CHAPTER VI
PATH DOUBLE COVERING NUMBER OF GRAPHS

6.1 Introduction

The Concept of path Double covering was introduced by Bondy [12]. The parameter path double covering number of a graph $\eta_{PD}$ was introduced by S. Arumugam and S.Meena [5] and they found $\eta_{PD}$ for trees, Unicyclic graphs, complete graphs, wheel and rectangular grid.

This Chapter is focused on finding the parameter path double covering number $\eta_{PD}$ of bicyclic graphs containing $U(l,m), D(l,m,i), C_m(l,i)$ and for various types of graphs like Triangular snake, Double Triangular snake, alternate double Triangular snake, Triple Triangular snake, Web graph, Gear graph, Double wheel, Triangular Cactus, Helm graph, Mobious Ladder, Flower graph, Mongolian Tent, $P_m \times P_n$, Shell graph, Multple shell graph, Ladder graph, Book graph, t-ply, Fan graph, $P_m(Q_{s_n})$ and $C_m(Q_{s_n})$.

A path double cover of a graph $G$ is a collection $P$ of paths in $G$ such that every edge of $G$ belongs to exactly two paths in $P$. The minimum cardinality of a path double cover is called the path double covering number of $G$ and is denoted by $\eta_{PD}(G)$. S.Arumugam and S.Meena proved the following results.

(i) For any graph $G$, $\eta_{PD}(G) \leq 2q$ and equality holds iff $G$ is isomorphic to $qK_2$.

(ii) Let $P$ be any path double cover of a graph $G$. Then $|P| = 2q - i_p$ where $i_p = \sum_{p \in P} i(p)$ and $i(p)$ is the number of internal vertices of $P$.

(iii) For any tree $T$, $\eta_{PD}(T) = n$ where $n$ is the number of pendent vertices of $T$. 

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(iv) For any graph $G$, $\eta_{pd}(T) \geq \Delta$. Further for any tree $T$, $\eta_{pd}(T) = \Delta$ if and only if $T$ is homeomorphic to a star.

(v) Let $G$ be a graph with $\delta = 1$, if there exists a path double cover $P$ such that every non-pendant vertex of $G$ is an internal vertex of $d(v)$ paths in $P$ then $P$ is minimum path double cover and $\eta_{pd} = |P|$

6.2 Path double covering of bicyclic graphs

Theorem 6.2.1

Let $G$ be a bicyclic graph with $n$ pendant vertices containing a $U(l,m)$ and let $j$ be the number of vertices of degree greater than 2 on $U(l,m)$ except $u_0$. Then the path double covering number of $G$ is

$$\eta_{pd} = \begin{cases} 
4 & \text{if } G = U(l,m) \\
n + 4 & \text{if } j = 0 \\
n + 3 & \text{if } j_1 = 0 \text{ or } j_2 = 0 \text{ and } j = 1 \text{ and } n \geq j \\
n + 2 & \text{if } j_1 = 0 \text{ or } j_2 = 0 \text{ and } j \geq 2 \text{ and } n > j \\
n + 2 & \text{if } j_1 \geq 1 \text{ or } j_2 \geq 1 \text{ and } j = 2, 3 \text{ and } n = j \\
n + 1 & \text{if } j_1 \geq 1 \text{ or } j_2 \geq 1 \text{ and } j = 2 \text{ and } n > j \\
n & \text{ otherwise}
\end{cases}$$

where $j_1$ and $j_2$ are the number of vertices of degree greater than 2 on $C_l$ and $C_m$ respectively and $j = j_1 + j_2$.

Proof:

Let $V(U(l,m)) = \{u_0, u_1, u_2, \ldots, u_{l-2}, u_l, u_{l+1}, \ldots, u_{l+m-2}\}$

Case 1: $G = U(l,m)$

The path double cover of $G$ is as follows

$$P_1 = \{u_1, u_2, u_3, \ldots, u_{l-2}, u_l, u_{l+1}, u_{l+2}, \ldots, u_{l+m-2}\}$$

$$P_2 = \{u_{l-1}, u_{l-2}, \ldots, u_2, u_1, u_0, u_{l+m-2}, u_{l+m+1}, \ldots, u_{l+1}, u_l\}$$
\[ P_3 = \{u_1, u_0, u_{i+m-2}\} \]

\[ P_4 = \{u_{i-1}, u_0, u_i\} \]

\[ P = \{P_1, P_2, P_3, P_4\} \] is a path double cover of \( G \)

\[ \eta_{pd}(G) = 4 = \Delta \]

**Case 2:**

Either \( j_1 = 0 \) or \( j_2 = 0 \) and \( j = 1 \)

we prove this by induction on \( n \).

when \( j = 1 \) and \( n = 1 \)

without loss of generality assume that \( j_1 = 1 \), \( \deg u_i = 3 \), \( u_i \) lies on \( C \) then \( G \) is isomorphic to the graph consisting of \( U(l, m) \) together with a path

\[ P = \{w_1, w_2, \ldots, w_r (= u_i)\} \]

The path double cover is as follows

\[ P_1 = \{w_1, w_2, w_3, \ldots, w_r = u_1, u_{i+1}, u_{i+2}, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_{i+m-2}\} \]

\[ P_2 = \{u_1, u_{i+1}, \ldots, u_{i+m-2}, u_{i-1}, u_i, u_{i+1}, \ldots, u_i\} = w_r, w_{r-1}, w_{r-2}, \ldots, w_1\} \]

\[ P_3 = \{u_{i-1}, u_0, u_i\} \]

\[ P_4 = \{u_{i+m-2}, u_0, u_1, \ldots, u_{i-1}, u_i\} \]

\[ P = \{P_1, P_2, P_3, P_4\} \] is a path double cover of \( G \)

\[ \Rightarrow \eta_{pd} \leq 4 \]

Since \( \eta_{pd} \geq \Delta = 4 \)

Now assume that the result is true for all bicyclic graphs with \( n-1 \) pendant vertices

and \( j = 1 \).

Let \( G \) be any bicyclic graph with \( n \) pendant vertices and \( j = 1 \)
Let \( w \) be any pendant vertex of \( G \). Choose a vertex \( v \) disjoint from \( u \) such that \( \deg v \geq 3 \) and \( d(w, v) \) is maximum. Let \( Q \) denote the \((w, v)\) path. Since \( \deg v \geq 3 \) there exist pendant vertices \( w_1, w_2 \) such that \((w, w_1)\) path \( Q_1 \) and \((w, w_2)\) path \( Q_2 \) both contain \( Q \).

Now let \( P_1 \) and \( P_2 \) denote the \((w_1, v)\) section of \( Q_1 \), \((w_2, v)\) section of \( Q_2 \) respectively. Let \( P = P_1 \circ P_2^{-1} \)

Clearly \( v \) is the only vertex of degree greater than two on \( P \).

Let \( P = (w_1 = v_0, v_1, v_2, \ldots, v_r = v, \ldots, v_k = w_2) \)

Consider the graph \( G_1 = G - \{v_0, v_1, v_2, \ldots, v_{r-1}, v_{r+1}, \ldots, v_k\} \)

**Case 2a:**

Let \( \deg_{G_1} v_r = 3 \)

Therefore \( G_1 \) is a bicyclic graph with \( n-1 \) pendant vertices.

Then \( \deg_{G_1} v_r = 3 \)

Therefore \( v_r \) is exterior to two paths in \( P_1 \), say for \( R_1 \) and \( R_2 \) in \( P_1 \).

Let \( P_1^1 = R_1 \circ P_1 \) and \( P_2^1 = R_2 \circ P_2 \)

Let \( P = P_1 \cup (P_1', P_2, P) - \{R_1, R_2\} \) is a path double cover of \( G \).

\[ n_{PD}(G) = n + 2 + 1 = n + 3 \]

Thus \( n_{PD}(G) \leq n + 3 \)

In any path double cover of \( G \) all the \( n \) pendant vertices and at least three vertices in \( U(l, m) \) are exterior points.

\[ (i.e) \ n_{PD}(G) \geq n + 3 \]

Hence \( n_{PD}(G) = n + 3 \)

**Sub Case 2b:**

\( \deg_G v_r = 3 \)
Then $\deg_{G_i} v_i \geq 2$

Then $G_1$ is bicyclic graph with $n$-2 pendant vertices.

Using induction hypothesis $\eta_{PD}(G_1) = (n - 2) + 3 = n + 1$

Let $P_1$ be a minimum path double cover of $G_1$

Then $P_1 \cup \{P, P\}$ is a minimum path double cover of $G$.

$\eta_{PD}(G) = n + 1 + 2 \leq n + 3$

In any path double cover of $G$ all the pendant vertices and at least three vertices in $U(l, m)$ are exterior points.

Thus $\eta_{PD}(G) \geq n + 3$

Hence $\eta_{PD}(G) = n + 3$

Case 3:

Sub case 3a:

Let $j_1 \geq 1, j_2 \geq 1, j = 2, 3$ and $j = n$

Let $j = 2$ and $n = j$

Since $j = 2$, clearly $j_1 = 1$ and $j_2 = 1$

Let $u_i$ be the vertex on $C_j$ and $u_k$ be the vertex on $C_m$ are of degree greater than 2.

$u_i$ and $u_k$ is incident with pendant vertices $w_1$ and $w_2$ respectively.

The path double cover is as follows

$P_1 = \{u_1, u_2, \ldots, u_j, u_0, u_1, \ldots, u_{j+m-2}\}$

$P_2 = \{(w_1 =)w_{11}, w_{12}, w_{13}, \ldots, w_{1n} (= u_i), u_{i+1}, \ldots, u_{i-1}, u_0, u_i, \ldots, u_{k}, w_{2n} (= u_k), w_{2(n-i)} \ldots (w_2 =)w_{21}\}$

$P_3 = \{(w_1 =)w_{11}, w_{12}, w_{13}, \ldots, w_{1n} (= u_i), u_{i-1}, \ldots, u_i, u_0, u_{i+m-2}, \ldots, u_k, w_{2n} (= u_k), w_{2(n-1)} \ldots (w_2 =)w_{21}\}$

$P_4 = \{u_1, u_0, \ldots, u_{j+m-2}\}$

$P = \{P_1, P_2, P_3, P_4\}$ is a path double cover of $G$
\[ \Rightarrow \eta_{PD}(G) \leq 4 = n + 2 = \Delta \]

We know that \( \eta_{PD}(G) \geq \Delta \)

\[ \therefore \eta_{PD}(G) = 4 = n + 2 = \Delta \]

If \( j = 3 \) and \( n = j \) then the proof is similar to the case 3a.

**Sub case 3b:**

Either \( j_1 = 0 \) or \( j_2 = 0 \) and \( n > j \) and \( j \geq 2 \) \((n > j)\)

We prove by induction on \( n \)

Let \( j = 2, n = 3 \) (since \( j \geq 2 \) and \( n > j \))

Without loss of generality assume that \( u_k \) on \( C_j \) is incident with the pendant vertices \( w_1 \) and \( u_i \) on \( C_j \) \((i < k)\) incident with the pendant vertices \( w_2 \) and \( w_3 \) respectively.

The path double cover is as follows

\[ P_1 = \{(w_2 =)w_{21}, w_{22}, w_{23}, \ldots, w_{2n} (= u_i), u_{i-1}, u_{i-2}, \ldots, u_0, u_{i+m-2}, \ldots, u_i\} \]

\[ P_2 = \{(w_3 =)w_{31}, w_{32}, w_{33}, \ldots, w_{3n} (= u_i), u_{i+1}, \ldots, u_k, u_{i-1}, u_0, u_{i+m-2}, \ldots, u_i\} \]

\[ P_3 = \{(w_1 =)w_{i1}, w_{i2}, w_{i3}, \ldots, w_{in} (= u_k), u_{k+1}, \ldots, u_{i-1}, u_0, u_i\} \]

\[ P_4 = \{(w_1 =)w_{i1}, w_{i2}, w_{i3}, \ldots, w_{in} (= u_k), u_{k-1}, \ldots, u_{i-1}, u_1, u_0, u_{i+m-2}\} \]

\[ P_5 = \{(w_2 =)w_{21}, w_{22}, w_{23}, \ldots, w_{2n} (= u_i), w_{3n} (= u_i), \ldots, w_{33}, w_{32}, (w_5 = w_{31})\} \]

\[ P = \{P_1, P_2, P_3, P_4, P_5\} \text{ is a path double cover of } G \]

\[ \therefore \eta_{PD} = n + 2 \]

Let us assume that the result is true for all bicyclic graph with \( n-1 \) pendant vertices \( j_1 = 0 \) or \( j_2 = 0 \) and \( n > j \) and \( j \geq 2 \) and

Let \( G \) be a bicyclic graph with \( n \) pendant vertices \((w_1, w_2, \ldots, w_n)\) and \( j_1 = 0 \) or \( j_2 = 0 \) and \( n > j \) and \( j \geq 2 \)

Using induction method as in case 2 we can prove that \( \eta_{PD}(G) = n + 2 \)
**Sub case 4c:** $j = 2$ and $n > j$

We prove this by induction on $n$.

Let $j = 2$ and $n = 4$

$u_i$ is incident with $w_1$ and $w_2$ on $C_l$

$u_k$ is incident with $w_3$ and $w_4$ on $C_m$

The path double cover is as follows

$P_1 = \{w_1, u_1, u_{i+1}, ..., u_{j-1}, u_0, u_{i-1}, ..., u_k, w_4\}$

$P_2 = \{w_4, u_k, u_{k-1}, ..., u_0, u_{i-1}, ..., u_i, w_1\}$

$P_3 = \{w_2, u_i, ..., u_{i+1}, u_0, u_{j+m-2}, ..., u_k, w_3\}$

$P_4 = \{w_2, u_i, u_{i+1}, ..., u_0, u_{j+m-2}, ..., u_k, w_3\}$

$P = \{P_1, P_2, P_3, P_4\}$ is a path double cover of $G$

$\therefore \eta_{PD}(G) = 4 = \Delta$

For any bicyclic graph with $n$ pendant vertices, using induction method as in case 2 we can prove that $\eta_{PD}(G) = n + 2$

**Case 4:**

Let $j = 4$, $n = j = 4$

Let $u_i, u_k$ be the two vertices on $C_l$ of degree greater than two incident with the pendant vertices $w_1$ and $w_2$ respectively.

Let $u_m, u_n$ be the two vertices on $C_m$ of degree greater than two incident with the pendant vertices $w_3$ and $w_4$ respectively. [$k < i$ & $m < n$]

The path double cover is as follows

$P_1 = \{w_1, u_i, u_{i+1}, u_{i+2}, u_{j-1}, u_0, u_{i-1}, ..., u_n, w_4\}$

$P_2 = \{w_1, u_i, ..., u_{i+1}, u_{i+2}, u_{j-1}, u_0, u_{i-1}, ..., u_m, w_3\}$

$P_3 = \{w_2, ..., u_k, u_{k+1}, ..., u_i, u_{i+1}, u_{j+1}, ..., u_{j+m-2}, u_j, u_{j+1}, ..., u_n, w_4\}$
\[ P_4 = \{ w_2, \ldots, u_{j-1}, u_0, u_{j+m-2}, \ldots, u_{\mu-2}, \ldots, u_{\mu}, \ldots, w_3 \} \]

\[ P = \{ P_1, P_2, P_3, P_4 \} \] is a path double cover of \( G \)

\[ \therefore \eta_{PD}(G) = n = \Delta \]

If \( n > j \) then we can use induction method as in case 2 to prove

\[ \eta_{PD}(G) = n + 1 \]

**Case 5:**

Let \( j > 5, \ n > j \)

Here the number of pendant vertices is greater than 5, using induction method as in case 2, we can find the path double covering such that all the vertices are internal vertices except the pendant vertices.

\[ \therefore \eta_{PD}(G) = n \]

**Case 5:**

If \( j = 0 \) then \( G \) is abicyclic graph containing \( U(l, m) \) with a tree \( T \) attached at \( u_0 \) so that

\[ \deg u_0 \geq 5 \]

Let \( T \) be a tree with \( n \) pendant vertices.

We prove this by induction on \( n \).

If \( n = 1 \) then \( G \) contains \( U(l, m) \) with a path \( u_0 = w_1, w_2, \ldots, w_m \) attached at \( u_0 \)

\[ P_1 = \{ u_1, u_2, \ldots, u_{j-1}, u_0, u_1, \ldots, u_{j+m-2} \} \]

\[ P_2 = \{ u_1, u_0 = w_1, w_2, w_3, \ldots, w_m \} \]

\[ P_3 = \{ u_1, u_0 = w_1, w_2, w_3, \ldots, w_m \} \]

\[ P_4 = \{ u_1, u_0, u_2 \} \]

Then is \( P = \{ P_1, P_2, P_3, P_4 \} \) a path double cover of \( G \)

\[ \therefore \eta_{PD} = 5 = n + 4 \]

For \( n \geq 2 \) we prove that the result using induction method as in case 2.
Theorem 6.2.2

Let $G$ be a bicyclic graph with $n$ pendant vertices containing a $D(l,m,i)$ as the unique bicycle in $G$ and let $j$ be the number of vertices of degree greater than 2 on $D(l,m,i)$ other than $u_{i-1}$ & $u_{i+1}$ . Then the path double covering number of $G$ is

$$
\eta_{PD} = \begin{cases} 
4 & \text{if } G = D(l,m,i) \\
n+4 & \text{if } \begin{cases} 
 j = 0, \deg u_{i-1} & \deg u_{i+1} > 3 \\
 j_1 = j_2 = 1 & \text{and } j_3 = 0 \\
 j_1 = j_2 = 0 & \text{and } j_3 \geq 1 
\end{cases} \\
n+3 & \text{if } \begin{cases} 
 j_1 = 1, 2, j_2 = j_3 = 0 \\
 j_1 = 2 & \text{or } j_2 = 1 & \text{and } j_3 = 0 \\
 j_1 = j_2 = j_3 = 1 \\
 j_1 & \text{or } j_2 = 1, j_3 = 0 \\
 j_1 \geq 3, j_2 = 0, 1, j_3 = 0 \\
 j_1 = j_2 = 2, j_3 = 0 & \text{or } 2 \\
 j_1 = j_2 = 1, j_3 = 0 \\
 j_1 \geq 3, j_2 = 2, j_3 = 0 \\
n+1 & \text{if } j_1 \geq 3, j_2 = 2, j_3 = 0 \\
n & \text{otherwise}
\end{cases}
\end{cases}
$$

where $j_1$ and $j_2$ are the number of vertices of degree greater than 2 on $C_1$ and $C_m$ and $j_3$ is the the number of vertices of degree greater than 2 on the path respectively.

Proof:

Let $V(D(l,m,i)) = \{u_0, u_1, u_2, u_3, \ldots, u_{i-1}, u_i, u_{i+1}, u_{i+2}, \ldots, u_{i+m+i-2}\}$

Case 1: $j = 0$

Case 1a:

$G = D(l,m,i)$

The path double cover of $G$ is as follows

$$P_1 = \{u_0, u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, u_{i+2}, \ldots, u_{i+m+i-2}\}$$

$$P_2 = \{u_{i+1}, u_{i+2}, \ldots, u_{i+m+i-2}, u_{i+1}, u_i, u_{i-1}, u_0, u_1, u_2, \ldots, u_{i-2}\}$$

$$P_3 = \{u_0, u_{i-1}, u_{i-2}\}$$

$$P_4 = \{u_{i+1}, u_{i+2}, u_{i+m+i-2}\}$$
$P = \{P_1, P_2, P_3, P_4\}$ is a path double cover of $G$

$\therefore \eta_{pd}(G) = 4 = \Delta$

**Case 1b:**

$G = D(l, m, i)$ with $\deg u_{r-1} \& \deg u_{r+1} \geq 4$

We prove this result by induction on $n$.

Without loss of generality assume that, $\deg u_{r-1} = 4$ then there is a path $(p_1, p_2, \ldots, p_t)$ attached with $u_{r-1}$

The path double cover of $G$ is as follows

$P_1 = \{u_0, u_1, u_2, \ldots, u_{r-1}, u_r, \ldots, u_{s+1-r}, u_{s+2-r}, \ldots, u_{m+i-2}\}$

$P_2 = \{u_{r+1}, u_{r+2}, \ldots, u_{s+2-r}, u_{s+3-r}, u_{s+4-r}, \ldots, u_{m+i-2}\}$

$P_3 = \{u_0, u_{r-1}, p_1, p_2, \ldots, p_t\}$

$P_4 = \{u_{s+1-r}, u_{s+2-r}, \ldots, u_{m+i-2}\}$

$P_5 = \{u_{t-2}, u_{t-1}, p_1, p_2, \ldots, p_t\}$

$P = \{P_1, P_2, P_3, P_4, P_5\}$ is a path double cover of $G$

$\therefore \eta_{pd} = 5 = n + 4$

For $n \geq 2$ we can use the induction method as in case 2 of theorem 6.2.2 to prove $\eta_{pd} = n + 4$

**Case 2:**

**Sub Case 2a:**

Let $j_1 = j_2 = 1$ and $j_3 = 0$

$j = j_1 + j_2 = 2$

We prove this by induction on $n$

Let $n = 2$
Let \( v_0,v_1,v_2,\ldots,v_r \) be a path attached with \( u_i \) in \( C_l \) and \( w_0,w_1,w_2,\ldots,w_r \) be a path attached with \( u_t \) in \( C_m \).

Define the path double cover as follows

\[
P_1 = \{v_0,v_1,\ldots,v_r = u_j, u_{j-1}, \ldots, u_i, u_0, u_{l-1}, u_{l-2}, \ldots, u_{l+i-2}, \ldots u_r, w_0\}
\]

\[
P_2 = \{v_0,v_1,\ldots,v_r = u_i, u_{l+i+1}, \ldots, u_{l+i+2}, u_{l+i+3}, u_{l+i+4}, \ldots, u_r, w_0\}
\]

\[
P_3 = \{v_r = u_i, u_{l+i+1}, \ldots, u_{l+i-1}\}
\]

\[
P_4 = \{v_r = u_i, u_{l-1}, \ldots, u_{l-2}\}
\]

\[
P_5 = \{w_r = u_r, u_{r-1}, \ldots, u_{r+i-1}\}
\]

\[
P_6 = \{w_r = u_r, u_{r+1}, \ldots, u_{r+i-1}\}
\]

\[
P = \{P_1, P_2, P_3, P_4, P_5, P_6\} \text{ is a path double cover of } G
\]

Here, the two pendant vertices are exterior vertices and exactly two vertices on both \( C_l \) and \( C_m \) are exterior points.

\[\therefore \eta_{pd} (G) = n + 4\]

Assume that the result is true for \( G \) with less than \( n-1 \) pendant vertices.

Let \( G \) be a graph containing \( L \) with \( n \) number of pendant vertices and \( j = 2 \).

The proof is similar to case 2 of the Theorem 6.2.1

**Sub Case 2b:**

Let \( j_1 = j_2 = 0 \) and \( j_3 \geq 1 \).

Let \( j = n = 1 \).

\( G \) is isomorphic to the graph consisting of \( D(l,m,i) \) together with a path \( P = \{p_0, p_1, \ldots, p_l (= u_i)\} \) attached with the vertex \( u_l \).

The path double cover is as follows

\[
P_1 = \{u_i, u_{l+1}, u_{l+2}, u_0, u_{l-1}, u_{l-2}, \ldots, u_{l+i-1}, u_{l+i}, \ldots, u_{l+i+2}, \ldots, u_r\}
\]

\[
P_2 = \{p_0, p_1, \ldots, p_l (= u_i), u_{l+1}, \ldots, u_{l+i+1}, u_{l+i+2}, \ldots, u_r\}
\]
Here one pendant vertex and two vertices in $C_l$ and $C_m$ are exterior vertices.

$P_3 = \{ p_0, p_1, \ldots, p_i = u_i, u_{i-1}, \ldots, u_1, u_0 \}$

$P_4 = \{ u_0, u_{i-3}, u_{i-2} \}$

$P_5 = \{ u_{i+1}, u_{i+3}, u_{i+m+i-2} \}$

$P = \{ P_1, P_2, P_3, P_4, P_5 \}$ is a path double cover of $G$

$\therefore \eta_{PD} = n + 4$

If $j = 1$ and $n > 1$ we prove this by using the technique used in case 2 of Theorem 6.2.1.

Case 3:

Sub case 3a:

$j_1 = 1, 2; j_2 = j_3 = 0; j = n$ or $j_2 = 1, 2; j_1 = j_3 = 0, j = n$

In this case we have $n$ pendant vertices (exterior vertices) and 2 vertices on $C_m$ and one vertex on $C_l$ are exterior vertices.

Using induction method as in case 2 of the Theorem 6.2.1, we can prove that

$\eta_{PD} = n + 3$

Sub case 3b:

$j_1 = j_2 = j_3 = 1; j = n$

In this case we have $n$ pendant vertices (exterior vertices), 2 vertices are exterior on $C_m$ and one vertex is exterior on $C_l$

Using induction method as in case 2 of the Theorem 6.2.1, we can prove that

$\eta_{PD} = n + 3$
**Sub case 3c:**

\( j_1 = 1, j_2 = j_3 = 0; j < n \)

In this case we have \( n \) pendant vertices (exterior vertices), 2 vertices are exterior on \( C_m \) and one vertex is exterior on \( C_l \)

Using induction method as in case 2 of the Theorem 6.2.1, we can prove that \( \eta_{PD} = n + 3 \)

The proof for the remaining cases is similar to the above cases.

**Theorem 6.2.3**

Let \( G \) be a bicyclic graph with \( n \) pendant vertices and \( G \) containing a \( C_m(i,l) \) which is the unique bicycle in \( G \) and let \( j \) be the number of vertices greater than 3 on \( C_m(i,l) \) except \( u_0 \) & \( u_i \). Then the path double covering number of \( G \) is

\[
\eta_{PD} = \begin{cases} 
3 & \text{if } G = C_m(i,l) \\
3 + j & \text{if } j = 1 \text{ and } \deg u_k \geq 6 \text{ for some } k \\
n + 2 & \text{if } j_1 \geq 1, j_2 = 0 \text{ or } j_2 \geq 1, j_1 = 0 \\
n & \text{otherwise}
\end{cases}
\]

Where \( j_1 \) and \( j_2 \) are the number of vertices of degree greater than 2 on the upper half and lower half on \( C_m(i,l) \) respectively and \( j = j_1 + j_2 \)

If \( \deg u_i \) or \( \deg u_0 \geq 4 \) then this will count into \( j_1 \).

**Proof:**

Let \( V(C_m(i,l)) = \{u_0, u_1, u_2, \ldots, u_{i-1}, u_j, u_{j+1}, \ldots, u_{m-1}, u_m, u_{m+1}, \ldots, u_{i+m-2}\} \)

\( j = j_1 + j_2 \)

**Case 1:** \( G = C_m(i,l) \)

The path double cover of \( G \) is as follows

\[ P_1 = \{u_{i+m-2}, \ldots, u_m, u_0, u_1, u_2, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_{m-1}\} \]
\[ P_2 = \{u_m, u_{m+1}, \ldots, u_{j+m-2}, u_j, u_{j+1}, \ldots, u_{j+m-1}, u_0, u_1, u_2, \ldots, u_{j-1}\} \]

\[ P_3 = \{u_m, u_0, u_{m-1}\} \]

\[ P_4 = \{u_{j+m-2}, u_j, u_{j-1}\} \]

\[ P_\{P, P_2, P_3, P_4\}\] is a path double cover of \( G \)

\[ \therefore \eta_{PD} (G) = 4 \]

**Case 2:** If \( j = 1 \) and \( \deg u_k \geq 6 \) for some \( k \)

Without loss of generality assume that \( \deg u_j = 7 \)

The path double cover of \( G \) is as follows

\[ P_1 = \{u_1, u_2, \ldots, u_{j-1}, u_j, u_{j+m-2}, u_{j+m-1}, u_m, u_0, u_{m-1}, \ldots, u_{j+1}\} \]

\[ P_2 = \{w_{11}, w_{12}, \ldots, u_i, u_{i+1}, \ldots, u_0, u_1, u_2, \ldots, u_{j-1}\} \]

\[ P_3 = \{w_{21}, w_{22}, \ldots, u_i, u_{i+m-2}, \ldots, u_m, u_0, u_1\} \]

\[ P_4 = \{w_{31}, w_{32}, \ldots, u_i, u_{i+1}\} \]

\[ P_5 = \{w_{41}, w_{42}, \ldots, u_i, u_{i+1}\} \]

\[ P_6 = \{w_{51}, w_{52}, \ldots, \} \]

\[ P_7 = \{w_{61}, w_{62}, \ldots, \} \]

\[ P_\{P, P_2, P_3, P_4, P_5, P_6, P_7\}\] is a path double cover of \( G \)

\[ \therefore \eta_{PD} (G) = 7 = n + 3 = \Delta \]

For the different types of paths the proof is similar to case 2 of the Theorem 6.2.1.

For the case \( \deg u_j \geq 7 \), the proof is by induction method as in case 2 of Theorem 6.2.1.

\[ \therefore \eta_{PD} (G) = n - 4 + 7 = n + 3 \]

**Case 3:** Let \( j_1 \geq 1, j_2 = 0 \) or \( j_2 \geq 1, j_1 = 0 \)

We prove by induction on \( j_1 \)
Suppose \( n = 1 \) then \( j_1 = 1 \), without loss of generality assume that the path \((w_1, w_2, \ldots, w_n = u_1)\) attached with \( u_1 \).

The path double cover of \( G \) is as follows:

\[
P_1 = \{w_1, w_2, \ldots, u_1, u_0, u_m, u_{m+1}, \ldots, u_{j+m-2}, u_{j+1}, u_{j+2}, \ldots, u_{m-1}\}
\]

\[
P_2 = \{w_1, w_2, \ldots, u_2, u_1, u_{j+m-2}, u_{j+1}, u_{j+2}, \ldots, u_{m}, u_0, u_{m-1}, u_{m+1}, \ldots, u_{j+1}\}
\]

\[
P_3 = \{u_{m-1}, u_0, u_1, u_2, \ldots, u_j, u_{j+1}\}
\]

\[
P = \{P_1, P_2, P_3\} \text{ is a path double cover of } G
\]

\[
\therefore \eta_{PD}(G) = 3 = \Delta
\]

The proof is similar to the case 2 of the Theorem 6.2.1.

**Case 4:** For the remaining cases we can find a path double covering such that all the vertices are internal vertices except the pendant vertices using induction method.

\[
\therefore \eta_{PD}(G) = n
\]

### 6.3 Path double covering number of triangular graphs

**Theorem 6.3.1**

Let \( G \) be a triangular snake graph \( 2n-1 \) vertices. Then \( \eta_{PD}(G) = 4 \)

**Proof:**

Let \( V(G) = \{v_1, v_2, v_3, \ldots, v_n, w_1, w_2, \ldots, w_{n-1}\}. \)

The path double cover of \( G \) is as follows.

\[
P_1 = \{v_1, v_2, v_3, \ldots, v_n\}
\]

\[
P_2 = \{v_1, w_1, v_2, w_2, \ldots, v_{n-1}, w_{n-1}, v_n\}
\]

\[
P = \{2P_1 \cup 2P_2\} \text{ is a path double cover of } G
\]

\[
\eta_{PD}(G) \leq 4 = \Delta
\]
Since $\eta_{PD}(G) \geq \Delta = 4$

$\eta_{PD}(G) = \Delta = 4$ is a minimum path double covering number of $G$.

**Theorem 6.3.2**

Let $G$ be a double triangular snake graph $3n-2$ vertices. Then $\eta_{PD}(G) = 6 = \Delta$

**Proof:**

Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n, w_1, w_2, \ldots, w_n, u_1, u_2, \ldots, u_n\}$.

The path double cover of $G$ is as follows.

$P_1 = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$

$P_2 = \{v_1, u_1, v_2, u_2, v_3, \ldots, v_{n-1}, u_{n-1}, v_n\}$

$P_3 = \{v_1, w_1, v_2, w_2, v_3, \ldots, w_{n-1}, v_n\}$

$P = \{2P \cup 2P_2 \cup 2P_3\}$ is a path double cover of $G$

$\Rightarrow \eta_{PD}(G) \leq 6$

Since $\eta_{PD}(G) \geq \Delta = 6$

$\therefore \eta_{PD}(G) = \Delta = 6$

**Theorem 6.3.3**

Let $G$ be a triple triangular snake graph with $4n-3$ vertices. Then $\eta_{PD}(G) = 8$

**Proof:**

Let $V(G) = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_{n-1}, u_1, u_2, \ldots, u_{n-1}, z_1, z_2, \ldots, z_{n-1}\}$

$w_i, u_i, z_i$ is adjacent to $v_i$ and $v_{i+1}$

The path double cover of $G$ is as follows.

$P_1 = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$
Theorem 6.3.4

Let $G$ be alternate double triangular snake graph. Then

$$
\eta_{PD} (G) = \begin{cases} 
\frac{3n+4}{2} & \text{if } n \text{ is even} \\
\frac{3n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
$$

Proof:

Let $V(G) = \{v_i, v_{i+1}, w_i, w_{i+1}, w_{i+2}, w_{i+3}, \ldots, w_{n-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}, \ldots, u_{n-1} \}$ if $n$ is odd

Or $V(G) = \{v_i, v_{i+1}, w_i, w_{i+1}, w_{i+2}, w_{i+3}, \ldots, w_{n-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}, \ldots, u_{n-1} \}$ if $n$ is even

$w_i$ is adjacent to $v_i$ and $v_{i+1}, i = 1, 3, 5, \ldots$

$u_i$ is adjacent to $v_i$ and $v_{i+1}, i = 1, 3, 5, \ldots$

The path double cover of $G$ is as follows

$$
P_1 = \{v_1, v_2, \ldots, v_{n-1}, v_n\}
$$

$$
P_2 = \left\{v_1, w_1, v_2, w_2, v_3, w_3, \ldots, w_{\frac{n}{2}+1}, v_{n-1}, v_n\right\} \text{if } n \text{ is odd (or)}
$$
\[ P_2 = \left\{ v_1, w_1, v_2, w_2, v_3, w_3, \ldots, w_{\frac{n}{2}}, v_n \right\} \text{ if } n \text{ is even} \]

\[ S_i = \left\{ v_j, u_j, v_{j+1}, w_i \right\}, i = 1, 2, 3, \ldots, j = 1, 3, 5, \ldots \]

\[ Q_i = \left\{ w_i, v_j, v_{j+1} \right\}, i = 1, 2, 3, \ldots, j = 1, 3, 5, \ldots \]

\[ R_i = \left\{ v_i, u_i, v_{i+1} \right\}, i = 1, 3, 5, \ldots \]

In all the paths

\[ i = 1, 2, 3, \ldots, \frac{n}{2} \text{ if } n \text{ is even, } \frac{n-1}{2} \text{ if } n \text{ is odd } \]

\[ j = 1, 3, 5, \ldots, \frac{n}{2} \text{ if } j \text{ is even, } \frac{n-1}{2} \text{ if } j \text{ is odd } \]

\[ P = \left\{ P_1, P_2, S, Q, R \right\} \text{ is a minimum path double cover of } G \]

\[ \therefore \eta_{PD} = \begin{cases} \frac{3n+4}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd} \end{cases} \]

6.4 Path double covering number of some classes of graphs

Theorem 6.4.1:

Let \( G \) be a triangular cactus graph. Then \( \eta_{PD}(G) = 2n \), where \( n \) is the number of triangles in the graph.

Proof:

Let \( V(G) = \left\{ v_0, v_1, v_{12}, v_{22}, \ldots, v_{n1}, v_{n2} \right\} \)

Case 1: \( n \) is even and \( n > 4 \)

The path double cover of \( G \) is as follows.

\[ P_i = \left\{ v_{i1}, v_{i2}, v_{i0}, v_{i+12}, v_{i+11} \right\}, \quad i = 1, 3, 5, \ldots, n-1 \]

\[ Q_j = \left\{ v_{j+1,2}, v_{j+1,1}, v_{j0}, v_{j1}, v_{j2} \right\}, \quad j = 1, 3, 5, 7, \ldots, n-1 \]

\[ R_k = \left\{ v_{k1}, v_{k0}, v_{k2} \right\}, \quad k = 1 \text{ to } n \]
$P = \{P_i, Q_j, R_k\}[i, j = 1, 3, 5, \ldots, n - 1 \& 1 \leq k \leq n]$ is a path double cover of $G$

$|P| = \frac{n}{2} + \frac{n}{2} + n = 2n$

$\eta_{PD}(G) \geq 2n = \Delta$

Since $\eta_{PD}(G) \geq \Delta = 2n$

$\eta_{PD}(G) = \Delta = 2n$ is a minimum path double covering number of $G$.

**Case 2:** $n$ is odd and $n > 5$

The path double cover of $G$ is as follows.

$P_1 = \{v_{11}, v_{12}, v_0, v_{n1}, v_{n2}\}$

$P_2 = \{v_{21}, v_{22}, v_0, v_{n2}, v_{n1}\}$

$P_3 = \{v_{12}, v_{11}, v_0, v_{21}, v_{22}\}$

$S_i = \{v_{i1}, v_{i2}, v_0, v_{i+1,2}, v_{i+1,1}\}, i = 3, 5, 7, \ldots, n - 2$

$Q_j = \{v_{j+1,2}, v_{j+1,1}, v_0, v_{j+1}, v_{j+2}\}, j = 3, 5, 7, \ldots, n - 2$

$R_k = \{v_{k1}, v_{0}, v_{k2}\}, k = 1, 2, \ldots, n$

$P = \{P_1, P_2, P_3, S_i, Q_j, R_k\}[i, j = 1, 3, 5, \ldots, n - 2 \& 1 \leq k \leq n]$ is a path double cover of $G$

$|P| = 3 + \frac{n - 1}{2} - 1 + \frac{n - 1}{2} - 1 + n = 2n$

$\eta_{PD}(G) \leq 2n = \Delta$

Since $\eta_{PD}(G) \geq \Delta = 2n$

$\eta_{PD}(G) = \Delta = 2n$ is a minimum path double covering number of $G$.

**Theorem 6.4.2**

Let $G$ be a flower graph with $n$ petals. Then $\eta_{PD}(G) = 2n$
Proof:

Let $V(G) = \{v_1, v_{i1}, v_{i2}, v_{13}, v_{21}, v_{22}, \ldots, v_{i1}, v_{i2}, v_{i3}\}$

Case 1: $n$ is even.

The path double cover of $G$ is as follows.

$P_i = \{v_{i1}, v_{i2}, v_{i3}, v_0, v_{i+1}, v_{i+2}, v_{i+3}\}, i = 1, 3, 5, \ldots, n-1$

$Q_i = \{v_{i+1}, v_{i+2}, v_{i+3}, v_0, v_{i1}, v_{i2}, v_{i3}\}, i = 1, 3, 5, \ldots, n-1$

$R_k = \{v_{k1}, v_{k2}, v_{k3}\}, k = 1 \text{ to } n$

$P = \{P_i, Q_i, R_k\}, i = 1, 3, 5, \ldots, n-1 \text{ to } k \leq n$ is a minimum path double cover of $G$.

$\eta_{PD}(G) = 2n = \Delta$ is a minimum path double covering number of $G$.

Case 2: $n$ is odd.

The path double cover of $G$ is as follows.

$P_1 = \{v_{11}, v_{12}, v_{13}, v_0, v_{21}, v_{22}\}$

$P_2 = \{v_{n1}, v_{n2}, v_{n3}, v_0, v_{21}, v_{22}\}$

$P_3 = \{v_{13}, v_{12}, v_{11}, v_{n3}, v_{n2}, v_{n1}\}$

$S_i = \{v_{i1}, v_{i2}, v_{i3}, v_0, v_{i+1}, v_{i+2}\}, i = 3, 5, 7, \ldots, n-2$

$Q_i = \{v_{i+1}, v_{i+2}, v_{i+3}, v_0, v_{i1}, v_{i2}, v_{i3}\}, i = 3, 5, \ldots, n-2$

$R_k = \{v_{k1}, v_{k2}, v_{k3}\}, k = 1 \text{ to } n$

$P = \{P_1, P_2, P_3, S_i, Q_i, R_k\}, i = 1, 3, 5, \ldots, n-2 \text{ to } k \leq n$ is a minimum path double cover of $G$.

$\eta_{PD}(G) = 2n = \Delta$ is a minimum path double cover of $G$. 
Theorem 6.4.3

Let $G$ be a $P_m(QS_n)$ graph. Then $\eta_{PD}(G) = 2m + 2$

Proof:

Let $V(G) = \{u_1, \ldots, u_m, L_{j_1}, \ldots, L_{j_n}, R_1, \ldots, R_m, p_1, \ldots, p_m\}, i = 1, 2, \ldots, n$

The path double cover of $G$ is as follows.

$P_i = \{u_{i_1}, L_{i_1}, u_{i_2}, L_{i_2}, \ldots, u_{i_1}, L_{i_1}, P_i, P_{i+1}, L_{i+1}, u_{i+1}, L_{i+1}, u_{i+2}, \ldots, L_{i+1}, u_{i+1}\}i = 1, 2, \ldots, m - 1$

$Q_j = \{u_{j_1}, R_{j_1}, u_{j_2}, R_{j_2}, u_{j_1}, P_j, P_{j+1}, R_{j+1}, u_{j+1}, R_{j+1}, u_{j+2}, \ldots, R_{j+1}, u_{j+1}\}j = 1, 2, \ldots, m - 1$

$R_k = \{u_{k_1}, L_{k_1}, u_{k_2}, L_{k_2}, \ldots, u_{k_1}, L_{k_1}, P_k, P_{k+1}, R_{k+1}, u_{k+1}, R_{k+1}, u_{k+2}, \ldots, R_{k+1}, u_{k+1}\}k = 1, m$

$S_r = \{u_{r_1}, R_{r_1}, u_{r_2}, R_{r_2}, u_{r_1}, P_r, P_{r+1}, R_{r+1}, u_{r+1}, R_{r+1}, u_{r+2}, \ldots, R_{r+1}, u_{r+1}\}r = 1, m$

$P = \{P_i, S_j, Q_j, R_k\}[1 \leq i, j \leq m - 1 \& r, k = 1, m]$ is a minimum path double cover of $G$

$\eta_{PD}(G) = m + 1 + m - 4 = 2m + 2$

Theorem 6.4.4

Let $G$ be a $C_m(QS_n)$ graph. Then $\eta_{PD}(G) = 2m$

Proof:

$V(G) = \{u_1, \ldots, u_m, L_{j_1}, \ldots, L_{j_n}, R_1, \ldots, R_m, C_1, \ldots, C_m\}, i = 1, 2, \ldots, n$

The path double cover of $G$ is as follows.

$P_i = \{u_{i_1}, L_{i_1}, u_{i_2}, L_{i_2}, \ldots, u_{i_1}, L_{i_1}, C_i, C_{i+1}, L_{i+1}, u_{i+1}, L_{i+1}, u_{i+2}, \ldots, L_{i+1}, u_{i+1}\}i = 1 \text{ to } m - 1$

$Q_j = \{u_{j_1}, R_{j_1}, u_{j_2}, R_{j_2}, u_{j_1}, C_j, C_{j+1}, R_{j+1}, u_{j+1}, R_{j+1}, u_{j+2}, \ldots, R_{j+1}, u_{j+1}\}j = 1 \text{ to } m - 1$

$U = \{u_{m_1}, L_{m_1}, u_{m_2}, L_{m_2}, \ldots, u_{m_1}, L_{m_1}, C_{m}, C_{1}, u_{1}, L_{1}, \ldots, u_{m}\}$

$W = \{u_{m_1}, R_{m_1}, L_{m_1}, u_{m_2}, L_{m_2}, \ldots, R_{m_1}, u_{m_1}, R_{m_1}, C_{m}, C_{1}, u_{1}, R_{1}, \ldots, u_{m}\}$

$P = \{P_i, Q_j, U, W\}[1 \leq i, j \leq m - 1]$ is a minimum path double cover of $G$
\[ \eta_{PD}(G) = m - 1 + m - 1 + 2 = 2m \]

**Theorem 6.4.5**

Let \( G \) be a ladder graph. Then \( \eta_{PD}(G) = 3 \)

**Proof:**

Let \( V(G) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\} \)

The path double cover of \( G \) is as follows.

\[
P_1 = \begin{cases} 
  y_1x_1x_2y_2x_3y_3x_4 \ldots y_{n-1}x_{n-1}x_ny_n, & \text{if } n \text{ is even} \\
  y_1x_1y_2x_2y_3x_3 \ldots y_{n-1}x_{n-1}y_{n-1}y_n, & \text{if } n \text{ is odd}
\end{cases}
\]

\[
P_2 = \begin{cases} 
  x_1y_1y_2x_2x_3y_3 \ldots y_{n-1}y_n, & \text{if } n \text{ is even} \\
  y_1x_1y_2x_2y_3x_3 \ldots y_{n-1}x_{n-1}y_{n-1}y_n, & \text{if } n \text{ is odd}
\end{cases}
\]

\[
P_3 = \begin{cases} 
  x_1x_2x_3 \ldots x_ny_n \ldots y_{n-1}y_n, & \text{if } n \text{ is even} \\
  x_nx_{n-1}x_{n-2} \ldots x_2x_1y_2 \ldots y_{n-1}y_n, & \text{if } n \text{ is odd}
\end{cases}
\]

\[ P = \{P_1, P_2, P_3\} \] is a minimum path double cover of \( G \)

\( \eta_{PD}(G) = \Delta = 3 \) is a minimum path covering number of \( G \).

**Theorem 6.4.6**

Let \( G \) be a fan graph with \( n+1 \) vertices. Then \( \eta_{PD}(G) = n \)

**Proof:**

Let \( V(G) = \{v_0, v_1, \ldots, v_n\} \) Where \( v_0 \) is adjacent to \( v_2, \ldots, v_n \)

The path double cover of \( G \) is as follows.

Let \( V(G) = \{v_0, v_1, \ldots, v_n\} \) Where \( v_0 \) is adjacent to \( v_2, \ldots, v_n \)

The path double cover of \( G \) is as follows.

\[ P_1 = \{v_0, v_1, v_2, v_3, \ldots, v_n\} \]
\( P_2 = \{v_2, v_3, \ldots, v_n, v_0\} \)

\( P_3 = \{v_2, v_1, v_0, v_n\} \)

\( P_4 = \{v_2, v_0, v_{n-1}\} \)

\( Q_i = \{v_i, v_0, v_{i+1}\} / i = 2, 3, \ldots, n-2 \)

\( P = \{P_1, P_2, P_3, P_4, Q_i\}[i = 2, 3, \ldots, n-2] \) is a minimum path double cover of \( G \)

\( \eta_{PD}(G) = n + 1 \) is a minimum path covering number of \( G \).

**Theorem 6.4.7**

Let \( G \) be a mobius graph. Then \( \eta_{PD}(G) = 4 \)

**Proof:**

Let \( V(G) = \{i_1, i_2, \ldots, i_n, e_1, e_2, \ldots, e_n\} \)

The path double cover of \( G \) is as follows.

\( P_1 = \{i_1, i_2, \ldots, i_{n-1}, i_n, e_n, e_{n-1}, \ldots, e_2, e_1\} \)

\( P_2 = \{i_n, e_1, i_1, i_2, e_2, i_3, I_4, \ldots, I_{n-2}, I_{n-1}, \ldots, e_{n-1}, e_n\} \)

\( P_3 = \{e_n, i_1, e_1, i_2, i_3, \ldots, e_{n-2}, e_{n-1}, i_{n-1}, i_n\} \)

\( P_4 = \{e_1, i_n, e_n, i_1\} \)

\( P = \{P_1, P_2, P_3, P_4\} \) is a minimum path double cover of \( G \)

\( \eta_{PD}(G) = 4 \)

**Theorem 6.4.8**

Let \( G \) be a shell graph with \( n + 1 \) vertices. Then \( \eta_{PD}(G) = \begin{cases} n, \text{if } n \text{ is even} \\ n + 1, \text{if } n \text{ is odd} \end{cases} \)
Proof:

Let \( V(G) = \{v_0, v_1, v_2, \ldots, v_n\} \)

Case 1: \( n \) is even

The path double cover of \( G \) is as follows.

\[ P_1 = \{v_0, v_1, v_2, \ldots, v_{n-1}, v_n\} \]
\[ P_2 = \{v_{n-1}, v_n, v_0, v_1, v_2, \ldots, v_{n-1}\} \]
\[ P_3 = \{v_{n-1}, v_n, v_0, v_2\} \]
\[ Q_i = \{v_i, v_0, v_{i+1}\} / i = 2, 3, \ldots, n - 2 \]

\( P = \{P_1, P_2, P_3, Q_i\}[i = 2, 3, \ldots, n - 2] \) is a minimum path double cover of \( G \)

\[ \therefore \eta_{PD}(G) = n \]

Case 1: \( n \) is odd

The path double cover of \( G \) is as follows.

\[ P_1 = \{v_0, v_1, v_2, \ldots, v_{n-1}, v_n\} \]
\[ P_2 = \{v_{n-1}, v_n, v_0, v_1, v_2, \ldots, v_{n-1}\} \]
\[ P_3 = \{v_{n-1}, v_n, v_0, v_2\} \]
\[ P_4 = \{v_0, v_{n-1}\} \]
\[ Q_i = \{v_i, v_0, v_{i+1}\} / i = 2, 3, \ldots, n - 2 \]

\( P = \{P_1, P_2, P_3, P_4, Q_i\}[i = 2, 3, \ldots, n - 2] \) is a minimum path double cover of \( G \)

\[ \therefore \eta_{PD}(G) = n + 1 \]

Theorem 6.4.9

Let \( G \) be a gear graph with \( 2n + 1 \) vertices. Then \( \eta_{PD}(G) = n + 2 \)
Proof:

Let $V(G) = \{v_0, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$

The path double cover of $G$ is as follows.

$P_1 = \{v_0, v_1, v_2, w_2, v_3, w_3, \ldots, v_n, w_n\}$

$P_2 = \{v_0, v_1, v_n, v_n, v_n-1, v_n-1, \ldots, v_2, w_1\}$

$P_3 = \{w_n, v_1, w_1\}$

$P_4 = \{v_n, v_0, v_2\}$

$Q_i = \{v_i, v_0, v_{i+1}\} / i = 2, 3, \ldots, n-1$

$P = \{P_1, P_2, P_3, P_4, Q_i\} [i = 2, 3, \ldots, n-1]$ is a minimum path double cover of $G$

$\therefore \eta_{pd}(G) = n - 2 + 4 = n + 2$

Theorem 6.4.10

Let $G$ be a web graph with $3n + 1$ vertices. Then $\eta_{pd}(G) = n + 2$

Proof:

Let $V(G) = \{v_0, v_1, v_{12}, v_{21}, v_{22}, v_3, v_{13}, v_{23}, \ldots, v_n, v_{n1}, v_{n2}\}$

Her $v_1, v_2, \ldots, v_n$ are pendant vertices.

$v_{i1}$ is adjacent to $v_i$ and $v_{i2}$

The path double cover of $G$ is as follows.

$P_1 = \{v_0, v_{12}, v_{n2}, v_{n-12}, \ldots, v_{32}, v_{22}, v_{21}, v_{13}, v_{n1}, v_{n-11}, v_{n-21}, \ldots, v_{41}, v_{31}, v_{3}\}$

$P_2 = \{v_0, v_{22}, v_{12}, v_{n2}, v_{n-12}, \ldots, v_{42}, v_{32}, v_{31}, v_{21}, v_{11}, v_{n1}, v_{n-11}, v_{n-21}, \ldots, v_{41}, v_{4}\}$

$P_3 = \{v_0, v_{32}, v_{22}, v_{12}, v_{11}, v_{1}\}$

$P_4 = \{v_0, v_{42}, v_{41}, v_{31}, v_{21}, v_{2}\}$
\[ P_5 = \{v_{11}, v_{12}, v_{02}, v_{01}, v_n \} \]
\[ P_6 = \{v_{21}, v_{22}, v_0, v_{31}, v_3 \} \]
\[ Q_i = \{v_i, v_{i+2}, v_0, v_{i+1}, v_{i+3}, v_{i+5}\} i = 4, 5, \ldots, n-1 \]
\[ P = \{P_1, \ldots, P_6, Q_i\} [i = 4, 5, \ldots, n-1] \text{ is a minimum path double cover of } G \]
\[ \eta_{PD} = 6 + n - 4 = n + 2 \]

**Theorem 6.4.11**

Let \( G \) be a t-ply graph. Then \( \eta_{PD}(G) = 2m - 1 \)

**Proof:**

Let \( V(G) = \{x, v_{i1}, v_{i2}, \ldots, v_{im}, y\} \quad i = 1, 2, \ldots, m \)

The path double cover of \( G \) is as follows.

\[ P_i = \{x, v_{i1}, v_{i2}, v_{i3}, \ldots, v_{in}, y, v_{i+1,1}, v_{i+1,2}, \ldots, v_{i+1,m}, 1\} / i = 1, 2, \ldots, m-1 \]

\[ Q_j = \{v_{i1}, x, v_{i+1,2}\}, j = 1, 3, \ldots, m-2 \]

\[ R = \{x, v_{m1}, v_{m2}, \ldots, v_{m1}, v_{m2}, \ldots, v_{11}\} \]

\[ P = \{P_i, Q_j, R\} [1 \leq i \leq m-1, j = 1, 3, \ldots, m-2] \text{ is a minimum path double cover of } G \]

\[ \therefore \eta_{PD}(G) = 2m - 1 \]

**Theorem 6.4.12**

Let \( G \) be a helm graph \( 2n+1 \) vertices. Then \( \eta_{PD}(G) = n + 4 \)

**Proof:**

Let \( V(G) = \{v_0, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\} \)

\( v_i \) is adjacent to \( w_j \) and \( v_0 \)
The path double cover of $G$ is as follows.

$P_1 = \{w_i, v_i, v_{i+1}, v_{i+2}, \ldots, v_n, w_n\}$

$P_2 = \{w_n, v_n, v_{n-1}, v_{n-2}, \ldots, v_1, w_1\}$

$P_3 = \{w_1, v_1, v_n, v_{n-1}, w_{n-1}\}$

$P_4 = \{v_1, v_0, v_n\}$

$P_5 = \{v_1, v_0, v_{n-1}\}$

$P_6 = \{v_n, v_0, v_{n-1}\}$

$Q_i = \{w_{i+i}, v_{i+i}, v_0, v_{i+i+1}, v_{i+i+2}, \ldots, v_n\} \quad i = 2, 3, \ldots, n-2$

$R = \{w_2, v_2, v_0, v_{n-1}, w_{n-1}\}$

$P = \{P_1, P_2, P_3, \ldots, P_6, Q_i, R\} [2 \leq i \leq n-2]$ is a minimum path double cover of $G$

**Theorem 6.4.13**

Let $G$ be a multiple shell graph with $mn-m+1$ vertices. Then $\eta_{PD} = mn$

**Proof:**

Let $V(G) = \{v_0, v_1, v_2, \ldots, v_n\}$ $i = 1, 2, \ldots, m$

$P_i = \{v_{i+i}, v_{i+i+1}, v_{i+i+2}, \ldots, v_n, v_0, v_{i+i+1}, v_{i+i+2}, \ldots, v_{i+i+n}\} \quad i = 1, 2, \ldots, m-1$

$P = \{v_{i+i}, v_{i+i+1}, v_{i+i+2}, \ldots, v_n, v_0, v_{i+i+1}, v_{i+i+2}, \ldots, v_{i+i+n}\}$

$Q_i = \{v_{i+i}, v_0, v_{i+i+1}\} \quad i = 1, 2, \ldots, m$

Consider $G_i = G - \{P, P_i, Q_i\}$ is a tree with $m(n-2)$ pendant vertices.

$\eta_{PD}(G_i) = m(n-2)$

$\therefore \eta_{PD}(G) = m(n-2) + 2m = mn$
Theorem 6.4.14

Let $G$ be a mongolian tent graph with $mn+1$ vertices. Then $\eta_{PD} = mn$

Proof:

Let $V(G) = \{v_0, v_1, v_2, \ldots, v_m\} / i = 1, 2, \ldots, m$

Case 1: $m$ and $n$ odd

$P_1 = \{v_{11}, v_{12}, \ldots, v_{1n}, v_{2n}, \ldots, v_{11}, v_{21}, \ldots, v_{mn}, \ldots, v_{mn}\}$

$P_2 = \{v_{11}, v_{12}, \ldots, v_{mn}, v_{m2}, \ldots, v_{12}, v_{mn}, \ldots, v_{mn}\}$

$P_3 = \{v_{11}, v_{12}, v_{22}, \ldots, v_{m2}, v_{m3}, \ldots, v_{13}, v_{(n-1)}, \ldots, v_{mn}\}$

$P_4 = \{v_{11}, v_{12}, v_{12}, v_{2n}, v_{3n}, \ldots, v_{41}, \ldots, v_{4n}, \ldots, v_{(m-1)n}, \ldots, v_{mn}\}$

$Q_i = \{v_{ii}, v_{ii}, v_{jj+1}\} / i = 1, 2, \ldots, n-1$

$Q_n = \{v_{11}, v_{0}, v_{1n}\}$

$P = \{P_1, \ldots, P_4, Q_i\} / i = 1, 2, \ldots, n$ is a minimum path double cover of $G$

$\therefore \eta_{PD}(G) = n + 4$

Case 2: $m$ and $n$ even

$P_1 = \{v_{11}, v_{12}, \ldots, v_{1n}, v_{2n}, \ldots, v_{11}, v_{21}, \ldots, v_{mn}, \ldots, v_{ml}\}$

$P_2 = \{v_{11}, v_{12}, \ldots, v_{ml}, v_{m2}, \ldots, v_{12}, v_{mn}, \ldots, v_{1n}\}$

$P_3 = \{v_{11}, v_{12}, v_{22}, \ldots, v_{m2}, v_{m3}, \ldots, v_{13}, v_{(n-1)}, \ldots, v_{1n}\}$

$P_4 = \{v_{11}, v_{12}, v_{22}, \ldots, v_{2n}, v_{3n}, \ldots, v_{31}, \ldots, v_{(m-1)n}, \ldots, v_{ml}\}$

$Q_i = \{v_{ii}, v_{0}, v_{jj+1}\} / i = 1, 2, \ldots, n-1$

$Q_n = \{v_{11}, v_{0}, v_{1n}\}$
\[ P = \{P_1, \ldots, P_4, Q_i\} \] is a minimum path double cover of \( G \)

\[ \therefore \eta_{PD}(G) = n + 4 \]

**Case 3:** \( m \) odd and \( n \) even

\[ P_1 = \{v_{1i}, v_{12}, \ldots, v_{1n}, v_{21}, v_{23}, \ldots, v_{3n}, \ldots, v_{m1}, \ldots, v_{mn}\} \]

\[ P_2 = \{v_{1i}, v_{21}, \ldots, v_{m1}, v_{m2}, \ldots, v_{2n}, \ldots, v_{mn}\} \]

\[ P_3 = \{v_{1i}, v_{12}, v_{22}, v_{23}, \ldots, v_{m2}, v_{m3}, \ldots, v_{3n}, v_{(n-1)\ell}, \ldots, v_{mn}\} \]

\[ P_4 = \{v_{1i}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{(n-1)\ell}, \ldots, v_{mn}\} \]

\[ Q_i = \{v_{1i, v_0}, v_{1i, 1}\} / i = 1, 2, \ldots, n - 1 \]

\[ Q_n = \{v_{1i, v_0}, v_{1i, n}\} \]

\[ P = \{P_1, \ldots, P_4, Q_i\} \] is a minimum path double cover of \( G \)

\[ \therefore \eta_{PD}(G) = n + 4 \]

**Case 4:** \( m \) even and \( n \) odd

\[ P_1 = \{v_{1i}, v_{12}, \ldots, v_{1n}, v_{21}, v_{23}, \ldots, v_{3n}, \ldots, v_{m1}, \ldots, v_{mn}\} \]

\[ P_2 = \{v_{1i}, v_{21}, \ldots, v_{m1}, v_{m2}, \ldots, v_{2n}, \ldots, v_{mn}\} \]

\[ P_3 = \{v_{1i}, v_{12}, v_{22}, v_{23}, \ldots, v_{m2}, v_{m3}, \ldots, v_{3n}, v_{(n-1)\ell}, \ldots, v_{mn}\} \]

\[ P_4 = \{v_{1i}, v_{21}, v_{22}, \ldots, v_{2n}, v_{3n}, \ldots, v_{(n-1)\ell}, \ldots, v_{mn}\} \]

\[ Q_i = \{v_{1i, v_0}, v_{1i, 1}\} / i = 1, 2, \ldots, n - 1 \]

\[ Q_n = \{v_{1i, v_0}, v_{1i, n}\} \]

\[ P = \{P_1, \ldots, P_4, Q_i\} \] is a minimum path double cover of \( G \)

\[ \therefore \eta_{PD}(G) = n + 4 \]
Theorem 6.4.15

Let $G$ be a book graph with $2n+2$ vertices ($n$ even). Then $\eta_{pd}(G) = \frac{3n+2}{2}$

Proof:

Let $V(G) = \{v_1, v_2, v_{i1}, v_{i2}\}, i = 1, 2, \ldots, n$

Where $v_{i1}$ is adjacent to $v_1$ and $v_{i2}$ is adjacent to $v_2$

The path double cover of $G$ is as follows

$P_1 = \{v_{i1}, v_{i2}, v_2, v_{1}, v_{21}, v_{22}\}$

$P_2 = \{v_{i2}, v_{i1}, v_1, v_{22}, v_2, v_{21}\}$

$P_3 = \{v_{i2}, v_2, v_{22}\}$

$P_4 = \{v_{i1}, v_1, v_{21}\}$

$Q_i = \{v_{i1}, v_{i2}, v_2, v_{i+12}, v_{i+1,1}\}, i = 3, 5, 7, \ldots, n$

$R_j = \{v_{j1}, v_{j2}, v_2, v_{j+12}, v_{j+1,1}\}, j = 3, 5, 7, \ldots, n-1$

$S_k = \{v_{k1}, v_1, v_{k+1,1}\}[k = 3, 5, \ldots, n]$

$P = \{P_1, \ldots, P_4, Q_i, R_j, S_k\}[i, k = 3, 5, \ldots, n \& j = 3, 5, \ldots, n-1]$ is a minimum path double cover of $G$

$\therefore \eta_{pd} = 4 + 3\left(\frac{n}{2} - 1\right) = \frac{3n+2}{2}$

Theorem 6.4.16

Let $G$ be a double wheel graph with $2n+1$ vertices. Then $\eta_{pd} = \frac{3n+2}{2}$

Proof:

Let $V(G) = \{v_{i1}, v_{i2}, v_{in}, v_0, v_{21}, v_{22}, v_{2n}\}$
The Path double Cover of $G$ is as follows

$$P_1 = \{v_{i1}, v_{i2}, \ldots, v_{in}, v_0, v_{21}, v_{22}, \ldots, v_{2n}\}$$

$$P_2 = \{v_{i1}, v_{i2}, v_{2n-1}, \ldots, v_{2n}, v_0, v_{n1}, v_{n2}, \ldots, v_{12}\}$$

$$P_3 = \{v_{2n}, v_{21}, v_{22}, v_0, v_{n1}, v_{n2}\}$$

$$P_4 = \{v_{21}, v_0, v_{12}\}$$

$$Q_i = \{v_{i1}, v_0, v_{i+1}\} / i = 2, 3, \ldots, n-2$$

$$Q_i = \{v_{i2}, v_0, v_{i,n-1}\}$$

$$R_j = \{v_{j1}, v_0, v_{j+1}\} / j = 3, 4, \ldots, n-1$$

$$R_n = \{v_{2n}, v_0, v_{2n}\}$$

$$P = \{P_1, \ldots, P_4, Q_i, R_j\} / i = 1, 2, \ldots, n-2 \& j = 3, 4, \ldots, n$$ is a path double cover of $G$

$$\Rightarrow \eta_{PD}(G) \leq 2n$$

Since $\eta_{PD}(G) \geq \Delta = 2n$

$$\therefore \eta_{PD}(G) = \Delta = 2n$$

6.5 Conclusion

This Chapter is focused on finding the parameter path double covering number $\eta_{PD}$ of bicyclic graphs containing $U(l, m), D(l, m, i), C_m(l, i)$ and for various types of graphs like Triangular snake, Double Triangular snake, alternate double Triangular snake, Triple Triangular snake, Web graph, Gear graph, Double wheel, Triangular Cactus, Helm graph, Mobious Ladder, Flower graph, Mongolian Tent, $P_m \times P_n$, Shell graph, Multiple shell graph, Ladder graph, Book graph, t-ply, Fan graph, $P_m(Q_n)$
and $C_m(Q_n)$. Similarly we can find the geodesic path double covers for the above graphs. Also we post the following open problems related to this chapter:

(i) In general $\eta_{PD} \leq 2\pi$ where $\pi$ is the path partition number characterize the class of graphs for which $\eta_{PD} = 2\pi$
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