ldots, v_m \} \ i = 1, 2, \ldots, m \)
The Geodesic graphoidal path covering of G is as follows
\[ P_i = \{v_{i+1}, v_i, v_{i-1}, \ldots, v_n\} \quad i = 1, 2, \ldots, m-1 \]
\[ P_n = \{v_1, v_2, v_3, \ldots, v_m\} \]
\[ P_{n+1} = \{v_{n+1}, v_0, v_1\} \]
\[ S = \text{The remaining edges} \]

From above paths all the vertices are exterior points except \( v_1 \) and \( v_{n!} \)
\[ \psi = P_1 \cup P_2 \cup \ldots \cup P_n \cup P_{n+1} \cup S \] is minimum geodesic graphoidal covering of G
\[ \Rightarrow \eta_g = q - p + 2 \]

For book graph the graphoidal coveing number is also same as geodesic graphoidal covering number

**Theorem 2.13:** Let G be double wheel graph with \( 2n+1 \) vertices
\[ \eta_g (G) = \begin{cases} 3n+2 & \text{if } n \text{ is odd} \\ 3n & \text{if } n \text{ is even} \end{cases} \]

**Proof:** Let \( V(G) = \{v_0, c_{11}, c_{12}, \ldots, c_{1n}, c_{21}, c_{22}, \ldots, c_{2n}\} \)
\[ P_1 = (c_{21}, v_0, c_{11}) \]
\[ P_j = \{c_{i,j}, c_{i,j+1}, c_{i,j+2}, i = 1, 2 \land j = 1, 3, 5, \ldots, n - 2 \} \]
\[ Q_{ij} = \begin{cases} \{c_{j,n-1}, c_{j,n}, c_{i,j}\}, & i = 1, 2 \land n \text{ is odd} \\ \{c_{i,n-1}, c_{i,n}, c_{i,j}\}, & i = 1, 2 \land n \text{ is even} \end{cases} \]
\[ \Rightarrow \eta_g = 2n - 2 + n - 1 + 1 = 3n - 2 \text{ if } n \text{ is odd} \]
\[ \eta_g = 2n - 2 + n + 1 + 1 = 3n \text{ if } n \text{ is even} \]

**Theorem 2.14:** \( \eta_g (A_5^3) = q - p + 2 \).

**Proof:** Let \( V(G) = \{x, c_{11}, c_{12}, \ldots, c_{1n}, c_{21}, c_{22}, \ldots, c_{2n}, c_{31}, c_{32}, \ldots, c_{3n}, y\} \)

The Geodesic graphoidal path covering of G is as follows
\[ P_i = \{x, c_{i}, c_{i+1}, c_{i+j}, y\} \quad i = 1, 2, \ldots, n \]

Clearly all the vertices are internal vertices except x and y.
\[ \psi = \{P_i\} \cup \text{Remaining Edges} \] is minimum geodesic graphoidal covering of G
\[ \Rightarrow \eta_g = q - p + 2 \]

**Theorem 2.15:** \( \eta_g (D_7^3) = q - p + 2 \).

**Proof:** Let \( V(G) = \{y, v_{i1}, v_{i2}, \ldots, v_{in}\} \quad i = 1, 2, \ldots, m \)

The geodesic graphoidal cover of \( D_7^3 \) is as follows:
\[ P_i = \{v_{i+1}, v_i, v_{i-1}, \ldots, v_n\} \quad i = 1, 2, \ldots, m-1 \]
\[ P_n = \{v_1, v_2, v_3, \ldots, v_m\} \]
\[ Q = \{v_{i+1}, y, v_{in}\} \]
\[ S = \text{The remaining edges} \]
\[ \psi = \left\{ P_i \right\} \cup Q \cup S \] is minimum geodesic graphoidal covering of \( G \) and all the vertices are internal vertices except \( v_{11} \) and \( v_{1n} \).

\[ \therefore \eta_G = q - p + 2 \]

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