Chapter 2

A Study of Modified Hermite Polynomials of One and Two Variables

2.1 Introduction

The present chapter deals with modified Hermite polynomials of one and two variables, denoted by $H_n(x; a)$ and $H_n(x, y; a)$. The aim of the present chapter is to investigate these two polynomials by finding some important results such as generating functions, recurrence relations, Rodrigues formula, orthogonality conditions, expansion formulae, integrals, fractional integrals, fractional derivatives, operator representations and other properties of the above polynomials.

2.2 The Modified Hermite Polynomials of One Variable $H_n(x; a)$

The modified Hermite polynomials $H_n(x; a)$ are defined by means of the generating relation

$$a^{2x^2-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x; a)t^n}{n!}, \quad a > 0, \ a \neq 1 \quad (2.2.1)$$

It follows from (2.2.1) that

$$H_n(x; a) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n!(2x)^{n-2k}(\log a)^{n-k}}{k!(n-2k)!} \quad (2.2.2)$$
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For \( a = e \), (2.2.2) reduces to Hermite polynomials \( H_n(x) \).

It may be remarked that \( H_n(x; a) \) is an even function of \( x \) for even \( n \), an odd function of \( x \) for odd \( n \).

\[
H_n(-x; a) = (-1)^n H_n(x; a)
\]

Also,

\[
H_{2n}(0; a) = (-1)^n 2^{2n} \left( \frac{1}{2} \right)_n (\log a)^n
\]

\[
H_{2n+1}(0; a) = 0
\]

and

\[
H'_{2n+1}(0; a) = (-1)^n 2^{2n+1} \left( \frac{3}{2} \right)_n (\log a)^{n+1}
\]

The first few modified Hermite polynomials are listed below:

\[
\begin{align*}
H_0(x; a) &= 1, \\
H_1(x; a) &= 2x \log a, \\
H_2(x; a) &= 4x^2(\log a)^2 - 2 \log a, \\
H_3(x; a) &= 8x^3(\log a)^3 - 12x(\log a)^2, \\
H_4(x; a) &= 16x^4(\log a)^4 - 48x^2(\log a)^3 + 12(\log a)^2, \\
H_5(x; a) &= 32x^5(\log a)^5 - 160x^3(\log a)^4 + 120x(\log a)^3, \\
H_6(x; a) &= 64x^6(\log a)^6 - 480x^4(\log a)^5 + 720x^2(\log a)^4 - 120(\log a)^3.
\end{align*}
\]

2.3 Generating Function for \( H_n(x; a) \)

Some other generating functions for \( H_n(x; a) \) are as given below:

\[
\sum_{n=0}^{\infty} \frac{(c)_n H_n(x; a)t^n}{n!} = (1 - 2xt \log a)^{-c} \left[ \frac{c}{2} \left( \frac{c}{2} \right)^2 \right] \frac{1}{\left( 1 - 2xt \log a \right)^2} \] (2.3.1)

\[
\sum_{n=0}^{\infty} \frac{H_{n+k}(x; a)t^n}{n!} = a^{2xt - t^2} H_k(x - t; a) \] (2.3.2)

\[
\sum_{n=0}^{\infty} \frac{H_n(x; a)H_n(y; a)t^n}{n!} = \left( 1 - 4t^2(\log a)^2 \right)^{-\frac{1}{2}} a \left\{ \frac{y^2 - (y - 2xt \log a)^2}{(1 - 4t^2(\log a)^2)} \right\} \] (2.3.3)

Replacing \( t \) by \( t/2 \) in (2.3.3), we get

\[
\sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)_n H_n(x; a)H_n(y; a)}{n!} = \left( 1 - t^2(\log a)^2 \right)^{-\frac{1}{2}} a \left\{ \frac{2xt \log a - (x^2 + y^2)^2(\log a)^2}{(1 - t^2(\log a)^2)} \right\} \] (2.3.4)
The above result for modified Hermite polynomials is similar to the result given by Brafman [4] for Hermite polynomials.

\[ \sum_{n=0}^{\infty} \frac{ \text{I}_n}{n!} H_n(x; a) t^n \cong a^{2x-t^2} [1 + 2ty(x - t) \log a]^{-c} \]

\[ \times \text{I}_0 \left( \frac{\frac{c}{2}, \frac{c}{2} + \frac{1}{2}}{-; \left(1 + 2xyt \log a - 2t^2y \log a \right)^2} \right) \] (2.3.5)

2.4 Recurrence Relations

The following recurrence relations hold for \( H_n(x; a) \):

\[ xH'_n(x; a) = nH'_{n-1}(x; a) + nH_n(x; a) \quad (2.4.1) \]

\[ H'_n(x; a) = 2n \log a \ H_{n-1}(x; a) \quad (2.4.2) \]

\[ D^s H_n(x; a) = \frac{(2 \log a)^n! H_{n-s}(x; a)}{(n-s)!} ; \quad D \equiv \frac{d}{dx} \quad (2.4.3) \]

\[ H_n(x; a) = 2x \log a \ H_{n-1}(x; a) - H'_{n-1}(x; a) \quad (2.4.4) \]

\[ H_n(x; a) = 2 \log a \{ xH_{n-1}(x; a) - (n-1)H_{n-2}(x; a) \} \quad (2.4.5) \]

\[ H''_n(x; a) = 4n(n-1)(\log a)^2 H_{n-2}(x; a) \quad (2.4.6) \]

Also the modified Hermite differential equation is

\[ H''_n(x; a) - 2x \log a \ H'_n(x; a) + 2n \log a \ H_n(x; a) = 0. \quad (2.4.7) \]

2.5 Rodrigues Formula for \( H_n(x; a) \)

The Rodrigues formula for modified Hermite polynomials \( H_n(x; a) \) is given by the following relation:

\[ H_n(x; a) = (-1)^n a^x D^n a^{-x} , \quad D \equiv \frac{d}{dx} \quad (2.5.1) \]

The proof of (2.5.1) is same as that of Rodrigues formula for \( H_n(x) \).
2.6 Integral Representations

Several integrals involving modified Hermite polynomials $H_n(x; a)$ are as follows:

\[ P_n(x) = \frac{2}{n!} \sqrt{\frac{\log a}{\pi}} \int_0^\infty a^{-t^2} t^n H_n(xt; a) \, dt \]  \hspace{1cm} (2.6.1)

\[ H_n(x; a) = (2 \log a)^{n+1} x^2 \int_{-\infty}^\infty a^{-t^2} t^{n+1} P_n(x/t) \, dt \]  \hspace{1cm} (2.6.2)

\[ \int_{-\infty}^{+\infty} a^{-x^2} x^n H_{n-2k}(x; a) \, dx = \frac{2^{-2k} n!}{k! (\log a)^k} \sqrt{\frac{\pi}{\log a}} \]  \hspace{1cm} (2.6.3)

\[ \int_0^\infty a^{-x^2} H_{2k}(x; a) H_{2s+1}(x; a) \, dx = \frac{(-1)^{k+s} 2^{2k+2s} \left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_s (\log a)^{k+s}}{(2s + 1 - 2k)} \]  \hspace{1cm} (2.6.4)

\[ \int_0^{+\infty} a^{-t^2} H_n(t; a) \, dt = H_{n-1}(x; 0) - a^{-x^2} H_{n-1}(x; a) \]  \hspace{1cm} (2.6.5)

\[ \int_0^{+\infty} a^{-t^2} H_{2n}(x; a) \, dt = \frac{\sqrt{\pi} (2n)!}{n!} (x^2 - 1) (\log a)^{n+\frac{1}{2}} \]  \hspace{1cm} (2.6.6)

\[ \int_{-\infty}^{+\infty} a^{-t^2} H_{2n+1}(x; a) \, dt = \frac{\sqrt{\pi} (2n + 1)!}{n!} x (x^2 - 1)^n (\log a)^{n+\frac{1}{2}} \]  \hspace{1cm} (2.6.7)

\[ \int_{-\infty}^{+\infty} a^{-t^2} t^n H_n(xt; a) \, dt = \frac{\sqrt{\pi} n!}{\sqrt{\log a}} P_n(x) \]  \hspace{1cm} (2.6.8)

\[ \int_{-\infty}^{+\infty} a^{-x^2} H_{2n}(\sqrt{2} x; a) \, dx = \frac{(2n)! \sqrt{\pi}}{n!} (\log a)^{n+\frac{1}{2}} \]  \hspace{1cm} (2.6.9)

\[ \int_{-\infty}^{+\infty} a^{-x^2} H_{2n+1}(\sqrt{2} x; a) \, dx = 0 \]  \hspace{1cm} (2.6.10)

\[ \int_{-\infty}^{+\infty} a^{-\frac{1}{2} x^2} H_{2n}(x; a) \, dx = \frac{(2n)! \sqrt{\pi}}{n!} \sqrt{\frac{\pi}{2}} (\log a)^{n+\frac{1}{2}} \]  \hspace{1cm} (2.6.11)

The second result (2.6.11) is trivial and the first (2.6.10) may be written
\[ \int_0^\infty a^{-t^2} [H_n(t; a)]^2 \cos \left( \sqrt{2 \log a \, x t} \right) dt = \sqrt{\pi} 2^{n-1} n! L_n(x^2)(\log a)^{n-\frac{1}{2}} \]  

(2.6.13)

\[ \Gamma(n + \mu + 1) \int_{-1}^{+1} (1 - t^2)^{\mu - \frac{1}{2}} H_{2n}(\sqrt{xt}; a) dt \]

\[ = (-1)^n \sqrt{\pi} (2n)! \Gamma \left( \mu + \frac{1}{2} \right) (\log a)^n L_n^\mu(x \log a), \left( \Re(\mu) > -\frac{1}{2} \right) \]  

(2.6.14)

\[ \Gamma(n + \mu + 1) \int_{-\frac{1}{\sqrt{\log a}}}^{\frac{1}{\sqrt{\log a}}} (1 - t^2 \log a)^{\mu - \frac{1}{2}} H_{2n}(\sqrt{xt}; a) dt \]

\[ = (-1)^n \sqrt{\pi} (2n)! \Gamma \left( \mu + \frac{1}{2} \right) (\log a)^n \frac{1}{2} L_n^\mu(x), \left( \Re(\mu) > -\frac{1}{2} \right) \]  

(2.6.15)

where \( L_n^\mu(x) \) is the generalized Laguerre polynomial.

### 2.7 Fractional Integrals and Derivatives

By using the definitions of fractional integrals and fractional derivatives given in section (1.7) of chapter one we have obtained the following fractional integrals and fractional derivatives for modified Hermite polynomials \( H_n(x; a) \) as given below:

\[ I^n \{ H_n(x; a) \} = \frac{1}{(1 + n)\mu (2 \log a)^\mu} H_{n+\mu}(x; a) \]  

(2.7.1)

Riemann-Liouville left sided fractional integral

\[ bI_x \{ H_n(x - b; a) \} = \frac{1}{(1 + n)_\alpha (2 \log a)^\alpha} H_{n+\alpha}(x - b; a) \]  

(2.7.2)

Riemann-Liouville right sided fractional integral

\[ cI_x \{ H_n(c - x; a) \} = \frac{1}{(1 + n)_\alpha (2 \log a)^\alpha} H_{n+\alpha}(c - x; a) \]  

(2.7.3)

The Weyl integral of \( H_n(x; a) \) of order \( \alpha \)

\[ xW_\infty \{ H_n(x; a) \} = \frac{(-1)^\alpha}{(1 + n)_\alpha (2 \log a)^\alpha} H_{n+\alpha}(x; a) \]  

(2.7.4)
Erdelyi-Kober operator of first kind for $H_n(x; a)$,

$$I[\alpha, \eta; H_n(x; a)] = \frac{(2x \log a)^n}{(n + \eta)_\alpha} \left[ \frac{\triangle(2, -n), \triangle(2, -n - \alpha - \eta)}{\triangle(2, -n - \eta)} \right] - \frac{1}{x^2 \log a}$$  \hspace{1cm} (2.7.5)

Erdelyi-Kober operator of second kind

$$I[\alpha, \eta; H_n(x; a)] = \frac{(2x \log a)^n}{(-n + \eta)_\alpha} \left[ \frac{\triangle(2, -n + \eta), \triangle(2, -n + \eta + 1)}{\triangle(2, -n + \alpha + \eta)} \right] - \frac{1}{x^2 \log a}$$  \hspace{1cm} (2.7.6)

Saigo integral operator of first kind

$$I_{0+}^{\alpha, \beta, \eta}[H_n(x; a)] = \frac{(2 \log a)^n \Gamma(1 + n) \Gamma(1 + n + \eta - \beta)x^{n-\beta}}{\Gamma(1 + n - \beta) \Gamma(1 + n + \alpha + \eta)} \times 4F_2 \left[ \begin{array}{l} \triangle(2, -n + \beta), \triangle(2, -n - \alpha - \eta); \\ \triangle(2, -n + \beta - \eta); \end{array} \right] - \frac{1}{x^2 \log a}$$  \hspace{1cm} (2.7.7)

Saigo integral operator of second kind

$$I_{-}^{\alpha, \beta, \eta}[H_n(x; a)] = \frac{(2 \log a)^n \Gamma(1 - n) \Gamma(1 - n - \beta + \eta)x^{n-1}}{\Gamma(1 - n - \beta) \Gamma(1 - n + \alpha + \eta)} \times 6F_4 \left[ \begin{array}{l} \triangle(2, -n), \triangle(2, -n + 1), \triangle(2, -n - \beta + \eta + 1); \\ \triangle(2, -n - \beta + 1) \triangle(2, -n + \alpha + \eta + 1); \end{array} \right] - \frac{1}{x^2 \log a}$$  \hspace{1cm} (2.7.8)

The left sided Riemann-Liouville fractional derivative of order $\alpha$

$$bD_x^\alpha \{H_n(x - b; a)\} = \frac{(2 \log a)^\alpha \Gamma(1 + n) \Gamma(n - \alpha)}{\Gamma(1 + n - \alpha)} H_{n-\alpha}(x - b; a)$$  \hspace{1cm} (2.7.9)

The right sided Riemann-Liouville fractional derivative of order $\alpha$

$$xD_c^\alpha \{H_n(c - x; a)\} = \frac{(2 \log a)^\alpha \Gamma(1 + n) \Gamma(n - \alpha)}{\Gamma(1 + n - \alpha)} H_{n-\alpha}(c - x; a)$$  \hspace{1cm} (2.7.10)

The Weyl fractional derivative of $f(x)$ of order $\alpha$,

$$xD_\infty^\alpha \{H_n(x; a)\} = (-n)_\alpha (2 \log a)^\alpha H_{n-\alpha}(x; a)$$  \hspace{1cm} (2.7.11)
2.8 Hypergeometric Form of $H_n(x; a)$

Hypergeometric representation of modified Hermite polynomials $H_n(x; a)$ is as given below:

$$H_n(x; a) = (2x \log a)^n \left[ \frac{-\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}; 1}{-; \frac{x^2}{2} \log a} \right]$$  \hspace{1cm} (2.8.1)

2.9 Orthogonality

The orthogonality condition for modified Hermite polynomials $H_n(x; a)$ is as follows:

$$\int_{-\infty}^{+\infty} a^{-x^2} H_m(x; a) H_n(x; a) \, dx = \begin{cases} 0, & \text{for } m \neq n \\ 2^n n! (\log a)^n \sqrt{2}, & \text{for } m = n \end{cases} \hspace{1cm} (2.9.1)$$

Further, the following result holds for $H_n(x; a)$:

**Theorem:** For the modified Hermite polynomials $H_n(x; a)$

(a) $\int_{-\infty}^{+\infty} a^{-x^2} x^k H_n(x; a) \, dx = 0$, \hspace{1cm} $k = 0, 1, 2, \ldots, (n-1)$.

(b) The zeros of $H_n(x; a)$ are real and distinct.

(c) $\sum_{k=0}^{n} \frac{H_n(x; a) H_k(y; a)}{(2 \log a)^{k+1} k!} = \left\{ \frac{H_{n+1}(y; a) H_n(x; a) - H_{n+1}(x; a) H_n(y; a)}{2^{n+1} n! (y-x)} \right\} (\log a)^n$

2.10 Addition Theorem

The following addition theorems hold for $H_n(x; a)$:

$$\frac{(\lambda^2 + \mu^2)^{\frac{n}{2}}}{n!} H_n \left\{ \frac{\lambda z_1 + \mu z_2}{(\lambda^2 + \mu^2)^{\frac{1}{2}}}; a \right\} = \sum_{r+s=n} \frac{\lambda^r \mu^s}{r! s!} H_r(z_1; a) H_s(z_2; a) \hspace{1cm} (2.10.1)$$

$$\left( \sum_{r=1}^{m} \lambda_r z_r \right) \frac{n}{2} H_n \left\{ \frac{\sum_{r=1}^{m} \lambda_r z_r}{\left( \sum_{r=1}^{m} \lambda_r^2 \right)^{\frac{1}{2}}}; a \right\} = n! \sum_{p_r=0}^{m} \left\{ \prod_{r=1}^{m} \left( \frac{\lambda_r}{p_r} \right) ! H_{p_r}(z_r; a) \right\} \hspace{1cm} (2.10.2)$$

Putting $\lambda = \mu$, in (2.10.1) then

$$2^n H_n \left\{ \frac{z_1 + z_2}{\sqrt{2}}; a \right\} = \sum_{r=0}^{n} {n \choose r} H_r(z_1; a) H_{n-r}(z_2; a) \hspace{1cm} (2.10.3)$$
Further with \( z_1 = z_2 = x \), we have
\[
2^n H_n(\sqrt{2}x; a) = \sum_{r=0}^{n} \binom{n}{r} H_r(x; a) H_{n-r}(x; a)
\] (2.10.4)

### 2.11 Summation Formulae

A number of summation formulae for modified Hermite polynomials \( H_n(x; a) \) are as given below:

\[
\sum_{k=0}^{n} \left\{ \frac{2^k k!(\log a)^k}{(k + 1)!} \right\}^{-1} [H_k(x; a)]^2 = \left\{ \frac{2^n n!(\log a)^n}{(n + 1)!} \right\}^{-1} [H_{n+1}(x; a)]^2 - H_n(x; a) H_{n+2}(x; a)
\] (2.11.1)

\[
\sum_{k=0}^{\min(m,n)} (-2 \log a)^k k! \binom{m}{k} \binom{n}{k} H_{m-k}(x; a) H_{n-k}(x; a) = H_{m+n}(x; a)
\] (2.11.2)

\[
\sum_{k=0}^{\min(m,n)} (2 \log a)^k k! \binom{m}{k} \binom{n}{k} H_{m+n-2k}(x; a) = H_{m}(x; a) H_{n}(x; a)
\] (2.11.3)

\[
\sum_{k=0}^{n} \binom{n}{k} H_k(\sqrt{2}x; a) H_{n-k}(\sqrt{2}y; a) = 2^n H_n(x + y; a)
\] (2.11.4)

\[
\sum_{k=0}^{n} \binom{2n}{2k} H_{2k}(\sqrt{2}x; a) H_{2n-2k}(\sqrt{2}y; a) = 2^{n-1} \left\{ H_{2n}(x + y; a) + H_{2n}(x - y; a) \right\}
\] (2.11.5)

\[
\sum_{k=0}^{\min(m,n)} \binom{n}{k} H_{2k}(x; a) H_{2n-2k}(y; a) = (-1)^n n!(\log a)^n L_n \left\{ (x^2 + y^2) \log a \right\}
\] (2.11.6)

### 2.12 Expansion of Legendre Polynomials in a Series of \( H_n(x; a) \)

In terms of \( H_n(x; a) \) Legendre polynomial can be written as

\[
P_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\binom{n}{k} 2^{n-k} F_0[-k, \frac{1}{2} + n - k; -; \frac{1}{\log a}](1)^k (\frac{1}{2})_{n-k} (\log a)^{2k-n}}{k!(n-2k)!} H_{n-2k}(x; a)
\] (2.12.1)
Also, in terms of Legendre polynomials $P_n(x)$, modified Hermite polynomials $H_n(x; a)$ can be written as

$$H_n(x; a) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{I_n[-k; \frac{3}{2} + n - 2k; \log a]n!(1)^k(2n - 4k + 1)(\log a)^{n-k}}{k!(\frac{3}{2})_n^{2k}} \times P_{n-2k}(x) \quad (2.12.2)$$

### 2.13 Binomial and Trinomial Operator Representations

In this section we have obtained certain results of binomial and trinomial operator representation type for $H_n(x; a)$ by using their Rodrigues formula. The results obtained are as given below:

Let $\frac{d}{dx} \equiv D_x$ and $\frac{d}{dy} \equiv D_y$, then

$$(D_x + D_y)^n \{a^{-x^2} b^{-y^2} \} = (-1)^n a^{-x^2} b^{-y^2} \sum_{r=0}^{n} \binom{n}{r} H_{n-r}(x; a) H_r(y; b) \quad (2.13.1)$$

Again let $\frac{d}{dx} \equiv D_x$, $\frac{d}{dy} \equiv D_y$ and $\frac{d}{dz} \equiv D_z$, then

$$(D_x + D_y + D_z)^n \{a^{-x^2} b^{-y^2} c^{-z^2} \} = (-1)^n a^{-x^2} b^{-y^2} c^{-z^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \binom{n}{r+s}(-1)^{r+s} \frac{r!s!}{r!s!} \times H_{n-r-s}(x; a) H_r(y; b) H_s(z; c) \quad (2.13.2)$$

and

$$(D_1 D_2 + D_1 D_3 + D_2 D_3)^n \{a^{-x^2} b^{-y^2} c^{-z^2} \} = a^{-x^2} b^{-y^2} c^{-z^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \binom{n}{r+s}(-1)^{r+s} \frac{r!s!}{r!s!} \times H_{n-s}(x; a) H_{n-r}(y; b) H_{r+s}(z; c) \quad (2.13.3)$$

### 2.14 Gauss Transforms

The following results hold when we apply Gauss transform defined by equation (1.4.7) on modified Hermite polynomials

$$g_x^n \{H_n(t; a)\} = (\log a)^{\frac{n-1}{2}} \left\{1 - 2a \log a\right\}^\frac{n}{2} H_n \left\{1 - 2a \log a\right\}^{-\frac{1}{2}} \frac{x}{\sqrt{\log a}} \quad (2.14.1)$$
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2.15 Laplace Transforms

Using the definition (1.4.4), the Laplace transform of modified Hermite polynomials \( H_n(x; a) \) is as follows:

\[
\mathcal{L}\{H_n(x; a) : s\} = \left(\frac{2 \log a}{s^{n+1}}\right)^n e^{-\frac{x^2}{4 \log a}}
\]  

(2.15.1)

2.16 Mellin Transforms

Also, the Mellin transform of \( H_n(x; a) \) is given by

\[
M\{e^{-x}H_n(x; a) : s\} = (2 \log a)^n \Gamma(n + s) \times \text{}_2F_2\left[\begin{array}{c}
-\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}, \\
-n-s+1, -n-s+2
\end{array} ; \frac{1}{4 \log a}\right]
\]

(2.16.1)

2.17 Limiting Relationships

The following relationships between Gegenbauer and Laguerre polynomials with \( H_n(x; a) \) exist:

\[
H_n(x; a) = n!(\log a)^{\frac{n}{2}} \lim_{|\nu| \to \infty} \left\{ \nu^{-\frac{n}{2}} C_n^\nu \left( \frac{x \sqrt{\log a}}{\sqrt{\nu}} \right) \right\},
\]

(2.17.1)

\[
H_n\left(\frac{x}{\sqrt{2}} ; a\right) = (-1)^n 2^n n!(\log a)^{\frac{n}{2}} \lim_{|\alpha| \to \infty} \left\{ \alpha^{-\frac{n}{2}} L_n^{(\alpha)} \left( \alpha + x \sqrt{\alpha \log a} \right) \right\}.
\]

(2.17.2)

2.18 Other Results for \( H_n(x; a) \)

Using the identity

\[
a^{2xt-t^2} = a^{(2xt-t^2)x}.a^{[t^2(x^2-1)]},
\]
we get the following result for $H_n(x; a)$

$$H_n(x; a) = \sum_{k=0}^{[n/2]} \frac{n! H_{n-2k}(x; a) x^{n-2k} (x^2 - 1)^k (\log a)^k}{k!(n-2k)!}. \quad (2.18.1)$$

Also, we have

$$\sum_{n=0}^{\infty} \frac{H_{2n}(x; a) t^n}{n!} = a^{-t} \, {}_0F_1 \left[ \begin{array}{c} -t \\ \frac{1}{2} \end{array} ; x^2 t (\log a)^2 \right], \quad (2.18.2)$$

and

$$\sum_{n=0}^{\infty} \frac{H_{2n+1}(x; a) t^n}{(2n+1)!} = 2 x \log a a^{-t} \, {}_0F_1 \left[ \begin{array}{c} -t \\ \frac{3}{2} \end{array} ; x^2 t (\log a)^2 \right]. \quad (2.18.3)$$

### 2.19 Modified Hermite Polynomials of Two Variables

In this section a study has been made of modified Hermite polynomials of two variables, denoted by $H_n(x, y; a)$ which for $a = e$, reduces to Hermite polynomials of two variables $H_n(x, y)$ defined and studied by Khan, M.A. and Abukhammash, G.S. [23]. They defined $H_n(x, y)$ by means of the following generating relation:

$$e^{2xt-(y+1)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x, y) t^n}{n!} \quad (2.19.1)$$

Here the definition (2.19.1) has been modified and some results such as generating functions, recurrence relations, Rodrigues formula, relationship with Legendre polynomials, expansion of polynomials and other properties for the modified Hermite polynomials of two variables $H_n(x, y; a)$ have been obtained.

The modified Hermite polynomials $H_n(x, y; a)$ of two variables is defined by means of the generating relation:

$$a^{2t-(y+1)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x, y; a) t^n}{n!}, \quad a > 0, \quad a \neq 1 \quad (2.19.2)$$

It follows from (2.19.2) that

$$H_n(x, y; a) = \sum_{r=0}^{[n/2]} \frac{n! H_{n-2r}(x; a) (-y)^r (\log a)^r}{r!(n-2r)!} \quad (2.19.3)$$
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where \( H_n(x; a) \) stands for the modified Hermite polynomial of one variable [25].

The definition (2.19.3) is equivalent to the following explicit representation of \( H_n(x, y; a) \)

\[
H_n(x, y; a) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{s=0}^{n-r} \frac{(-n)_{2r+2s}(2x)^{n-2r-2s}(-y)^r(-1)^s(\log a)^{n-r-s}}{r!s!}(2.19.4)
\]

In terms of double hypergeometric function, modified Hermite polynomials of two variables can be written as

\[
H_n(x, y; a) = (2x \log a)^n F_{2:0;0}^{2:0;0} \left[ \begin{array}{c} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2} : -; \\ -; -; \end{array} \right] - \frac{y}{x^2 \log a}, -\frac{1}{x^2 \log a} \right]
\]

(2.19.5)

For \( a = e \), (2.19.3), (2.19.4) and (2.19.5) reduces to Hermite polynomials of two variables \( H_n(x, y) \) due to Khan, M.A. and Abukhammash, G.S. [23].

It may be remarked that \( H_n(x, y; a) \) is an even function of \( x \) for even \( n \), an odd function of \( x \) for odd \( n \).

\[
H_n(-x, y; a) = (-1)^n H_n(x, y; a)
\]

Also,

\[
\begin{align*}
H_{2n}(0, y; a) &= (-1)^n (y + 1)^{\frac{n}{2}} 2^{2n} \left( \frac{1}{2} \right)_n (\log a)^n, \\
H_{2n+1}(0, y; a) &= 0
\end{align*}
\]

(2.19.6)

\[
\begin{align*}
H_{2n}(0, 0; a) &= (-1)^n 2^{2n} \left( \frac{1}{2} \right)_n (\log a)^n, \\
H_{2n+1}(0, 0; a) &= 0
\end{align*}
\]

(2.19.7)

and

\[
\begin{align*}
\frac{\partial}{\partial x} H_{2n}(0, y; a) &= 0, \\
\frac{\partial}{\partial x} H_{2n+1}(0, y; a) &= (-1)^n (2 \log a)^{n+1} \left( \frac{3}{2} \right)_n (y + 1)^{\frac{n+1}{2}}
\end{align*}
\]

(2.19.8)
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2.20 Recurrence Relations for $H_n(x, y; a)$

We have

$$a^{2xt-(y+1)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^n}{n!}$$

(2.20.1)

Differentiating (2.20.1) partially w.r.t. ‘$x$’, we get

$$(2t \log a) a^{2xt-(y+1)t^2} = \sum_{n=0}^{\infty} \frac{\partial \partial x H_n(x, y; a)t^n}{n!}.$$  (2.20.2)

or

$$\left(2 \log a\right) \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{\partial \partial x H_n(x, y; a)t^n}{n!}.$$  (2.20.5)

Equating the coefficients of $t^n$ on both sides, we get

$$\frac{\partial}{\partial x} H_n(x, y; a) = 2n \log a H_{n-1}(x, y; a)$$  (2.20.3)

Iteration of (2.20.3) gives

$$\frac{\partial^s}{\partial x^s} H_n(x, y; a) = \frac{(2 \log a)^s n! H_{n-s}(x, y; a)}{(n-s)!}$$  (2.20.4)

Again differentiating (2.20.1) partially w.r.t. ‘$y$’, we get

$$(-t^2 \log a) a^{2xt-(y+1)t^2} = \sum_{n=0}^{\infty} \frac{\partial \partial y H_n(x, y; a)t^n}{n!}$$  (2.20.5)

or,

$$(- \log a) \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^{n+2}}{n!} = \sum_{n=0}^{\infty} \frac{\partial \partial y H_n(x, y; a)t^n}{n!}.$$  (2.20.5)

Equating the coefficients of $t^n$ on both sides, we obtain

$$\frac{\partial}{\partial y} H_n(x, y; a) = -n(n-1) \log a H_{n-2}(x, y; a)$$  (2.20.6)

Iteration of (2.20.6) gives

$$\frac{\partial^r}{\partial y^r} H_n(x, y; a) = \frac{(-1)^r n!}{(n-2r)!} H_{n-2r}(x, y; a)$$  (2.20.7)
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Now differentiating (2.20.1) partially w.r.t. ‘t’, we get

\[
2(x - y t - t) \log a a^{2x -(y+1)t^2} = \sum_{n=1}^{\infty} \frac{H_n(x, y; a)t^{n-1}}{(n - 1)!} \quad (2.20.8)
\]

or,

\[
(2x \log a) \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^n}{n!} - (2y \log a) \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^{n+1}}{n!} - (2 \log a) \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{H_n(x, y; a)t^{n-1}}{(n - 1)!}.
\]

Now equating the coefficients of \(t^n\) on both sides, we obtain

\[
H_{n+1}(x, y; a) = (2x \log a) \left\{ xH_n(x, y; a) - n(y - 1)H_{n-1}(x, y; a) \right\} \quad (2.20.9)
\]

Also, by multiplying (2.20.2), (2.20.5) and (2.20.8) by \((x - t), 2y\) and \(-t\) respectively and adding, we get

\[
(x - t) \sum_{n=0}^{\infty} \frac{\partial}{\partial x} H_n(x, y; a)t^n \left/ n! \right. + 2y \sum_{n=0}^{\infty} \frac{\partial}{\partial y} H_n(x, y; a)t^n \left/ n! \right. - \sum_{n=0}^{\infty} \frac{nH_n(x, y; a)t^n}{n!} = 0
\]

Now equating the coefficients of \(t^n\) on both sides, we get

\[
x \frac{\partial}{\partial x} H_n(x, y; a) - nH_n(x, y; a) + 2y \frac{\partial}{\partial y} H_n(x, y; a) = n \frac{\partial}{\partial y} H_{n-1}(x, y; a). \quad (2.20.10)
\]

Using (2.20.3), (2.20.6) and (2.20.10) we get

\[
2n \log a H_{n-1}(x, y; a) - nH_n(x, y; a) - 2n(y - 1) \log a H_{n-2}(x, y; a)
\]

\[
= n \frac{\partial}{\partial x} H_{n-1}(x, y; a) \quad (2.20.11)
\]

Now from (2.20.3) and (2.20.11), we get

\[
2n \log a H_{n-1}(x, y; a) - nH_n(x, y; a) = 2n(n - 1)(y + 1)H_{n-2}(x, y; a) \quad (2.20.12)
\]

which is a pure recurrence relation.
2.21 **Relation Between** \( H_n(x, y; a) \) **and** \( H_n(x; a) \)

From our earlier section, we have

\[
a^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x; a) \frac{t^n}{n!}
\]

(2.21.1)

Replacing \( x \) by \( \frac{x}{\sqrt{y}+1} \) and \( t \) by \( t\sqrt{y}+1 \) in (2.21.1) we get

\[
a^{2xt-(y+1)t^2} = \sum_{n=0}^{\infty} H_n(\frac{x}{\sqrt{y}+1}; a)(t\sqrt{y}+1)^n \frac{t^n}{n!}
\]

\[
\sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^n}{n!} = \sum_{n=0}^{\infty} \frac{H_n(\frac{x}{\sqrt{y}+1}; a) (y+1)^{\frac{n}{2}}}{n!}
\]

Equating the coefficients of \( t^n \) on both sides, we get

\[
H_n(x, y; a) = (y+1)^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{y}+1}; a\right)
\]

(2.21.2)

2.22 **More Generating Function for** \( H_n(x, y; a) \)

Consider the sum

\[
\sum_{n=0}^{\infty} \frac{(c)_n H_n(x, y; a)t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(c)_n (-n)_{2r+2s}(2x)^{n-2r-2s}(-y)^r(-1)^s (\log a)^n-r-s t^n}{r!s!n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (c)_{n+2r+2s}(2x)^{n}(-y)^r(-1)^s (\log a)^{n+r+s} t^{n+2r+2s}
\]

\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(c)_{2r+2s}(-y)^r(-1)^s (\log a)^{r+s} t^{2r+2s}}{r!s!} \sum_{n=0}^{\infty} \frac{(c+2r+2s)n(2xt \log a)^n}{n!}
\]

\[
= (1 - 2xt \log a)^{c-2r-2s} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{2^{2r+2s} \left(\frac{c}{2}\right)_r \left(\frac{c+1}{2}\right)_s}{r!s!} \frac{(-1)^{r+s} y^r (t^2 \log a)^{r+s}}{r!s!}
\]

\[
= (1-2xt \log a)^{-c} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(\frac{c}{2} + \frac{1}{2}\right)_{r+s}}{r!s!} \left\{ \frac{-4yt \log a}{(1-2xt \log a)^2} \right\}^r \left\{ \frac{-4t^2 \log a}{(1-2xt \log a)^2} \right\}^s
\]
We thus arrived at the (divergent) generating function
\[
\sum_{n=0}^{\infty} \frac{(c)_n H_n(x, y; a)t^n}{n!} 
\]
\[
\cong (1 - 2xt \log a)^{-c} F_{2:0;0}^{2:0;0} \left[ \begin{array}{c}
\frac{c}{2}, \frac{c}{2} + \frac{1}{2} ; -; -; \\
-; -; \frac{4yt^2 \log a}{(1 - 2xt \log a)^2}, \frac{4t^2 \log a}{(1 - 2xt \log a)^2}
\end{array} \right]
\]
(2.22.1)

Consider the Sum
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{H_{n+k}(x, y; a)t^{n+k}}{k!n!} = \sum_{n=0}^{\infty} \frac{H_n(x, y; a)}{n!} \sum_{k=0}^{n} \frac{n!t^{n-k}v^k}{k!(n-k)!} 
\]
\[
= \sum_{n=0}^{\infty} \frac{H_n(x, y; a)(t + v)^n}{n!} 
\]
\[
= a^{2x(t+v)-(y+1)(t+v)^2} 
\]
\[
= a^{2x-(y+1)^2} a^{2v(x-ty-t)-(y+1)v^2} 
\]
or
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{H_{n+k}(x, y; a)t^{n+k}}{k!n!} = a^{2x-(y+1)^2} \sum_{k=0}^{\infty} \frac{H_k(x - ty - t, y; a)v^k}{k!}. 
\]

By equating the coefficients of \( \frac{v^k}{k!} \), we obtain
\[
\sum_{n=0}^{\infty} \frac{H_{n+k}(x, y; a)t^n}{n!} = a^{2x-(y+1)^2} H_k(x - ty - t, y; a) 
\]  
(2.22.2)

Again consider the sum
\[
\sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{H_{n+r+s}(x, y; a)t^{n}u^rv^s}{r!s!n!} = \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^{n-r-s}u^rv^s}{r!s!(n-r-s)!} 
\]
\[
= \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^n}{n!} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-1)^{r+s}\left(\frac{u}{7}\right)^r\left(\frac{v}{7}\right)^s}{r!s!} 
\]
\[
= \sum_{n=0}^{\infty} \frac{H_n(x, y; a)(t + u + v)^n}{n!} 
\]
\[
= a^{2x(t+u+v)-(y+1)(t+u+v)^2} 
\]
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\[ a^{2x^2 - (y+1)t^2} a(x^2 - ty - t - uy - u) \sum_{s=0}^{\infty} \frac{H_s(x - ty - t - uy - u; a) v^s}{s!} \]

or

\[ \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{H_{n+r+s}(x, y; a) t^n u^r v^s}{r! s! n!} \]

\[ = a^{2x^2 - (y+1)t^2} a(x^2 - ty - t - uy - u; a) v^s \]

Equating the coefficients of \( \frac{v^s}{s!} \), we get

\[ \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{H_{n+r+s}(x, y; a) t^n u^r}{n! r!} \]

\[ = a^{2x^2 - (y+1)t^2} a(x^2 - ty - t - uy - u; a) H_r(x - ty - t - uy - u; a) \]

### 2.23 Rodrigues Formula for \( H_n(x, y; a) \)

From equation (2.19.2) we have

\[ a^{2x^2 - (y+1)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x, y; a) t^n}{n!} \]

(2.23.1)

Examination of (2.23.1) in the light of Maclaurin’s theorem gives us at once

\[ H_n(x, y; a) = \left[ \frac{d^n}{dt^n} a^{2x^2 - (y+1)t^2} \right]_{t=0} \]

The function \( a^{-\frac{x^2}{\sqrt{y+1}}} \) is independent of \( t \), so we can write

\[ a^{-\frac{x^2}{\sqrt{y+1}}} H_n(x, y; a) = \left[ \frac{d^n}{d\omega^n} a^{-\left(\frac{x}{\sqrt{y+1}} - \sqrt{y+1}t\right)} \right]_{\omega=\frac{x}{\sqrt{y+1}}} \]

Now put \( \frac{x}{\sqrt{y+1}} - \sqrt{y+1}t = \omega \), then

\[ a^{-\frac{x^2}{\sqrt{y+1}}} H_n(x, y; a) = (-1)^n (y+1)^{\frac{n}{2}} \left[ \frac{d^n}{d\omega^n} a^{-\omega^2} \right]_{\omega=\frac{x}{\sqrt{y+1}}} \]

\[ = (-1)^n (y+1)^{\frac{n}{2}} \frac{d^n}{d \left( \frac{x}{\sqrt{y+1}} \right)^2} a^{-\frac{x^2}{\sqrt{y+1}}} \]

or,

\[ H_n(x, y; a) = (-1)^n (y+1)^{\frac{n}{2}} a^{-\frac{x^2}{(y+1)}} D^n a^{-\frac{x^2}{(y+1)}} , \quad D \equiv \frac{d}{dx} \]

(2.23.2)

a formula of the same nature as Rodrigues’s formula for modified Hermite polynomial of one variable \( H_n(x) \).
2.24 Special Properties for \( H_n(x, y; a) \)

Here we explain some special properties of modified Hermite polynomials \( H_n(x, y; a) \) by the following manner. Consider the identity

\[
a^{2xt-(y+1)t^2} = a^{2(xt)-(y+1)(xt)^2} a^{(y+1)x^2-(y+1)t^2} = a^{2(xt)-(y+1)(xt)^2} a^{(y+1)(x^2-1)t^2}
\]

or,

\[
\sum_{n=0}^{\infty} \frac{H_n(x, y; a) t^n}{n!} = \sum_{n=0}^{\infty} \frac{H_n(1, y; a) (xt)^n}{n!} \sum_{k=0}^{\infty} \frac{(y+1)^k (x^2-1)^k (\log a)^k t^{2k}}{k!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{H_{n-2k}(1, y; a) x^{n-2k} (y+1)^k (x^2-1)^k (\log a)^k t^n}{k! (n-2k)!}
\]

Equating the coefficients of \( t^n \), we get

\[
H_n(x, y; a) = \sum_{k=0}^{\infty} \frac{n! H_{n-2k}(1, y; a) x^{n-2k} (y+1)^k (x^2-1)^k (\log a)^k}{k! (n-2k)!}.
\] (2.24.1)

Next, by considering the identity

\[
a^{2(x_1+x_2)t-(y+1)t^2} = a^{2x_1t-(y+1)t^2} a^{2x_2t-(y+1)t^2} a^{(y+1)t^2}
\]

or,

\[
\sum_{n=0}^{\infty} \frac{H_n(x_1+x_2, y; a) t^n}{n!} = \sum_{n=0}^{\infty} \frac{H_n(x_1, y; a) t^n}{n!} \sum_{r=0}^{\infty} \frac{H_r(x_2, y; a) t^r}{r!} \sum_{s=0}^{\infty} \frac{(y+1)^s t^{2s} (\log a)^s}{s!} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{H_{n-r-2s}(x_1, y; a) H_r(x_2, y; a) (y+1)^s (\log a)^s t^n}{r! s! (n-r-2s)!}
\]

Equating the coefficients of \( t^n \), we obtain

\[
H_n(x_1+x_2, y; a) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{n! H_{n-r-2s}(x_1, y; a) H_r(x_2, y; a) (y+1)^s (\log a)^s}{r! s! (n-r-2s)!}.
\] (2.24.2)

Now considering the identity

\[
a^{2xt-(y_1+y_2+1)t^2} a^{2xt-t^2} = a^{2xt-(y_1+1)t^2} a^{2xt-(y_2+1)t^2},
\]
we get
\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{H_n(x, y_1 + y_2; a) H_k(x; a) t^{n+k}}{n!k!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{H_n(x, y_1; a) H_k(x, y_2; a) t^{n+k}}{n!k!}. \]

By equating the coefficients of \( t^{n+k} \), we get
\[ H_n(x, y_1 + y_2; a) H_k(x; a) = H_n(x, y_1; a) H_k(x, y_2; a) \tag{2.24.3} \]
where \( H_k(x; a) \) is modified Hermite polynomial of one variable [4].

Again, by considering the identity
\[ a^{2\lambda xt - (y+1)t^2} = a^{2xt - (y+1)t^2} . a^{2(\lambda - 1)xt}, \]
we have
\[ \sum_{n=0}^{\infty} \frac{H_n(\lambda x, y; a) t^n}{n!} = \sum_{n=0}^{\infty} \frac{H_n(x, y; a) t^n}{n!} \sum_{k=0}^{\infty} \frac{2^k(\lambda - 1)^k x^k (\log a)^k}{k!} \]
\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{H_{n-k}(x, y; a) 2^k(\lambda - 1)^k x^k (\log a)^k t^n}{k!(n-k)!}. \]

By equating the coefficients of \( t^n \) we get
\[ H_n(\lambda x, y; a) = \sum_{k=0}^{n} \frac{H_{n-k}(x, y; a) n!2^k(\lambda - 1)^k x^k (\log a)^k}{k!(n-k)!}. \tag{2.24.4} \]

Similarly, by considering the identity
\[ a^{2xt - (\lambda y+1)t^2} = a^{2xt - (y+1)t^2} . a^{(1-\lambda)yt^2}, \]
and proceeding as above we obtain
\[ H_n(x, \lambda y; a) = \sum_{k=0}^{[n/2]} n! H_{n-2k}(x, y; a) [(1 - \lambda)y \log a]^k k!(n-2k)! \tag{2.24.5} \]

Also by considering the identity
\[ a^{2\lambda xt - (\mu y+1)t^2} = a^{2xt - (y+1)t^2} . a^{2(\lambda - 1)xt - (1-\mu)y+1} . a^{t^2}, \]
we have
\[
\sum_{n=0}^{\infty} \frac{H_n(\lambda x, \mu y; a)t^n}{n!} = \sum_{n=0}^{\infty} H_n(x, y; a)t^n \sum_{r=0}^{\infty} \frac{H_r \{(\lambda - 1)x, (1 - \mu)y; a\} t^r}{r!} \sum_{s=0}^{\infty} \frac{(t^2 \log a)^s}{s!}.
\]

By equating the coefficients of \(t^n\) we get
\[
H_n(\lambda x, \mu y; a) = \sum_{r=0}^{n} \sum_{s=0}^{\frac{n-r}{2}} \frac{n!H_{n-r-2s}(x, y; a)H_r \{(\lambda - 1)x, (1 - \mu)y; a\}(\log a)^s}{r!s!(n-r-2s)!}.
\]

(2.24.6)

2.25 Expansion of Polynomials

We have
\[
ad^{2xt-(y+1)t^2} = \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^n}{n!}
\]

it follows that
\[
ad^{2xt} = a^{(y+1)t^2} \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^n}{n!}
\]
or,
\[
\sum_{n=0}^{\infty} \frac{(2xt \log a)^n}{n!} = \sum_{k=0}^{\infty} \left[\frac{(y + 1)t^2}{k!}\right]^k \sum_{n=0}^{\infty} \frac{H_n(x, y; a)t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\frac{n}{2}} \frac{H_{n-2k}(x, y; a)(y + 1)^k(\log a)^k t^n}{k!(n - 2k)!}.
\]

Equating coefficients of \(t^n\), we get
\[
\frac{(2x)^n}{n!} = \sum_{k=0}^{\frac{n}{2}} \frac{H_{n-2k}(x, y; a)(y + 1)^k(\log a)^{k-n}}{k!(n - 2k)!}.
\]

(2.25.1)

Now we have
\[
\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\frac{n}{2}} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k} t^n}{k!(n - 2k)!}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} (2x)^n t^{n+2k}}{k!n!}
\]
By using (2.25.1) we may write it as

\[
\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} \right)_{n+k+2s} H_n(x, y; a)(y + 1)^s (\log a)^{-n-s} t^{n+2k+2s}}{k! s! n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{k (-k)_s (-1)^k \left( \frac{1}{2} + n + k \right)_s \left( \frac{1}{2} \right)_{n+k} H_n(x, y; a)(y + 1)^s (\log a)^{-n-s} t^{n+2k}}{s! k! n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} 2 F_0 \left[ -k, \frac{1}{2} + n + k, \frac{y+1}{\log a} \right] (-1)^k \left( \frac{1}{2} \right)_{n+k} H_n(x, y; a)(\log a)^{-n} t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} 2 F_0 \left[ -k, \frac{1}{2} + n - k, \frac{y+1}{\log a} \right] (-1)^k \left( \frac{1}{2} \right)_{n-k} H_{n-2k}(x, y; a)(\log a)^{2k-n} t^n
\]

Equating the coefficients of \(t^n\), we get

\[
P_n(x) = \sum_{k=0}^{\left[ \frac{n}{2} \right]} 2 F_0 \left[ -k, \frac{1}{2} + n - k, \frac{y+1}{\log a} \right] (-1)^k \left( \frac{1}{2} \right)_{n-k} H_{n-2k}(x, y; a)(\log a)^{2k-n} t^n
\]

(2.25.2)

We now employ (2.25.1) to expand the Legendre polynomial of two variables

\[
P_n(x, y) = \sum_{k=0}^{\left[ \frac{n}{2} \right]} (-1)^k \left( \frac{1}{2} \right)_{n-k} (2x)^{n-2k}(y + 1)^k
\]

(defined and studied by Khan, M.A. and Ahmed, S. [28]) in a series of modified Hermite polynomials of two variables. Consider the series

\[
\sum_{n=0}^{\infty} P_n(x, y) t^n = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} \right)_{n+k} (2x)^n(1 + y)^k t^{n+2k}}{k! (n - 2k)!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} \right)_{n+k} (1 + y)^k H_{n-2r}(x, y; a)(\log a)^{-n} t^{n+2k}}{k! r! (n - 2r)!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} \right)_{n+k+2r} (1 + y)^k H_n(x, y; a)(\log a)^{-r-n} t^{n+2k+2r}}{k! r! n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} \right)_{n+k+r} (1 + y)^k H_n(x, y; a)(\log a)^{-r-n} t^{n+2k}}{(k-r)! r! n!}
\]
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\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^k(-k)_r}{k!r!n!} \left( \frac{1}{2} \right)_{n+k} \left( \frac{1}{2} + n + k \right)_r (1+y)^k H_n(x, y; a)(\log a)^{-r-n}t^{n+2k} \]

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} 2F_0 \left[ \begin{array}{c} -k, \frac{1}{2} + n - k; \frac{1}{\log a} \\ -1 \end{array} \right] \frac{(-1)^k \left( \frac{1}{2} \right)_{n-k} (1+y)^k (\log a)^{2k-n} t^{n}}{k!(n-2k)!} \times H_{n-2k}(x, y; a). \]

The final result is

\[ P_n(x, y) = \sum_{k=0}^{\infty} 2F_0 \left[ \begin{array}{c} -k, \frac{1}{2} + n - k; \frac{1}{\log a} \\ -1 \end{array} \right] \frac{(-1)^k \left( \frac{1}{2} \right)_{n-k} (1+y)^k (\log a)^{2k-n} t^{n}}{k!(n-2k)!} \times H_{n-2k}(x, y; a). \] (2.25.3)

### 2.26 Binomial and Trinomial Operator Representations of \(H_n(x, y; a)\)

Here we have obtained certain results of binomial and trinomial operator representation type for modified two variables Hermite polynomials \(H_n(x, y; a)\) by using their Rodrigues formula. The results obtained are as follows:

Let \( \frac{d}{d \left( \sqrt{xy + 1} \right)} \equiv D_1 \) and \( \frac{d}{d \left( \sqrt{wz + 1} \right)} \equiv D_2 \), then

\[ (D_1 + D_2)^n a^{-\frac{x^2}{y+1}} b^{-\frac{w^2}{z+1}} = (-1)^n (y+1)^{-\frac{n}{2}} a^{-\frac{x^2}{y+1}} b^{-\frac{w^2}{z+1}} \sum_{r=0}^{n} \binom{n}{r} \left( \sqrt{\frac{y+1}{2}} \right)^r \times H_{n-r}(x, y; a) H_r(w, z; b) \] (2.26.1)

Again let \( \frac{d}{d \left( \sqrt{xy + 1} \right)} \equiv D_1 \), \( \frac{d}{d \left( \sqrt{wz + 1} \right)} \equiv D_2 \) and \( \frac{d}{d \left( \sqrt{uvw + 1} \right)} \equiv D_3 \), then
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\[(D_1 + D_2 + D_3)^n a^{-\frac{w^2}{x+y}} b^{-\frac{w^2}{x+z}} c^{-\frac{w^2}{x+t}} = (-1)^n a^{-\frac{w^2}{x+y}} b^{-\frac{w^2}{x+z}} c^{-\frac{w^2}{x+t}} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-1)^{r+s}}{r!s!} \]

\[\times (v + 1)^{-\frac{n}{2}} \left( \sqrt{\frac{v + 1}{x + 1}} \right)^r \left( \sqrt{\frac{v + 1}{z + 1}} \right)^s H_{n-r-s}(u, v; a) H_r(w, x; b) H_s(y, z; c) \quad (2.26.2)\]

and

\[(D_1 D_2 + D_1 D_3 + D_2 D_3)^n a^{-\frac{w^2}{x+y}} b^{-\frac{w^2}{x+z}} c^{-\frac{w^2}{x+t}} = (v + 1)^{-\frac{n}{2}} (x + 1)^{-\frac{n}{2}} a^{-\frac{w^2}{x+y}} b^{-\frac{w^2}{x+z}} c^{-\frac{w^2}{x+t}} \]

\[\times \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-1)^{r+s}}{r!s!} \left( \sqrt{\frac{v + 1}{z + 1}} \right)^r \left( \sqrt{\frac{x + 1}{z + 1}} \right)^s H_{n-r-s}(u, v; a) H_{n-r}(w, x; b) \]

\[\times H_{r+s}(y, z; c) \quad (2.26.3)\]

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(i) A Study of modified Hermite polynomials, accepted for publication in Pro. Mathematica, Peeru.

(ii) A study of modified Hermite polynomials of two variables, communicated for publication.