Chapter 4

Some Applications of Nice Submodules

Go down deep enough into anything and you will find mathematics.

-Dean Schlicter

4.1. Introduction

Naji [46] introduced the concepts of summable and $\sigma$-summable modules. Here we generalize these concepts to weakly summable and weakly $n$-summable modules. A $QTAG$-module $M$ of countable length is weakly $n$-summable if $H^n(M) = \bigcup_{i<\omega} M_i$, $M_i \subseteq M_{i+1} \subseteq H^n(M)$ and the heights of the elements of $M$ assume only finitely many values. If $n = 1$, these modules are summable. In fact $n$-summable modules are summable but the converse is not true in general. In section two we investigate $n$-summable and $n$-layered modules in terms of nice submodules which are very significant in the study. Section three deals with $\alpha$-modules. We also study pillared and strongly pillared $QTAG$-modules. Naji [46] defined $HT$-modules and here we extend this study in section four, find an easy equivalent definition of $HT$-modules and obtain interesting results. Last section is devoted to the study of $\omega_1$-separable and weakly $\omega_1$-separable modules. We find that the property of being $\omega_1$-separable or weakly $\omega_1$-separable module is shared by the finite direct sums of the modules with the same property. Direct summands also inherit these properties.

4.2 Weakly $n$-Summable and $n$-Layered $QTAG$-Modules

We start with the following:

**Definition 4.2.1.** A $QTAG$-module $M$ is weakly $n$-summable if $H^n(M) = \bigcup_{i<\omega} M_i$, $M_i \subseteq M_{i+1} \subseteq H^n(M)$ and the heights of the elements of $M_i$’s assume
only finitely many values. When \( n = 1 \), then \( M \) is summable if the length of \( M \) is countable.

The nice submodules are significant in the study of \( QTAG \)-modules. The following result emphasise this claim.

**Theorem 4.2.2.** Suppose the \( QTAG \)-module \( M \) has a nice submodule \( N \) of countable length such that \( M/N \) is a weakly \( n \)-summable module of countable length. Then \( M \) is a weakly \( n \)-summable module if and only if \( H^n(\sigma) = \bigcup_{i<\omega} N_i \subseteq N_{i+1} \subseteq H^n(M) \) and all \( N_i \)'s are height-finite in \( M \).

**Proof.** If \( H^n(M) = \bigcup_{i<\omega} M_i \) and all \( M_i \)'s are height-finite in \( M \), then \( H^n(M) = \bigcup_{i<\omega} N_i \) if \( N_i = M_i \cap N \). Now \( N_i \subseteq N_{i+1} \subseteq H^n(M) \) and all \( N_i \)'s are height-finite in \( M \).

Conversely, suppose \( M/N \) is of countable length. Now there exists an ordinal \( \sigma \) such that \( H_\sigma(M/N) = (H_\sigma(M) + N)/N = 0 \), i.e., \( H_\sigma(M) \subseteq N \). Again \( N \) is of countable length and there exists some ordinal \( \rho \) such that \( H_\rho(N) = 0 \). Thus \( H_{\rho+\sigma}(M) = 0 \). Since \( \rho + \sigma \) is again countable \( M \) is of countable length. Now, \( H^n(M/N) = \bigcup_{i<\omega} (K_i/N) \) where \( K_i \subseteq K_{i+1} \subseteq M \), \( H_n(K_i) \subseteq N \) and for every \( i \), \( (K_i/N) \)'s are height-finite in \( M/N \). Therefore there exist ordinals \( \mu_1, \ldots, \mu_i \) such that the nonzero elements of \( K_i/N \) are contained in \( [(H_{\mu_1}(M/N) \setminus H_{\mu_1+1}(M/N)) \cup \ldots \cup (H_{\mu_i}(M) \setminus H_{\mu_i+1}(M))] \) and we may write

\[
N_i \setminus \{0\} \subseteq [(H_{\lambda_1}(M) \setminus H_{\lambda_1+1}(M)) \cup \ldots \cup (H_{\lambda_i}(M) \setminus H_{\lambda_i+1}(M))]
\]

for some ordinals \( \lambda_1, \lambda_2, \ldots, \lambda_i \). Since \( \left( \frac{H^n(M) + N}{N} \right) \subseteq H^n \left( \frac{M}{N} \right) \), \( H^n(M) = \bigcup_{i<\omega} H^n(K_i) = \bigcup_{i<\omega} M_i \), where \( M_i = H^n(K_i) \).

Now we choose an ascending chain of submodules \( \{T_i\}_{i<\omega} \) of \( H^n(M) \) such that \( T_i \cap N = 0 \) and \( (T_i \oplus N)/N = (K_i/N) \cap (H^n(M) + N)/N \). Therefore \( T_i = H^n(K_i) + N \) because \( T_i \subseteq H^n(K_i) \). Since \( H^n(N) = \bigcup_{i<\omega} N_i \), \( N_i \subseteq N_{i+1} \subseteq H^n(N) \), we have to show that \( H^n(M) = (T_i \oplus N_i) \). For \( x \in H^n(M) \), \( x + N \in (H^n(M) + N)/N \subseteq H^n(M/N) \). Thus \( x + N \in (K_\ell/N) \cap ((H^n(M) + N)/N) \) for some \( \ell \) and we have \( x + N \subseteq T_\ell + N \). Therefore \( x \in T_\ell + H^n(N) \).
or \( x \in T_k \oplus N_k \) for some index \( k \) and \( H^n(M) = \bigcup_{i<\omega} (T_i \oplus N_i) \).

Since \( N \) is nice in \( M \), we have \(((T_i \oplus N)/N)\{N\} \subseteq (K_i/N)\{N\}\subseteq \frac{H_{\mu_1}(M) + N}{N} \cup \ldots \cup \frac{H_{\mu_i}(M) + N}{N} \cup \ldots \cup \frac{H_{\mu_{i+1}}(M) + N}{N} \).

Therefore
\[
(T_i \oplus N) \setminus N \subseteq [(H_{\mu_1}(M) + N) \setminus (H_{\mu_{i+1}}(M) + N)] \cup \ldots \cup [(H_{\mu_i}(M) \setminus (H_{\mu_{i+1}}(M)) + N] = (H_{\mu_i}(M) \setminus (H_{\mu_{i+1}}(M)) \cup \ldots \cup (H_{\mu_i}(M) \setminus (H_{\mu_{i+1}}(M)) + N).
\]

Also \( T_i \setminus N = T_i \setminus \{0\} \subseteq [(H_{\mu_1}(M) \setminus (H_{\mu_{i+1}}(M)) \cup \ldots \cup (H_{\mu_i}(M) \setminus (H_{\mu_{i+1}}(M)) + N]. \)

Since \( H^n(M) + N) = H^n(M) + H^n(N) \) for each ordinal \( \alpha \), we have
\[
T_i \setminus N \subseteq [(H_{\mu_i}(M) \setminus (H_{\mu_{i+1}}(M)) \cup \ldots \cup (H_{\mu_i}(M) \setminus (H_{\mu_{i+1}}(M)) + H^n(N).
\]

Now we select an ascending chain of submodules \( \{Q_i\}_{i<\omega} \) of \( H^n(M) \) such that \( Q_i \subseteq T_i \), for every \( i \), with \( \bigcup_{i<\omega} Q_i = \bigcup_{i<\omega} T_i \), and \( Q_i \setminus \{0\} \subseteq [(H_{\mu_1}(M) \setminus (H_{\mu_{i+1}}(M)) \cup \ldots \cup (H_{\mu_i}(M) \setminus (H_{\mu_{i+1}}(M)) + N_i \) with \((Q_i \oplus N_i + H_{\beta+1}(M)) \cap H^n(N) \subseteq N_i \) for every ordinal \( \beta \neq \mu_i, \lambda_i \). Therefore
\[
H^n(M) = \bigcup_{i<\omega} (Q_i + N_i) \text{ with } Q_i + N_i \subseteq Q+i+ N_i+1.
\]

Let \( x \in Q_i + N_i \) such that \( x = u_i + v_i, \ u_i \in Q_i, \ v_i \in N_i \). Thus \( u_i \in Q_i \setminus \{0\} \subseteq T_i \setminus \{0\} \subseteq (H_{\mu_1}(M) \setminus (H_{\mu_{i+1}}(M)) \cup \ldots \cup (H_{\mu_i}(M) \setminus (H_{\mu_{i+1}}(M)) \) and
\[
v_i \in N_i \setminus \{0\} \subseteq [(H_{\lambda_1}(M) \setminus (H_{\lambda_{i+1}}(M)) \cup \ldots \cup (H_{\lambda_i}(M) \setminus (H_{\lambda_{i+1}}(M))].
\]

If \( H_M(u_i) \neq H_M(v_i) \) then \( H_M(x) = \min\{H_M(u_i), H_M(v_i)\} \in \{\mu_1, \ldots, \mu_i, \lambda_1, \ldots, \lambda_i\} \). Otherwise, if \( H_M(u_i) = H_M(v_i) \) then suppose on contrary that \((Q_i \oplus N_i) \cap (H_{\beta}(M) \setminus (H_{\beta+1}(M)) \) is non-empty for some \( \beta \neq \mu_1, \ldots, \mu_i, \lambda_1, \ldots, \lambda_i \). Now \( H_M(x) = \beta \) for some \( x \). Since \( x + N \in [(K_i/N)\{N\} \cap (H_{\beta}(M) + N)/N = [(K_i/N)\{N\} \cap H_{\beta}(M/N), \) \( x + N \) must belong to \( H_{\beta+1}(M/N) = (H_{\beta+1}(M) + N)/N \). Therefore \( x \in H^n(H_{\beta+1}(M) + N) = H^n(H_{\beta+1}(M)) + H^n(N) \) and so \( x = u_\beta + v, \) for some \( u_\beta \in H^n(H_{\beta+1}(M)) \) and some non-zero \( v \in H^n(N) \setminus \{0\} \). Thus \( x - u_\beta = v \in (Q_i \oplus N_i + H_{\beta+1}(M)) \cap H^n(N) \subseteq N_i \) and \( v \in N_i \) with \( H_M(v) = \beta \) because \( H_M(x) = \beta \) and \( H_M(u_\beta) \geq \beta + 1 > \beta \).
Therefore \((N_i \setminus \{0\}) \cap (H_\beta(M) \setminus H_{\beta+1}(M))\) is not empty which is a contradiction. We finally infer that \((Q_i \oplus N_i) \setminus \{0\} \subseteq (H_{\mu_i}(M) \setminus H_{\mu_i+1}(M)) \cup \ldots \cup (H_{\nu_i}(M) \setminus H_{\nu_i+1}(M)) \cup (H_{\lambda_i}(M) \setminus H_{\lambda_i+1}(M)) \cup \ldots \cup (H_{\lambda_i}(M) \setminus H_{\lambda_i+1}(M))\), whence they are height-finite in \(M\) as required.

An immediate consequence of the above result can be stated as follows:

**Corollary 4.2.3.** Let \(N\) be a nice and isotype submodule of a \(QTAG\)-module \(M\) such that \(M/N\) is a weakly \(n\)-summable module. Then \(M\) is a weakly \(n\)-summable module if and only if \(N\) is a weakly \(n\)-summable module.

Now we investigate \(n\)-layered \(QTAG\)-modules and we start with the following:

**Definition 4.2.4.** A \(QTAG\)-module \(M\) is said to be \(n\)-layered if \(H^n(M) = \bigcup_{i<\omega} M_i\), \(M_i \subseteq M_{i+1} \subseteq H^n(M)\) and all the elements of \(M\) assume a finite number of finite heights. Equivalently \(M_i \cap H_i(M) \subseteq H_\omega(M)\) for every \(i\).

For \(n = 1\), these modules are \(\Sigma\)-modules. All the \(n\)-layered \(QTAG\)-modules are \(\Sigma\)-modules but the converse is not true if \(n \geq 2\). Moreover, each weakly \(n\)-summable module is \(n\)-layered module and the converse is not true in general.

**Proposition 4.2.5.** For \(n \geq 1\), every \(n\)-layered module of length \(< \omega\) is weakly \(n\)-summable.

**Proof.** For a \(n\)-layered module \(M\), we may write \(H^n(M) = \bigcup_{i<\omega} M_i\), \(M_i \subseteq M_{i+1} \subseteq H^n(M)\) and for every \(i < \omega\), \(M_i \cap H_i(M) \subseteq H_\omega(M)\). Since the length of \(H_\omega(M) < \omega\), \(H_\omega(M)\) is bounded by some \(k \geq 1\) and \(H_{\omega+k}(M) = 0\). Therefore \(M_i\)'s have only finitely many height values in \(M\) i.e., \(M_i\)'s are height-finite and \(M\) is weakly \(n\)-summable.

**Theorem 4.2.6.** Let \(N\) be a nice submodule of the \(QTAG\)-module \(M\) such that \(N \cap H_{\omega+n}(M) = H_{\omega+n}(N)\) and \(M/N\), \(n\)-layered. Then \(M\) is \(n\)-layered.
if and only if \( H^n(N) = \bigcup_{i<\omega} N_i \), \( N_i \subseteq N_{i+1} \subseteq H^n(N) \) and for all \( i < \omega \), \( N \cap H_i(M) \subseteq H_\omega(M) \).

**Proof.** If \( M \) is \( n \)-layered module then \( H^n(M) = \bigcup_{i<\omega} M_i \), \( M_i \subseteq M_{i+1} \subseteq H^n(M) \) and for every \( i < \omega \), \( M_i \cap H_i(M) \subseteq H_\omega(M) \). Thus \( H^n(N) = \bigcup_{i<\omega} N_i \), where \( N_i = N \cap M_i \). Since \( N_i \subseteq N_{i+1} \subseteq H^n(N) \) and
\[
N_i \cap H_i(M) \subseteq M_i \cap H_i(M) \subseteq H_\omega(M)
\]
for every \( i < \omega \), we are done.

For the converse, we may write \( H^n(M/N) = \bigcup_{i<\omega} (K_i/N) \), where \( K_i \subseteq K_{i+1} \subseteq M \) with \( H_n(B_i) \subseteq N \) and for every \( i < \omega \), \( (K_i/N) \cap H_i(M/N) \subseteq H_\omega(M/N) \). Hence \( K_i \cap H_i(M) \subseteq H_\omega(M) + N \). Since \( H^n(M) + N \subseteq \bigcup_{i<\omega} (K_i/N) \) and for every \( i < \omega \), \( (K_i/N) \cap H_i(M) \subseteq H_\omega(M) + N \). Therefore \( x + N \in (H^n(M) + N)/N \subseteq H^n(M/N) \) and therefore \( x + N \in (K_\ell/N) \cap [(H^n(M) + N)/N] \) for some \( \ell \). This implies that \( x + N \subseteq T_\ell \oplus N \) and \( x \in T_\ell \oplus N \). Therefore \( x \in T_\ell \oplus H^n(N) \) or \( x \in T_j \oplus N_j \) for some \( j \) and \( H^n(M) = \bigcup_{i<\omega} (T_i \oplus N_i) \). Again we choose an ascending chain \( \{Q_i\}_{i<\omega} \) of submodules of \( H^n(M) \) such that for every \( i \), \( Q_i \subseteq T_i \) with \( \bigcup_{i<\omega} Q_i = \bigcup_{i<\omega} T_i \) and \( (Q_i \oplus N_i) \cap H_i(M) \subseteq H_\omega(M) + N_i \). The selection of \( Q_i \)'s is ensured because \( T_i \oplus N_i \subseteq T_i \oplus N \subseteq K_i \) with \( (T_i \oplus N_i) \cap H_i(M) \subseteq K_i \cap H_i(M) \subseteq H^n(H_\omega(M) + N) = H^n(H_\omega(M)) + H^n(N) \subseteq H_\omega(M) + H^n(N) \). Now \( H^n(M) = \bigcup_{i<\omega} (Q_i \oplus N_i) \) where \( (Q_i \oplus N_i) \subseteq (Q_{i+1} \oplus N_{i+1}) \). Finally we have to show that \( (Q_i \oplus N_i) \cap H_i(M) \subseteq H_\omega(M) \). We have
\[
(Q_i \oplus N_i) \cap H_i(M) \subseteq (H_\omega(M) + N_i) \cap H_i(M) = H_\omega(M) + (N_i \cap H_i(M)) = H_\omega(M).
\]
Since \( N_i \cap H_i(M) \subseteq H_\omega(M) \), we are done.
Now we state an immediate consequence of the above discussion.

**Corollary 4.2.7.** Let \( N \) be a nice and isotype submodule of \( M \) such that \( M/N \) is \( n \)-layered. Then \( M \) is a \( n \)-layered module if and only if \( N \) is a \( n \)-layered module.

### 4.3 \( \alpha \)-Modules

Naji [46] defined \( \alpha \)-modules and obtained some interesting results. A \( QTAG \)-module \( M \) is an \( \alpha \)-module if \( M/H_\beta(M) \) is totally projective for each \( \beta < \alpha \). Now we further develop the study of \( \alpha \)-modules. We prove some basic results.

**Lemma 4.3.1.** Direct sums and direct summands of the \( QTAG \)-modules having nice systems also have nice systems.

**Proof.** Consider the \( QTAG \)-module \( M = \bigoplus M_i \) such that each \( M_i \) is a \( QTAG \)-module having a nice system. Let \( \{N_{ij}\}_{j \in J} \) be a system of nice submodules of \( M_i \), \( i \in I \) such that

(i) \( \{0\} \in \{N_{ij}\} \);

(ii) for any subset \( \{N_{ik}\} \) in \( \{N_{ij}\} \), \( \Sigma N_{ik} \in \{N_{ij}\} \);

(iii) for any \( N_{ij} \in \{N_{ij}\} \) and a countable subset \( \{x_\ell\} \) of \( M \), there exists \( N_{ik} \in \{N_{ij}\} \) such that \( N_{ij} + \Sigma x_\ell R \subseteq N_{ik} \) and \( N_{ik}/N_{ij} \) is countably generated.

Consider the set \( B \) of all submodules of \( M \) which are of the form \( N = \bigoplus N_{ij} \) with \( N_{ij} \in \{N_{ij}\} \). Now each \( N \) is nice in \( M \), thus (i) and (ii) are satisfied for \( \{N_{ij}\} \). For a countable subset \( \{x_\ell\} \subseteq M \) and \( N \in B \), there exist a countable subset \( I' \) of \( I \) such that \( \Sigma x_\ell R \subseteq \bigoplus_{i \in I'} M_i \) and for each \( i \in I' \), there is a submodule \( N_{ik} \in \{N_{ij}\} \) such that \( N_{ij} \subseteq N_{ik} \) and \( g \left( \frac{N_{ik}}{N_{ij}} \right) \leq \omega \). Let \( M = M' \oplus M'' \) be direct sum of \( QTAG \)-modules such that \( M \) has a nice system \( \{N_i\} \) satisfying the properties (i), (ii) and (iii). We may define \( \{N'_i\} \) as a set of all submodules \( M' \) such that for some \( N_i \in \{N_i\} \), \( N_i = N'_i \oplus (N_i \cap M'') \) holds. Now \( N'_i \) is nice in \( M' \). Thus \( \{N'_i\} \) satisfying (i) and (ii). Consider a countable
subset \( \{x_i'\} \subseteq M' \) and \( N_i' \in \{N_i'\} \). Now \( N_i \) satisfies \( N_i = N_i' \oplus (N_i \cap M'') \). We may choose \( N_i \in \{N_i\} \) such that \( N_i + \Sigma x_i R \subseteq N_i \) and \( g\left( \frac{N_i}{N_i'} \right) \leq \omega \). There are submodules \( K_1, K_2 \) such that \( N_i' \subseteq K_i \subseteq M' \), \( N_i \cap M'' \subseteq K_2 \subseteq M'' \) with \( g\left( \frac{K_1}{N_i'} \right) \leq \omega \) and \( g\left( \frac{K_2}{N \cap M''} \right) \leq \omega \) satisfying \( N_i \subseteq K_1 \oplus K_2 \). We may select \( N_{\nu} \in \{N_i\} \) with \( N_i + K_i \subseteq N_{\nu} \) and \( g\left( \frac{N_{\nu}}{N_i} \right) \leq \omega \). On repeating the process we get chains of modules \( N_i \in \{N_i\} \), \( K_{ii} \subseteq M' \) and \( K_{2i} \subseteq M'' \) with \( N_{ij} \subseteq K_{1j} \oplus K_{2j} \), \( N_j + K_{1j} = N_{j+1} \) and \( g\left( \frac{N_{ij}}{N_i} \right) \leq \omega \). Therefore \( \cup N_{ij} \in \{N_i\} \) satisfies \( \cup N_{ij} = (\cup N_{ij} \cap M') \oplus (\cup N_{ij} \cap M'') \) and \( \cup N_{ij} \cap M' \in \{N_i'\} \). Also \( N_i' + \Sigma x_i R \subseteq \cup N_{ij} \cap M' \) and \( g\left( \frac{\cup N_{ij} \cap M'}{N_i'} \right) \leq g\left( \frac{\cup N_{ij} \cap M''}{N_i} \right) \leq \omega \).

**Lemma 4.3.2.** Let \( N \) be a nice and isotype submodule of a \( QTAG \)-module \( M \). Then for each ordinal \( \beta \), the following hold:

(i) \( (H_{\beta}(M) + N)/H_{\beta}(M) \) is nice and isotype in \( M/H_{\beta}(M) \);

(ii) \( H_{\beta}(N) \) is nice in \( M \);

(iii) \( N/H_{\beta}(N) \) is nice and isotype in \( M/H_{\beta}(N) \).

**Proof.** Since \( N \) is isotype in \( M \), for each ordinal \( \rho < \beta \), we have

\[
[ (H_{\beta}(M) + N)/H_{\beta}(M) ] \cap H_{\rho}(M/H_{\beta}(M)) = [ (H_{\beta}(M) + N)/H_{\beta}(M) ] \cap H_{\rho}(M)/H_{\beta}(M) = [ (H_{\beta}(M) + N) \cap H_{\rho}(M) ]/H_{\beta}(M) = (H_{\beta}(M) + (N \cap H_{\rho}(M))) / H_{\beta}(M) = (H_{\beta}(M) + H_{\rho}(N)) / H_{\beta}(M) \subseteq (H_{\beta}(M) + H_{\rho}(N + H_{\beta}(M))) / H_{\beta}(M) \subseteq H_{\rho}((H_{\beta}(M) + N)/H_{\beta}(M)),
\]

therefore \( (H_{\beta}(M) + N)/H_{\beta}(M) \) is isotype in \( M/H_{\beta}(M) \). In order to prove that \( (H_{\beta}(M) + N)/H_{\beta}(M) \) is nice in \( M/H_{\beta}(M) \), it is sufficient to prove that

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\( H_\beta(M) + N \) is nice in \( M \). For any limit ordinal \( \sigma \), there may be two cases. If \( \sigma \leq \beta \), then
\[
\bigcap_{\rho < \sigma} (H_\beta(M) + N + H_\rho(M)) = \bigcap_{\rho < \sigma} (N + H_\rho(M)) = N + H_\sigma(M) = H_\beta(M) + N + H_\sigma(M).
\]
If \( \sigma > \beta \), then
\[
\bigcap_{\rho < \sigma} (H_\beta(M) + N + H_\rho(M)) = [\bigcap_{\rho < \sigma} (H_\sigma(M) + N)] \cap [\bigcap_{\beta \leq \rho < \sigma} (H_\beta(M) + N)] = \bigcap_{\rho < \beta} ((H_\rho(M) + N) \cap (H_\beta(M) + N)) = H_\beta(M) + N = H_\beta(M) + N + H_\sigma(M)
\]
and we are done.
Since \( N \) is nice in \( M \), for each limit ordinal \( \sigma \),
\[
\bigcap_{\rho < \sigma} (N + H_\rho(M)) = N + H_\sigma(M).
\]
Therefore for each limit ordinal \( \sigma > \beta \), we have
\[
\bigcap_{\rho < \sigma} (H_\beta(N) + H_\rho(M)) = [\bigcap_{\rho < \beta} (H_\beta(N) + H_\rho(M))] \cap [\bigcap_{\beta \leq \rho < \sigma} (H_\beta(N) + H_\rho(M))] \subseteq (N + H_\sigma(M)) \cap H_\beta(M) = H_\sigma(M) + (N \cap H_\beta(M)) = H_\sigma(M) + H_\beta(N)
\]
as required.
For \( \sigma \leq \beta \) we have
\[
\bigcap_{\rho < \sigma} (H_\beta(N) + H_\rho(M)) = \bigcap_{\rho < \sigma} H_\rho(M) = H_\sigma(M) = H_\sigma(M) + H_\beta(N).
\]

Thus in both cases \( \bigcap_{\rho<\sigma} (H_\beta(N) + H_\rho(M)) = H_\sigma(M) + H_\beta(N) \) as required.

Since \( H_\beta(N) \) is always nice in \( N \) and \( N \) is nice in \( M \), \( H_\beta(N) \) is always nice in \( M \). Now \( N \) is nice in \( M \), due to [46] \( N/H_\beta(N) \) is nice in \( M/H_\beta(N) \). Now for every ordinal \( \sigma \),

\[
(N/H_\beta(N)) \cap H_\sigma(M/H_\beta(N)) = [N \cap (H_\sigma(M) + H_\beta(N))]/H_\beta(N)
\]

\[
= (H_\beta(N) + H_\sigma(N))/H_\beta(N)
\]

\[
= H_\sigma(N/H_\beta(N))
\]

and \( N/H_\beta(N) \) is isotype in \( M/H_\beta(N) \).

**Remark 4.3.3.** let \( N \) be any submodule of a \( QTAG \)-module \( M \). For every \( k < \omega \), \( H_k(M) + N \) is nice in \( M \). In fact for every limit ordinal \( \sigma \),

\[
\bigcap_{\rho<\sigma} (H_k(M) + N + H_\rho(M)) = \bigcap_{k<\rho<\sigma} (H_k(M) + N) = H_k(M) + N + H_\sigma(M).
\]

Now we are able to prove the following result.

**Theorem 4.3.4.** Let \( N \) be a nice and isotype submodule of a \( QTAG \)-module \( M \) and \( M/N \) an \( \alpha \)-module. Then \( M \) is an \( \alpha \)-module if and only if \( N \) is an \( \alpha \)-module.

**Proof.** We have \( (H_\beta(M) + N)/H_\beta(M) \cong N/(N \cap H_\beta(M)) = N/H_\beta(N) \) and

\[
(M/N) \cong \frac{M}{(H_\beta(M) + N)/N} \cong \frac{M}{(H_\beta(M) + N)}/H_\beta(M)
\]

is totally projective. By Lemma 4.3.2, \( (H_\beta(M) + N)/H_\beta(M) \) is nice and isotype in \( M/H_\beta(M) \), for all ordinals \( \beta \). Again \( \frac{M}{H_\beta(M)} \cong \frac{N}{H_\beta(N)} \oplus \frac{M/N}{H_\beta(M/N)} \). Therefore \( M/H_\beta(M) \) must be totally projective for all \( \beta < \alpha \), if \( N/H_\beta(M) \) is totally projective. Thus \( M \) is an \( \alpha \)-module.

Conversely, if \( M/H_\beta(M) \) is totally projective then by Lemma 4.3.1, \( N/H_\beta(N) \) is also totally projective being isomorphic to a summand of \( M/H_\beta(M) \).
A QTAG-module $M$ is said to be pillared module, if $M/H_\omega(M)$ is a direct sum of uniserial modules. We extend this concept to strongly pillared modules.

**Definition 4.3.5.** A submodule $N$ of a QTAG-module is **strong** if it is contained in $K \subseteq M$, where $K$ is a direct sum of uniserial modules. Moreover $N$ is strongly pillared in $M$ if $N/H_\omega(M)$ is strong in $M/H_\omega(N)$.

**Theorem 4.3.6.** Let $M$ be a QTAG-module with a nice submodule $N$ such that $M/N$ is pillared. If $N$ is strongly pillared in $M$, then $M$ is pillared. The converse holds if $N \cap H_\omega(M) = H_\omega(N)$. Moreover, if $N$ is $h$-pure in $M$, then $M$ is pillared if and only if $N$ is pillared.

**Proof.** Since $M/N$ is pillared, $\frac{(M/N)}{H_\omega(M/N)}$ is a direct sum of uniserial modules. Now we have

$$\frac{(M/N)}{H_\omega(M/N)} = \frac{(M/N)}{(H_\omega(M) + N)/N} \cong \frac{M}{H_\omega(M) + N} \cong \frac{(M/H_\omega(M))}{(H_\omega(M) + N)/H_\omega(M)}$$

is a direct sum of uniserial modules. Now $N/H_\omega(N) = \bigcup_{k<\omega} (N_k/H_\omega(N))$ such that $H_\omega(N) \subseteq N_k \subseteq N_{k+1}$ and for every $k$, $N_k \cap H_k(M) = H_\omega(N)$. Therefore $N = \bigcup_{k<\omega} N_k$ and $[(H_\omega(M) + N)/H_\omega(M)] = \bigcup_{k<\omega} [(H_\omega(M) + N_k)/H_\omega(M)]$. Moreover,

$$[(H_\omega(M) + N_k)/H_\omega(M)] \cap H_k(M/H_\omega(M))$$

$$= [(H_\omega(M) + N_k)/H_\omega(M)] \cap \frac{H_k(M)}{H_\omega(M)}$$

$$= [(H_\omega(M) + N_k) \cap H_k(M)]/H_\omega(M)$$

$$= [H_\omega(M) + (N_k \cap H_k(M))]/H_\omega(M)$$

$$= 0.$$ 

Therefore $(H_\omega(M) + N)/H_\omega(M)$ is a submodule of a direct sum of uniserial modules and $M/H_\omega(M)$ is a direct sum of uniserial modules.

Now assume that $M/H_\omega(M)$ is a direct sum of uniserial modules. Since $(N + H_\omega(M))/H_\omega(M)$ is a submodule of $M/H_\omega(M)$, it is strong in $M/H_\omega(M)$.
Therefore \((N + H_\omega(M))/H_\omega(M) = \bigcup_{i<\omega} (K_i/H_\omega(M))\), where
\[H_\omega(M) \subseteq K_i \subseteq K_{i+1} \subseteq N + H_\omega(M)\]
and \(K_i \cap H_i(M) = H_\omega(M)\) for each \(i\). Thus \(N = \bigcup_{i<\omega} (K_i \cap N)\).

Also \(\frac{N + H_\omega(M)}{H_\omega(M)} \simeq \frac{N}{N \cap H_\omega(M)} = \frac{N}{H_\omega(N)}\) and this isomorphism preserve heights such that \(N/H_\omega(N) = \bigcup_{i<\omega} [(K_i \cap N)/H_\omega(N)]\). Therefore
\[K_i \cap N \cap H_i(M) = H_\omega(M) \cap N = H_\omega(N)\]
and \(N/H_\omega(N)\) is strong in \(M/H_\omega(N)\) and the result follows.

Now we study \(h\)-reduced \(QTAG\)-modules whose countably generated submodules are the direct sum of uniserial modules.

**Definition 4.3.7.** A \(QTAG\)-module is said to be \(\aleph_1\)-\textit{U-module} if its countably generated submodules are the direct sum of uniserial modules. A submodule \(N \subseteq M\) is said to be \textit{strong} \(\aleph_1\)-\textit{U-module} in \(M\) if every countably generated submodule is contained in \(K \subseteq M\), where \(K\) is the direct sum of uniserial modules.

**Theorem 4.3.8.** Let \(M\) be a \(QTAG\)-module with a submodule \(N\) such that \(M/N\) is \(\aleph_1\)-\textit{U-module}. Then \(M\) is \(\aleph_1\)-\textit{U-module} if and only if \(N\) is strong \(\aleph_1\)-\textit{U-module}.

**Proof.** Consider a submodule \(K \subseteq M\) such that \(g(K) = \aleph_0\). Now, \(\frac{K + N}{N} \simeq \frac{K}{K \cap N}\) and there may be two cases:

(i) If \(\frac{K}{K \cap N}\) is finitely generated, then \(g(K) = g(K \cap N)\) and since \(N\) is a \(\aleph_1\)-\textit{U-module}, \(K \cap N\) is a direct sum of uniserial modules. Therefore by [45], \(K\) is a direct sum of uniserial modules.

(ii) Otherwise, if \(g\left(\frac{K}{K \cap N}\right)\) is countable and \(M/N\) is a \(\aleph_1\)-\textit{U-module}, then \(\frac{K}{K \cap N}\) is a direct sum of uniserial modules. If \(g(K \cap N)\) is countable then \(K \cap N\) is strong in \(M\) because \(N\) is a \(\aleph_1\)-\textit{U-module} in \(M\). Thus \(K \cap N\) is
strong in $K$. Otherwise, if $g(K \cap N)$ is finite then it may be embedded in a countably generated submodule of $N$ which is strong in $M$. Thus $K \cap N$ is strong in $M$, hence in $K$, and $K$ is a direct sum of uniserial modules as required. The converse is trivial.

**Corollary 4.3.9.** Let $M$ be a $QTAG$-module with a submodule $N$ such that $M/N$ is bounded. Then $M$ is $\aleph_1$-$U$-module if and only if $N$ is $\aleph_1$-$U$-module.

**Proof.** Since $M/N$ is bounded, $H_n(M) \subseteq N$ for some $n \in \mathbb{N}$, therefore $H_\omega(N) = H_\omega(M)$. Let $K$ be a submodule of $M$ such that $K$ is countably generated. Then $\frac{K + N}{N} \cong \frac{K}{K \cap N}$ is bounded. Since $K \cap N$ is a submodule of $N$, $K \cap N$ is separable, therefore $K$ is also separable. Thus $K$ is a direct sum of uniserial modules. The converse follows from the fact that submodules of $\aleph_1$-$U$-module are also $\aleph_1$-$U$-modules.

### 4.4 $HT$-Modules

Naji [46] defined $HT$-modules with the help of small homomorphisms and large submodules. Here we study these modules with a different but equivalent definition of $HT$-modules.

**Definition 4.4.1.** A $QTAG$-module $M$ is a $HT$-module if and only if there exists some $k \in \mathbb{N}$, such that $Soc(H_k(M)) \subseteq K \subseteq M$ if $M/K$ is a direct sum of uniserial modules. Moreover, a submodule $N \subseteq M$ is strongly $HT$ in $M$ if there exists some $k \in \mathbb{N}$ such that $Soc(H_k(M)) \subseteq K$ for $K \subseteq N$ with $N/K$ a direct sum of uniserial modules.

**Proposition 4.4.2.** Let $N$ be a $h$-pure submodule of a $HT$-module $M$. Then $M/N$ is also a $HT$-module.

**Proof.** Let $K/N$ be a submodule of $M/N$ such that $\frac{(M/N)}{(K/N)} \cong \frac{M}{N}$ is a direct sum of uniserial modules. Therefore there exists some $k \in \mathbb{N}$ such that $Soc(H_k(M)) \subseteq K$. Now $(Soc(H_k(M)) + N) \subseteq K$ and $\frac{[Soc(H_k(M)) + N]}{N} \subseteq$
Since $N$ is $h$-pure in $M$, $\operatorname{Soc}(H_k(M) + N) = \operatorname{Soc}(H_k(M)) + \operatorname{Soc}(N)$ and 
$$\frac{[\operatorname{Soc}(H_k(M) + N) + N]}{N} \subseteq \frac{K}{N}$$. $N$ is also $h$-pure in $H_k(M) + N \subseteq M$, therefore
$$\operatorname{Soc}(H_k(M/N)) = \operatorname{Soc}([H_k(M) + N]/N) = [\operatorname{Soc}(H_k(M) + N) + N]/N.$$ and $\operatorname{Soc}(H_k(M/N)) \subseteq K/N$. Therefore $M/N$ is a $HT$-module.

**Proposition 4.4.3.** Let $N$ be a $h$-pure submodule of a $QTAG$-module $M$ such that $M/N$ is a direct sum of uniserial modules. Then the following conditions are equivalent:

(i) $M$ is a $HT$-module;

(ii) $N$ is strongly $HT$-module in $M$;

(iii) $N$ is a $HT$-module and $M/N$ is bounded.

**Proof.** (i) $\Rightarrow$ (ii) Let $N/K$ be a direct sum of uniserial modules for an arbitrary submodule $K$ of $N$. Now $N/K$ is $h$-pure in $M/K$, thus $N/K$ is strong in $M/K$ and \( \frac{(M/K)}{(N/K)} \simeq \frac{M}{N} \) is also a direct sum of uniserial modules.

Therefore $M/K$ is a direct sum of uniserial modules and so \( \frac{N}{K} \oplus \frac{M}{N} \simeq \frac{M}{K} \). The structure of $M$ ensures the existence of some natural number $k$ such that $\operatorname{Soc}(H_k(M)) \subseteq K$. Therefore $\operatorname{Soc}(H_k(N)) \subseteq K$ and we are done.

(ii) $\Rightarrow$ (i) Let $K$ be an arbitrary but fixed submodule of $M$ such that $M/K$ is a direct sum of uniserial modules. Then \( \frac{(N + K)}{K} \simeq \frac{N}{N \cap K} \) is also a direct sum of uniserial modules. The structure of $N$ ensures the existence of a natural number $k$ such that $\operatorname{Soc}(H_k(N)) \subseteq N \cap K \subseteq K$ and the result follows.

(iii) $\Leftrightarrow$ (ii) By Proposition 4.2.2, if $M$ is a $HT$-module then $M/N$ is a $HT$-module. As $M/N$ is a direct sum of uniserial modules, it is bounded and there exists some $k \in \mathbb{N}$ such that $H_k(M) \subseteq N$ and $N$ is strongly $HT$-module in $M$. Now we have to show that if $N$ is strongly $HT$-module in $M$ and $N$ is $h$-pure in $M$, then $M/N$ should be bounded. Now there exists a number $k \in \mathbb{N}$ such that $\operatorname{Soc}(H_k(M)) \subseteq N$. Since $N$ is $h$-pure in $M$,
$Soc(H_k(M)) = Soc(H_k(N))$. Again $H_k(N)$ is $h$-pure in $H_k(M)$, therefore $H_k(M) = H_k(N)$, hence $H_k(A) \subseteq N$ and $M/N$ is bounded.

**Proposition 4.4.4.** Finite direct sum of $HT$-modules is a $HT$-module.

**Proof.** We shall prove that direct sum of two $HT$-modules is a $HT$-module. Let $M$ be a $QTAG$-module such that $M = M' \oplus M''$, where $M'$ and $M''$ are $HT$-modules. Let $N$ be a submodule of $M$ such that $M/N$ is a direct sum of uniserial modules. Now $\frac{(M' + N)}{N} \cong \frac{M'}{M' \cap N}$ and $\frac{(M'' + N)}{N} \cong \frac{M''}{M'' \cap N}$ are submodules of $M/N$, therefore they are the direct sums of uniserial modules. Now there are integers $m$ and $k$ such that $H_k(M') \subseteq (M' \cap N) \subseteq N$ and $H_m(M'') \subseteq (M'' \cap N) \subseteq N$. If $n = \max(m, k)$, then

$$Soc(H_n(M)) = Soc(H_n(M')) \oplus Soc(H_n(M''))$$

$$\subseteq Soc(H_k(M')) \oplus Soc(H_m(M''))$$

$$\subseteq K$$

and the result follows.

**Proposition 4.4.5.** A direct summand of a $HT$-module $M$ is a $HT$-module.

**Proof.** Let $M$ be a $QTAG$ and $HT$-module. Let $M'$ be a direct summand of $M$ such that $M = M' \oplus M''$. We have to show that there exists a positive integer $k$ such that $Soc(H_k(M')) \subseteq N$ whenever $M'/N$ is a direct sum of uniserial modules. If $M'/N$ is a direct sum of uniserial modules then

$$\frac{M}{N \oplus M''} = \frac{M' \oplus M''}{N \oplus M''} \cong \frac{M'}{N \oplus M''}.$$ 

Therefore there exists an integer $k$ such that $Soc(H_k(M)) \subseteq (N \oplus M'')$ and

$$Soc(H_k(M')) \subseteq (N \oplus M'') \cap M' = N \oplus (M'' \cap M') = N$$

which is required.
Now we investigate those $QTAG$-modules which do not have an unbounded direct summand which is a direct sum of uniserial modules.

**Definition 4.4.6.** A $QTAG$-module $M$ is said to be $HE$-module if it has no unbounded direct summand which is a direct sum of uniserial modules.

**Proposition 4.4.7.** Let $M$ be a $HE$-module and $N$ a $h$-pure submodule of $M$. Then $M/N$ is a $HE$-module.

**Proof.** Suppose $M/N$ has a direct summand of $K/N$, which is a direct sum of uniserial modules. Therefore $(M/N) = (T/N) \oplus (K/N)$, $T \oplus K = M$ and $K \cap T = N$. Since $N$ is $h$-pure in $M$, it is $h$-pure in $K$, therefore $K = N \oplus K_1$. This implies that $A = T \oplus K_1$ because

\[
T \cap K_1 = T \cap (K \cap K_1) = (T \cap K) \cap K_1 = N \cap K_1 = 0.
\]

Since $K_1 \simeq K/N$ is a direct sum of uniserial modules, it is bounded and we are done.

**Proposition 4.4.8.** Let $M$ be a $QTAG$-module with $N$ as a $h$-pure submodule such that $M/N$ is a direct sum of uniserial modules. Then $M$ is a $HE$-module if and only if $N$ is a $HE$-module and $M/N$ is bounded.

**Proof.** By Proposition 4.4.7, if $M/N$ is $HE$-module and $M/N$ is a direct sum of uniserial modules then $M/N$ is bounded. Moreover, $M$ can be expressed as $M \simeq N \oplus M/N$. Again finite direct sums and direct summands of $HE$-modules are again $HE$-modules and the result follows.

**Proposition 4.4.9.** Let $N$ be a fully invariant $HE$-module of $M$. If $M/N$ is a $HE$-module then $M$ is also a $HE$-module.
Proof. Let $M = K \oplus T$. Suppose $T$ is a direct sum of uniserial modules and we may write $T = \bigoplus_{n=0}^{\infty} (\bigoplus_{i \in I_n} x_i R)$ where $x_i R$ is a uniserial module and $d(x_i R) = n$. Now $M = K \oplus \bigoplus_{n=0}^{\infty} (\bigoplus_{i \in I_n} x_i R)$ and

$$N = (N \cap K) \oplus (N \cap T)$$

$$= (N \cap K) \oplus \bigoplus_{n=0}^{\infty} \bigoplus_{i \in I_n} (N \cap x_i R).$$

Therefore $(N \cap T) = \bigoplus_{n=0}^{\infty} \bigoplus_{i \in I_n} (N \cap x_i R)$ and

$$\frac{T + N}{N} \simeq \frac{T}{N \cap T}$$

$$= \frac{\bigoplus_{n=0}^{\infty} \bigoplus_{i \in I_n} x_i R}{\bigoplus_{n=0}^{\infty} \bigoplus_{i \in I_n} (N \cap x_i R)}$$

$$\simeq \bigoplus_{n=0}^{\infty} \bigoplus_{i \in I_n} [x_i R/(N \cap x_i R)]$$

are direct sum of uniserial modules.

Now $M/N = [(K + N)/N] \oplus [(T + N)/N]$ and

$$(K + N) \cap (T + N) = N + K \cap (T + N)$$

$$= N + K \cap (T + N \cap K)$$

$$= N + (N \cap K) + (K \cap T)$$

$$= N.$$

Since $(T + N)/N$ is a direct sum of uniserial modules it is bounded hence $T$ is bounded as $T \cap N$ is bounded.

4.5 $\omega_1$-Separable Modules

The cardinality of the minimal generating set of a module $M$, $g(M)$ plays a very important role in the study of $QTAG$-modules. Here we study $\omega_1$-separable and weakly $\omega_1$-separable $QTAG$-modules. We start with the following:
**Definition 4.5.1.** A $QTAG$-module $M$ is said to be $\omega_1$-separable if each of its countably generated submodules is contained in a direct summand of $M$ which is a direct sum of uniserial modules.

**Proposition 4.5.2.** Let $N$ be a countably generated nice submodule of the $QTAG$-module $M$. If $M$ is $\omega_1$-separable then $M/N$ is $\omega_1$-separable.

**Proof.** Since a $\omega_1$-separable module is separable. $M$ is separable and by [32], $M/N$ is also separable. Let $K/N$ be a submodule of $M/N$ such that $g(K/N) = \aleph_0$. Then $g(K) = \aleph_0$ and there exists a direct summand $T$ of $M$ containing $K$ such that $g(T) = \aleph_0$ and $M = T \oplus M_1$. Now $\frac{M}{N} = \frac{T}{N} \oplus \frac{(M_1 + N)}{N}$ and $T \cap (M_1 + N) = N + (T \cap M_1) = N$. But $K/N \subseteq T/N$ and $g(T/N) = \aleph_0$ because $\aleph_0 = g(K/N) \leq g(T/N) \leq g(T) = \aleph_0$ and $T/N$ is a direct sum of uniserial modules.

We now generalize $\omega_1$-separable modules to weakly $\omega_1$-separable modules and investigate their properties.

**Definition 4.5.3.** A separable $QTAG$-module $M$ is said to be weakly $\omega_1$-separable if for all countably generated submodules $N$ of $M$, $g(\bigcap_{k<\omega} (H_k(M) + N)) = \aleph_0$.

**Theorem 4.5.4.** Let $N$ be a countably generated nice submodule of a separable module $M$. Then $M$ is weakly $\omega_1$-separable if and only if $M/N$ is weakly $\omega_1$-separable.

**Proof.** Consider the nice submodule $N$ in the separable $QTAG$-module $M$ such that $M/N$ is separable. Let $T/N$ be a submodule of $M/N$ such that $g(T/N) = \aleph_0$. Thus $g(T) = \aleph_0$ and $g(\bigcap_{k<\omega} (H_k(M) + T)) = \aleph_0$. 

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Now,

\[ \aleph_0 = g(T/N) \]

\[ \leq g(\bigcap_{k<\omega} (H_k(M/N) + T/N)) \]

\[ = g(\bigcap_{k<\omega} [(H_k(M) + T)/N]) \]

\[ = g(\bigcap_{k<\omega} [(H_k(M) + T)/T]) \]

\[ \leq g(\bigcap_{k<\omega} (H_k(M) + T)) = \aleph_0. \]

Therefore \( g(\bigcap_{k<\omega} (H_k(M/N) + T/N)) = \aleph_0 \) and \( M/N \) is weakly \( \omega_1 \)-separable.

For the converse, consider a countably generated submodule \( K \subseteq M \). Since \( M/N \) is separable, \( N \) is nice in \( M \) and \( H_\omega(M) \subseteq N \). If \( g\left(\frac{K + N}{N}\right) < \omega \), then it is nice in \( M/N \). Since \( N \) is nice in \( M \), \( K + N \) is nice in \( M \). Therefore \( \bigcap_{k<\omega} (H_k(M) + K + N) = H_\omega(M) + K + N = K + N \) and \( g(\bigcap_{k<\omega} (H_k(M) + K + N)) = g(K + N) = \aleph_0 \) and so \( g(\bigcap_{k<\omega} (H_k(M) + K)) = \aleph_0 \), since \( K \subseteq \bigcap_{k<\omega} (H_k(M) + K) \) and the result follows. Otherwise, if \( g\left(\frac{K + N}{N}\right) = \aleph_0 \), then

\[ g(\bigcap_{k<\omega} [H_k(M/N) + (K + N)/N]) = g(\bigcap_{k<\omega} [(H_k(M) + N)/N + (K + N)/N]) \]

\[ = g(\bigcap_{k<\omega} (H_k(M) + K + N)/N)) \]

\[ = \aleph_0 \text{ and } g(N) \]

\[ = g(\bigcap_{k<\omega} (H_k(M) + K + N)). \]

Since \( K \subseteq \bigcap_{k<\omega} (H_k(M) + K) \) and \( g(K) = g(N) = \aleph_0 \), implying that

\[ g(\bigcap_{k<\omega} (H_k(M) + K)) = \aleph_0. \]
We conclude with the remark that finite direct sums, direct summands and submodules of $\omega_1$-separable modules (or weakly $\omega_1$-separable modules) are again $\omega_1$-separable (or weakly $\omega_1$-separable).