Chapter 1
Preliminaries

Mathematics is the science of what is clear by itself.

-Carl Gustav Jacobi

1.1 Introduction

In this chapter we have collected some basic definitions and results, prerequisite to the study of QTAG-modules. In the theory of Abelian groups the concepts of neat, pure, basic, high subgroups, divisible and closed groups etc. are very significant. Some of these concepts were generalized for modules over Dedekind rings, hnp-rings and Artinian rings by R.B. Warfield, H. Marubayashi and S. Singh etc. Singh started the study of TAG-modules/ $S_2$-modules satisfying certain conditions. Later on K. Benabdullah, M.Z. Khan and A. Mehdi etc. generalized various results for $S_2$-modules. S. Singh called them $T A G$-modules and proved that the results which are true for $T A G$-modules are also true for $Q T A G$-modules [45]. These $Q T A G$-modules satisfy only one condition, namely “every finitely generated submodule of any homomorphic image of the module is a direct sum of uniserial modules.” Later on Khan, Mehdi, Abbasi, Sirohi and Naji contributed a lot to the study of these $Q T A G$-modules.

Throughout the thesis all the rings $R$ considered are associative with unity, and the modules are torsion and unital right $R$-modules.

In section two we state some definitions and properties of $Q T A G$-modules and their $h$-neat, $h$-pure, basic and high submodules etc. In section three the definition and properties of closed modules are stated. Some refer to them as torsion complete modules. It is found that closed modules are also $h$-dense with $h$-topology. The last section introduces certain homomorphisms, the cardinality of the generating set $g(M)$, final $g(M)$, $\sigma$-projective modules, totally projective modules, $Ulm$ submodules, $Ulm$ factors and invariants etc.
1.2 Some Basic Concepts

Definition 1.2.1. A module $M$ is said to be uniform, if the intersection of any two of its non-zero submodules is non-zero.

Definition 1.2.2. Let $M_R$ be a module. Then $x \in M$ is said to be uniform, if $xR$ is a uniform module.

Definition 1.2.3. Let $M$ be a non-zero module. Then a finite chain of submodules of $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$ is called the composition series of length $n$ for $M$, provided that $M_i/M_{i+1}$ is simple for every $i$.

Definition 1.2.4. A module $M$ is said to be uniserial, if it has a unique composition series and its decomposition length is denoted by $d(M)$.

Definition 1.2.5. A unital, torsion module $M$ is said to be the $TAG$-module (or $S_2$-module) if it satisfies the following:

(i) every finitely generated submodule of every homomorphic image of $M$ is a direct sum of uniserial modules;

(ii) given any two uniserial submodules $U$ and $V$ of a homomorphic image of $M$, for any submodule $W$ of $U$, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$ [44].

Definition 1.2.6. A module satisfying only the first condition of Definition 1.2.5, is said to be the $QTAG$-module [45].

Remark 1.2.7. The definition of $QTAG$-module implies that every element of $M$ may be written as a finite sum of uniform elements. Without loss of generality we may consider uniform elements only.

Remark 1.2.8. Singh [45] proved that the definitions and results which are true for $TAG$-modules hold good for $QTAG$-modules too.
Definition 1.2.9. A $QTAG$-module $M$ is said to be decomposable, if it is a direct sum of uniserial modules [3].

Definition 1.2.10. Let $x$ be a uniform element of $M$. Then $d(xR)$ is defined to be the exponent of $x$ and is denoted by $e(x)$.

Definition 1.2.11. A uniform element $y \in M$ is called predecessor of a uniform element $x \in M$ if $x \in yR$ and $d(yR/xR) = 1$.

Definition 1.2.12. Let $x$ be a uniform element of $M$. Then $\text{sup}\{d(U/xR)\}$, where $U$ runs through all the uniserial submodules of $M$ containing $x$, is defined to be the height of $x$ in $M$ and is denoted by $H_M(x)$ or simply by $H(x)$ [18].

Definition 1.2.13. For every $k \geq 0$, $H_k(M)$ denotes the submodule of $M$ generated by the uniform elements of height at least $k$ and $M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ [18].

Definition 1.2.14. For any arbitrary $x \in M$, $H_M(x) = k$ if $x \in H_k(M)$, $x \notin H_{k+1}(M)$ and $H(x) = \infty$, if $x \in M^1$.

Definition 1.2.15. Let $M$ be a $QTAG$-module. The submodule generated by the uniform elements of exponent at most $k$ is denoted by $H^k(M)$ [20].

Definition 1.2.16. Let $M$ be a module, then the sum of all simple submodules of $M$ is called the socle of $M$ and is denoted by $\text{Soc}(M)$. A submodule of $\text{Soc}(M)$ is called a subsocle of $M$.

Definition 1.2.17. A module $M$ is said to be bounded, if there exists an integer $n$ such that $H(x) \leq n$ for every uniform element $x \in M$ [47].

Proposition 1.2.18. Any bounded $QTAG$-module $M$ is a direct sum of uniserial modules [47, Corollary 1].
Proposition 1.2.19. If $M$ is a $QTAG$-module and $N$ is a submodule of $M$, then $N$ can be embedded in a bounded summand of $M$ if and only if the heights of the uniform elements of $N$ in $M$ are bounded [18].

Remark 1.2.20. The submodule of $M$ generated by the uniform elements of infinite height is denoted by $M^1$. Equivalently $M^1 = \bigcap_{k=0}^{\infty} H_k(M)$.

Definition 1.2.21. Let $M$ be a $QTAG$-module. Then $M$ is called separable if every finitely generated submodule of $M$ can be embedded in a summand of $M$.

Definition 1.2.22. A submodule $N \subset M$ is said to be high, if it is a complement of $M^1$ i.e., $M = N \oplus M^1$.

Definition 1.2.23. If $N$ is a submodule of a $QTAG$-module $M$, then a submodule $K$ of $M$ is $N$-high, if it is maximal with the property of being disjoint from $N$.

Definition 1.2.24. If $M$ is a $QTAG$-module, then $M$ is called a $\Sigma$-module, if all of its high submodules are direct sums of uniserial modules.

Definition 1.2.25. Let $N$ be a submodule of $M$. Then $N$ is said to be essential in $M$ if $N \cap T \neq 0$ for every non-zero submodule $T$ of $M$ and $M$ is said to be the essential extension of $N$.

Proposition 1.2.26. If $N$ is an essential submodule of $M$, then $Soc(N) = Soc(M)$.

Definition 1.2.27. A submodule $N \subset M$ is said to be $h$-neat in $M$, if $H_1(N) = N \cap H_1(M)$ [20].

Definition 1.2.28. A submodule $N$ of $M$ is said to be $h$-pure in $M$, if $H_k(N) = N \cap H_k(M)$ for every $k = 0, 1, 2, \ldots, \infty$ [20].
**Proposition 1.2.29.** Every direct summand of a $QTAG$-module $M$ is $h$-pure in $M$ [20].

**Proposition 1.2.30.** Every bounded $h$-pure submodule of a $QTAG$-module $M$ is a summand of $M$ [47].

**Proposition 1.2.31.** If $M$ is a $QTAG$-module such that $M/K = (N/K) \oplus (T/K)$, where $N$, $T$ and $K$ are the submodules of $M$ and $K$ is $h$-pure in $N$, then $T$ is also $h$-pure in $M$ [20, Corollary 2].

**Proposition 1.2.32.** If $N$ is a submodule of a $QTAG$-module $M$ and $H_N(x) = H_M(x)$ for every uniform element $x \in Soc(N)$, then $N$ is $h$-pure in $M$ [18, Lemma 1].

**Definition 1.2.33.** A $QTAG$-module $M$ is called $h$-pure-complete, if for every subsocle $S$ of $M$ there exists a $h$-pure submodule $N$ of $M$ such that $S = Soc(N)$ [18].

**Definition 1.2.34.** Let $M$ be a $QTAG$-module. A subset $\{x_i | i \in I\}$ of uniform elements of $M$ is called $h$-pure-independent, if it is independent in the sense that $\Sigma x_i R$ is direct and $\Sigma x_i R$ is an $h$-pure submodule of $M$.

**Definition 1.2.35.** For a $QTAG$-module $M$ and an ordinal $\sigma$, $H_\sigma(M)$ is defined as $H_\sigma(M) = \bigcap_{\rho<\sigma} H_\rho(M)$.

**Definition 1.2.36.** Let $M$ be a $QTAG$-module. The smallest ordinal $\sigma$ such that $H_\sigma(M) = 0$, is said to be the length of $M$.

**Remark 1.2.37.** Since $\omega$ is the first infinite ordinal, $H_\omega(M) = \bigcap_{k=0}^{\infty} H_k(M) = M^1$.

**Definition 1.2.38.** For an ordinal $\sigma$, a submodule $N$ of $M$ is said to be $\sigma$-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \sigma$ [45].
Definition 1.2.39. A submodule $N$ of $M$ is said to be isotype in $M$, if it is $\sigma$- pure for every ordinal $\sigma$ [45].

Definition 1.2.40. Let $M$ be a QTAG- module and $N$ a submodule of $M$. An element $x \in M$ is said to be proper with respect to $N$, if $H(x+N) \geq H(y)$ for every $y \in x+N$. In other words, if $H(x) = \sigma$, then $x \notin H_{\sigma+1}(M)+N$ [33].

Definition 1.2.41.: A submodule $N$ of a QTAG- module $M$ is nice, if for every ordinal $\sigma$, there exists an elements $x_\sigma \in N_{\sigma+1}/N_\sigma$ which is proper with respect to $N_\sigma$ [33].

Definition 1.2.42. A family $\mathcal{N}$ of submodules of $M$ is called a nice system in $M$ if

(i) $0 \in \mathcal{N}$;

(ii) if $\{N_i\}_{i \in I}$ is any subset of $\mathcal{N}$, then $\sum_{i \in I} N_i \in \mathcal{N}$;

(iii) given any $N \in \mathcal{N}$ and any countable subset $X$ of $M$, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that $K/N$ is countably generated [33].

Definition 1.2.43. A QTAG- module $M$ has a nice basis if it can be expressed as $M = \bigcup_{k<\omega} M_k$, $M_k \subseteq M_{k+1} \subseteq M$ and each $M_k$ is nice in $M$ and a direct sum of uniserial modules [46].

Proposition 1.2.44. A submodule $N$ of a QTAG- module $M$ is nice in $M$ if and only if $H_{\sigma}(M/N) = (H_{\sigma}(M) + N)/N$ for every ordinals $\sigma$ [33].

Definition 1.2.45. A QTAG- module $M$ is called h-divisible if $H_1(M) = M$ [21].

Proposition 1.2.46. A QTAG- module $M$ is h-divisible if and only if every uniform element of $Soc(M)$ is of infinite height [21, Lemma 2].
Theorem 1.2.47. A $QTAG$-module $M$ is $h$-divisible if and only if $M$ is a direct sum of infinite length uniform submodules [21, Theorem 3].

Theorem 1.2.48. If $M$ is a $QTAG$-module and $N$ is a $h$-divisible submodule of $M$, then $N$ is a direct summand of $M$ [21, Theorem 4].

Definition 1.2.49. A $QTAG$-module $M$ is said to be $h$-reduced if it is free from the elements of infinite height. Equivalently, it may be said that $M$ does not have a $h$-divisible submodule.

Definition 1.2.50. A $QTAG$-module is a $\Sigma$-module if its high submodules are the direct sum of uniserial modules.

Proposition 1.2.51. A $QTAG$-module $M$ is a $\Sigma$-module if and only if $\text{Soc}(M) = \bigcup_{k<\omega} M_k$, $M_k \subseteq M_{k+1}$ and for every $k \geq 1$, $M_k \cap H_k(M) = \text{Soc}(H_\omega(M))$ [22].

Proposition 1.2.52. Let $N$ be a submodule of a $QTAG$-module $M$ such that $M/N$ is a direct sum of uniserial modules. Then $M$ is a direct sum of uniserial modules if and only if $N = \bigcup_{k<\omega} N_k$, $N_k \subseteq N_{k+1}$ and $N_k \cap H_k(M) = 0$. Equivalently, if $\text{Soc}(N) = \bigcup_{k<\omega} S_k$, $S_k \subseteq S_{k+1}$ and $S_k \cap H_k(M) = 0$, for every $k \in \mathbb{Z}^+$.

Proposition 1.2.53. A $QTAG$-module $M$ is a $\Sigma$-module if and only if $H_k(M)$ is a $\Sigma$-module, for $k \geq 0$.

Proposition 1.2.54. If $M$ is a $QTAG$-module and $N$ is a submodule such that $N \supseteq H_k(M)$, then $N$ is a $\Sigma$-module provided that $M$ is a $\Sigma$-module [22, Corollary 17].

Proposition 1.2.55. Let $M$ be a $h$-reduced $QTAG$-module. Then $M$ is a $\Sigma$-module if and only if $H_k(M)$ is a $\Sigma$-module for all $k = 0, 1, 2, \ldots, \infty$ [22].

Proposition 1.2.56. Every $h$-divisible module is injective.
Basic submodules of a $QTAG$-module play a significant role in the study of $QTAG$-modules.

**Definition 1.2.57.** Let $M$ be a $QTAG$-module. A submodule $B$ of $M$ is called a basic submodule of $M$, if the following conditions hold:

(i) $B$ is a $h$-pure submodule of $M$.

(ii) $B$ is a direct sum of uniserial modules.

(iii) $M/B$ is a direct sum of uniform modules of infinite length i.e., $M/B$ is $h$-divisible [21].

**Remark 1.2.58.** Basic submodule $B$ of $M$ can be written as $B = \bigoplus_{i=1}^{\infty} B_i$, where each $B_i$ is a direct sum of uniserial modules of length $i$ [21].

**Theorem 1.2.59.** Every $QTAG$-module contains a basic submodule [21, Theorem 1].

**Theorem 1.2.60.** Let $M$ be a $QTAG$-module and $B$, a submodule of $M$ with $B = \bigoplus_{n=1}^{\infty} B_n$, where each $B_n$ is a direct sum of uniserial modules of length $n$. Then $B$ is a basic submodule of $M$ if and only if $M = B_1 \oplus B_2 \cdots \oplus B_n \oplus (B^*_n, H_n(M))$, where $B^*_n = B_{n+1} \oplus B_{n+2} \oplus \cdots$ [21, Theorem 2].

**Proposition 1.2.61.** Let $M$ be a $QTAG$-module and $B$ as in Theorem 1.2.60. Then $B$ is a basic submodule of $M$ if and only if $B_1 \oplus B_2 \cdots \oplus B_n$ is a direct summand of $M$ and is maximal with respect to the property $(B_1 \oplus B_2 \cdots \oplus B_n) \cap H_n(M) = 0$.

**Theorem 1.2.62.** Any two basic submodules of $M$ are isomorphic [21, Theorem 5].

**Theorem 1.2.63.** Let $B = \bigoplus_{i=1}^{\infty} B_i$ be a basic submodule of a $QTAG$-module ($S_2$-module) $M$. Then $M$ can be written as $M = S_k \oplus M_k$, where
\[ M_k = B_k^* + H_k(M), \quad B_k^* = B_{k+1} \oplus B_{k+2} \oplus \cdots \]

and \( S_k \) is the maximal summand of \( M \) bounded by \( k \) [18, Theorem 9].

**Definition 1.2.64.** A submodule \( L \) of a QTAG-module \( M \) is said to be large if \( L \) is fully invariant and \( M = L + B \), for every basic submodule \( B \) of \( M \).

### 1.3 Closed Modules and Topological Considerations

Towards the definition of a closed module we need the following:

**Definition 1.3.1.** Since every \( x \in M \) can be uniquely written as a finite sum of uniform elements. We define \( h \)-exponent of an element \( x \in M \) as follows:

\[
h\text{-exp}(x) = \max\{e(u_1), e(u_2), \ldots, e(u_n)\}
\]

where \( x = u_1 + u_2 + \ldots + u_n \) with \( u_i \) uniform [28].

**Definition 1.3.2.** A sequence \( \{x_n\} \) is said to be a Cauchy sequence if \( x_k - x_{k+1} \in H_k(M) \) for every \( k \) and \( h\text{-exp}(x_n) \) of \( x_n \) are bounded for every integer \( n \) [28].

**Definition 1.3.3.** An element \( x \in M \) is the limit of the Cauchy sequence \( \{x_n\} \), if \( x - x_k \in H_k(M) \), for all \( k = 0, 1, \ldots, \infty \) [28].

**Remark 1.3.4.** The sum and difference of two Cauchy sequences is also a Cauchy sequence.

**Definition 1.3.5.** A QTAG- module \( M \) without elements of infinite height is said to be closed, if every Cauchy sequence in \( M \) has a limit in \( M \) [27].

**Remark 1.3.6.** Intersection of two closed QTAG- modules is a closed QTAG- module [28].

**Theorem 1.3.7.** A QTAG- module \( M \) is closed if and only if \( M = \bar{B} \) where \( \bar{B} = \Sigma B_i \), the complete direct sum of \( B_i \)'s [28].
Corollary 1.3.8. Two closed $QTAG$-modules are isomorphic if and only if their basic submodules are isomorphic [28].

Theorem 1.3.9. Every direct summand of a closed $QTAG$-module is closed and direct sum of a finite number of closed $QTAG$-modules is closed [28].

Theorem 1.3.10. A $QTAG$-module without elements of infinite height is closed if and only if its socle is closed [28].

In $h$-topology the set of modules $H_k(M)$, $k = 0, 1, 2, \ldots, \infty$ forms a base for the neighborhood system of zero. We call the submodules of $M$ closed with respect to $h$-topology as complete modules.

Definition 1.3.11. A submodule $N$ of $M$ is said to be complete if $N = \bar{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$, and $\bar{N}$ is called the completion of $N$ with respect to the $h$-topology [2].

Remark 1.3.12. $M^1$ is the completion of zero module.

Definition 1.3.13. A submodule $N$ of $M$ is said to be $h$-dense in $M$ if $\bar{N} = M$.

1.4 Ulm Submodules and Totally Projective $QTAG$-Modules

Here we state some important definitions and results which are general in nature but significant for the next chapters.

Definition 1.4.1. Let $M$ be a $QTAG$-module. It defines a well ordered sequence of submodules $M = M^0 \supset M^1 \supset M^2 \supset \ldots \supset M^\tau = 0$ for some ordinal $\tau$. Here $M^1 = \bigcap_{k \in \omega} H_k(M)$, $M^{\sigma+1} = (M^\sigma)^1$ and $M^\sigma = \bigcap_{\rho < \sigma} M^\rho$, if $\sigma$ is a limit ordinal. $M^\sigma$ is said to be the $\sigma$-th Ulm submodule of $M$. 
Definition 1.4.2. The $\sigma$-th Ulm factor of a $QTAG$-module $M$ is the quotient $M^{\sigma}/M^{\sigma+1} = M_{\sigma}$. $M_0, M_1, \ldots, M_\sigma, \ldots$ ($\sigma < \tau$) is said to the Ulm sequence of $M$ and $\tau$ is the Ulm type of $M$.

Definition 1.4.3. The generalized height of $x$ in $M$ denoted by

$$H^*_M(x) = \begin{cases} 
\alpha, & \text{if } x \neq 0 \text{ and } \alpha + 1 \text{ is the first ordinal such that } x \notin H_{\alpha+1}(M), \\
\infty, & \text{if } x = 0.
\end{cases}$$

For any uniform element $x \in M$, there exist uniform elements $x_1, x_2, x_3, \ldots$ such that

$$xR \supseteq x_1R \supseteq x_2R \supseteq x_3R \supseteq \cdots \text{ and } d(x_iR/x_{i+1}R) = 1.$$ 

Now the Ulm- sequence of $x$ is defined as

$$U(x) = (H(x_1), H(x_2), H(x_3), \ldots).$$

This is analogous to the $U$-sequences in groups [12,13].

Definition 1.4.4. For any $QTAG$-module $M$, $g(M)$ denotes the smallest cardinal number $\lambda$ such that $M$ admits a generating set $X$ of uniform elements of cardinality $\lambda$ i.e., $|X| = \lambda$ [45].

Definition 1.4.5. The final $g(M)$ or $\text{fin } g(M)$ of a $QTAG$-module $M$ is defined as the infimum of $g(H_k(M))$ for $k = 0, 1, 2, \ldots, \infty$ i.e., $\text{fin } g(M) = \inf(g(H_k(M)))$.

Definition 1.4.6. For a $QTAG$-module $M$, the $\sigma$-th Ulm invariant of $M$, $f_M(\sigma)$ is the cardinal number $g(Soc(H_\sigma(M))/Soc(H_{\sigma+1}(M)))$ [35].

Definition 1.4.7. For a submodule $N$ of a $QTAG$-module $M$, the $\sigma$-th Ulm invariant of $N$ with respect to $M$ is defined as

$$f_\sigma(N, M) = g[Soc(H_\sigma(M))/((H_{\sigma+1}(M) + N) \cap Soc(H_\sigma(M)))][35].$$
Definition 1.4.8. For the ordinals $\alpha$ and $\beta$,

(i) $\beta \in \alpha$ means $0 \leq \beta < \alpha$;

(ii) $\omega$ is the first infinite ordinal and $\omega^* = \omega - \{0\}$;

(iii) $C$ denotes the cardinality of the set with the power of continuum.

Now we state some definitions and facts related to ordinals.

Definition 1.4.9. An ordinal $\beta$ is an initial ordinal if for all ordinals $\alpha < \beta$ implies $|\alpha| \leq |\beta|$ [11].

Definition 1.4.10. An ordinal $\alpha$ is cofinal in an ordinal $\beta$ if there is a one to one map $f : \alpha \rightarrow \beta$ such that for every $\gamma \in \beta$, there exists $\sigma \in \alpha$ with $\gamma \leq f(\sigma)$ [11].

Definition 1.4.11. The confinality of an ordinal $\beta$ is the least ordinal $\alpha$ where $\alpha$ is confinal in $\beta$ [11].

Definition 1.4.12. An ordinal $\alpha$ is cofinal with $\sigma$ if and only if there exists a one to one order preserving map $f : \sigma \rightarrow \alpha$ such that for every $\beta$ in $\alpha$, there exists $\rho \in \sigma$ and $\beta \leq f(\sigma)$ [1].

Remark 1.4.13. The cofinality of any limit ordinal is $\omega$ [1].

Definition 1.4.14. The cofinality of an ordinal $\alpha$ is the smallest ordinal $\beta$ which is the order type of $\alpha$ [1].

Remark 1.4.15. An ordinal which is equal to its cofinality, is always an initial ordinal [1].

Definition 1.4.16. $\omega_1$ is the smallest uncountable ordinal. Consider the set of all well orderings of natural numbers. Then each well ordering defines a countable ordinal and $\omega_1$ is the order type of this set.
Definition 1.4.17. Let $M$ and $M'$ be two $QTAG$-modules and $N$ a submodule of $M$. A homomorphism $f : N \to M'$ is height preserving homomorphism, if $H_{M'}(f(x)) \geq H_{M}(x)$ for all $x \in N$.

Definition 1.4.18. A $QTAG$-module $M$ is $(\omega + 1)$-projective, if there exists a submodule $N \subset H^1(M)$ such that $M/N$ is a direct sum of uniserial modules.

Definition 1.4.19. A $QTAG$-module $M$ is $(\omega + k)$-projective, if there exists a submodule $N \subset H^k(M)$ such that $M/N$ is a direct sum of uniserial modules.

Definition 1.4.20. Let $\sigma$ be a limit ordinal such that $\sigma = \omega + \beta$. A $QTAG$-module $M$ is called $\sigma$-projective if there exists a submodule $N \subset H^\beta(M)$ such that $M/N$ is a direct sum of uniserial modules.

Definition 1.4.21. A $h$-reduced $QTAG$-module $M$ is called totally projective, if it has a nice system.

Remark 1.4.22. $M$ is totally projective if and only if $M/H_{\sigma}(M)$ is $\sigma$-projective for every ordinal $\sigma$.

Remark 1.4.23. Direct sums and summands of totally projective $QTAG$-modules are totally projective.

Remark 1.4.24. A totally projective $QTAG$-module of Ulm type $\leq \sigma$ is $\sigma$-projective [27].

Remark 1.4.25. A $QTAG$-module $M$ is $k$-projective if and only if $H_k(M) = 0$.

Remark 1.4.26. Direct sums of uniserial modules are totally projective and $\omega$-projective [27].
Definition 1.4.27. Let $\alpha$ denote the class of all $QTAG$-modules $M$ such that $M/H_\beta(M)$ is totally projective for all ordinals $\beta < \alpha$. These modules are called $\alpha$-modules.

Definition 1.4.28. An $QTAG$-module is an $\omega$-elongation of a totally projective $QTAG$-module by a $(\omega + k)$-projective $QTAG$-module if and only if $H_\omega(M)$ is totally projective and $M/H_\omega(M)$ is $(\omega + k)$-projective.