CHAPTER 4

SYSTEM OF GENERALIZED VARIATIONAL INCLUSIONS

4.1. INTRODUCTION

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years. One of the most interesting and important problems in the theory of variational inclusions is the development of an efficient and implementable iterative algorithm. Various kinds of iterative methods have been studied to find the approximate solutions for variational inclusions. Among these methods, the proximal mapping method for solving variational inclusions has been widely used by many authors. For details, we refer to see [22,36,38,39,42-45,58,59,63,67,68,70,73,83,94,96,111,122,127,137,139,140] and the references therein.

In 2001, Huang and Fang [58] were the first to introduce the generalize m-accretive mapping and give the definition of the proximal mapping for the generalized m-accretive mapping in Banach spaces. Since then a number of researchers investigated several classes of generalized m-accretive mappings such as H-accretive, $(H, \eta)$-accretive, and $(A, \eta)$-accretive mappings, see for examples [22,35,36,38,42-45,58,68,70,73,127,137]. Sun et al. [122] introduced a new class of M-monotone mapping in Hilbert spaces. Recently, Zho and Huang [139,140] and Kazmi et al. [70] introduced and studied a class of $H(\cdot,\cdot)$-accretive mappings in Banach spaces, an natural extension of M-monotone mapping and studied variational inclusions involving these mappings. In recent past, the methods based on different classes of proximal mappings have been developed to study the existence of solutions and to discuss convergence and stability analysis of iterative algorithms for various classes of variational inclusions, see for example [22,35,36,38,39,42-45,58,68,70,73,111,122].

Very recently, Luo and Huang [83] introduced and studied a new class of B-monotone mappings in Banach spaces, an extension of H-monotone mappings [42]. They discussed some properties of the proximal mapping associated with B-monotone mapping and obtained some applications for solving variational inclusions in Banach spaces.
Motivated by recent work going in this direction, we consider a class of accretive mappings called generalized \( H(\cdot,\cdot) \)-accretive mappings, an natural generalization of accretive (monotone) mappings studied in [42-44,83,122,137,139], in Banach spaces. We prove that the proximal mapping of the generalized \( H(\cdot,\cdot) \)-accretive mapping is single-valued and Lipschitz continuous. Further, we consider a system of generalized variational inclusions involving generalized \( H(\cdot,\cdot) \)-accretive mappings in real \( q \)-uniformly smooth Banach spaces. Using proximal mapping method, we prove the existence and uniqueness of solution and suggest an iterative algorithm for the system of generalized variational inclusions. Furthermore, we discuss the convergence criteria of the iterative algorithm under some suitable conditions. Our results can be viewed as a refinement and improvement of some known results given in [42-44,83,122,137,139].

The remaining part of this chapter is organized as follows:

In Section 4.2, we define a class of generalized \( H(\cdot,\cdot) \)-accretive mappings and its associated class of proximal mappings in Banach spaces. Further, we prove that the proximal mapping is single-valued and Lipschitz continuous.

In Section 4.3, we consider a system of generalized variational inclusions (in short, SGVI) involving generalized \( H(\cdot,\cdot) \)-accretive mappings in real \( q \)-uniformly smooth Banach spaces. Using proximal and fixed-point mapping methods, we prove the existence and uniqueness of solution of SGVI.

In Section 4.4, we suggest an iterative algorithm for finding the approximate solution of SGVI and discuss its convergence analysis.

4.2. GENERALIZED \( H(\cdot,\cdot) \)-ACCRETIVE MAPPINGS

Throughout the chapter unless otherwise stated, we assume that \( E \) is \( q \)-uniformly smooth Banach space.

First, we recall the following concepts.

**Definition 4.2.1**[139]. Let \( A, B : E \to E \) be single-valued mappings and \( H : E \times E \to E \) be a nonlinear mapping. Then
(i) $H(A, \cdot)$ is said to be \textit{\(\alpha\)-strongly accretive} with respect to $A$ if there exists a constant $\alpha > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x - y) \rangle \geq \alpha \|x - y\|^q; \quad \forall x, y, u \in E;$$

(ii) $H(\cdot, B)$ is said to be \textit{\(\beta\)-relaxed accretive} with respect to $B$ if there exists a constant $\beta > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y) \rangle \geq -\beta \|x - y\|^q; \quad \forall x, y, u \in E;$$

(iii) $H(\cdot, \cdot)$ is said to be \textit{\(\alpha\beta\)-symmetric accretive} with respect to $A$ and $B$, if $H(A, \cdot)$ is \(\alpha\)-strongly accretive with respect to $A$ and $H(\cdot, B)$ is \(\beta\)-relaxed accretive with respect to $B$ with $\alpha \geq \beta$. and $\alpha = \beta$ if and only if $x = y, \forall x, y, u \in E$;

(iv) $H(\cdot, \cdot)$ is said to be \textit{\(\xi\)-Lipschitz continuous} with respect to the first argument if there exists a constant $\xi > 0$ such that

$$\|H(x, u) - H(y, u)\| \leq \xi \|x - y\|, \quad \forall x, y, u \in E;$$

(v) $H(\cdot, \cdot)$ is said to be \textit{\(\eta\)-Lipschitz continuous} with respect to the second argument if there exists a constant $\eta > 0$ such that

$$\|H(u, x) - H(u, y)\| \leq \eta \|x - y\|, \quad \forall x, y, u \in E.$$

**Definition 4.2.2**[83]. Let $M : E \times E \rightarrow 2^E$ be a set-valued mapping, and $f, g : E \rightarrow E$ be single-valued mappings.

(i) $M(f, \cdot)$ is said to be \textit{\(\alpha\)-strongly accretive} with respect to $f$ if there exists a constant $\alpha > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq \alpha \|x - y\|^q, \quad \forall x, y, w \in E, \forall u \in M(f(x), w), v \in M(f(y), w);$$

(ii) $M(\cdot, g)$ is said to be \textit{\(\beta\)-relaxed accretive} with respect to $g$ if there exists a constant $\beta > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq -\beta \|x - y\|^q, \quad \forall x, y, w \in E, \forall u \in M(w, g(x)), v \in M(w, g(y));$$

53
(iii) $M(., .)$ is said to be $\alpha \beta$-symmetric accretive with respect to $f$ and $g$ if $M(f, .)$ is said to be $\alpha$-strongly accretive with respect to $f$ and $M(., g)$ is said to be $\beta$-relaxed accretive with respect to $g$ with $\alpha \geq \beta$ and $\alpha = \beta$ if and only if $x = y$.

Now, we define the following concept.

**Definition 4.2.3.** Let $A, B, f, g : E \rightarrow E$ and $H : E \times E \rightarrow 2^E$ be single-valued mappings. Let $M : E \times E \rightarrow 2^E$ be a set-valued mapping. The mapping $M$ is said to be generalized $\alpha \beta$-$H(., .)$-accretive with respect to $A, B, f$ and $g$, if $M(f, g)$ is $\alpha \beta$-symmetric accretive with respect to $f$ and $g$, and $(H(A, B) + \lambda M(f, g))(E) = E$ for every $\lambda > 0$.

The following theorems give some properties of generalized $H(., .)$-accretive mappings.

**Theorem 4.2.1.** Let $A, B, f, g : E \rightarrow E$; let $H : E \times E \rightarrow E$ be $\alpha' \beta'$-symmetric accretive mapping with respect to $A$ and $B$ and $\alpha' > \beta'$, and let $M : E \times E \rightarrow E$ be a generalized $\alpha \beta$-$H(., .)$-accretive mapping with respect to mappings $A, B, f$ and $g$. If the following inequality: $\langle u - v, J_q(x - y) \rangle \geq 0$, holds for all $(v, y) \in \text{Graph}(M(f, g))$, then $(u, x) \in \text{Graph}(M(f, g))$, where $\text{Graph}(M(f, g)) := \{(u, x) \in E \times E : (u, x) \in M(f(x), g(x))\}$.

**Proof.** Suppose, on the contrary that there exists some $(u_0, x_0) \notin \text{Graph}(M(f, g))$ such that

$$\langle u_0 - v, J_q(x_0 - y) \rangle \geq 0, \quad \forall (v, y) \in \text{Graph}(M(f, g)). \quad (4.2.1)$$

Since $M$ is generalized $H(., .)$-accretive with respect to $A, B, f$ and $g$, we have that $(H(A, B) + \lambda M(f, g))(E) = E$ holds for every $\lambda > 0$, and hence there exists $(u_1, x_1) \in \text{Graph}(M(f, g))$ such that

$$H(Ax_1, Bx_1) + \lambda u_1 = H(Ax_0, Bx_0) + \lambda u_0 \in E. \quad (4.2.2)$$

It follows from (4.2.1) and (4.2.2) that

$$0 \leq \lambda \langle u_0 - u_1, J_q(x_0 - x_1) \rangle = -\langle H(Ax_0, Bx_0) - H(Ax_1, Bx_1), J_q(x_0 - x_1) \rangle$$

$$\leq -\langle (H(Ax_0, Bx_0) - H(Ax_1, Bx_0), J_q(x_0 - x_1)) - \langle (H(Ax_1, Bx_0) - H(Ax_1, Bx_1), J_q(x_0 - x_1)) \rangle$$

$$\leq - (\alpha' - \beta') ||x_0 - x_1||^p \leq 0,$$

54
which gives $x_1 = x_0$, since $\alpha' > \beta'$. By (4.2.2), we have $u_1 = u_0$, a contradiction. This completes the proof.

**Theorem 4.2.2.** Let $A, B, f, g : E \rightarrow E$ and let $H : E \times E \rightarrow E$ be $\alpha'\beta'$-symmetric accretive mapping with respect to $A$ and $B$. Let $M : E \times E \rightarrow 2^E$ be a generalized $\alpha\beta$-$H(., .)$-accretive mapping with respect to $A, B, f$ and $g$. Then the mapping $(H(A, B) + \lambda M(f, g))^{-1}$ is single-valued for all $\lambda > 0$.

**Proof.** For any $x^* \in E$, let $x, y \in (H(A, B) + \lambda M(f, g))^{-1}(x^*)$. It follows that

\[
\frac{1}{\lambda}(x^* - H(A, B)(x)) \in M(f, g)(x),
\]

and

\[
\frac{1}{\lambda}(x^* - H(A, B)(y)) \in M(f, g)(y).
\]

Since $M$ is $\alpha\beta$-symmetric accretive with respect to $f$ and $g$ and $H$ is $\alpha'\beta'$-symmetric accretive with respect to $A$ and $B$, we have

\[
(\alpha - \beta)||x - y||^q \leq \left( \frac{1}{\lambda}(x^* - H(A, B)(x)) - \frac{1}{\lambda}(x^* - H(A, B)(y)), J_q(x - y) \right)
\]

\[
= - \frac{1}{\lambda}(H(A, B)(x) - H(A, B)(y), J_q(x - y))
\]

\[
\leq - \frac{1}{\lambda}(\alpha' - \beta')||x - y||^q,
\]

which implies that

\[
[(\alpha - \beta) + \frac{1}{\lambda}(\alpha' - \beta')]||x - y||^q \leq 0.
\]

It follows from $\alpha \geq \beta$ and $\alpha' \geq \beta'$ that $x = y$ and so $(H(A, B) + \lambda M(f, g))^{-1}$ is single-valued. This completes the proof.

Based on Theorem 4.2.2, we can define the following proximal mapping $R_{H(., .), \lambda}^{M(., .), \lambda}$ for a generalized $\alpha\beta$-$H(., .)$-accretive mapping $M$ as follows:

**Definition 4.2.4.** Let $A, B, f, g : E \rightarrow E$ be single-valued mappings and let $H : E \times E \rightarrow E$ be $\alpha'\beta'$-symmetric accretive mapping with respect to $A$ and $B$. Let $M : E \times E \rightarrow 2^E$ be generalized $\alpha\beta$-$H(., .)$-accretive mapping with respect to the mappings $A, B, f$ and $g$. The proximal mapping $R_{H(., .), \lambda}^{M(., .), \lambda} : E \rightarrow E$ is defined by

\[
R_{H(., .), \lambda}^{M(., .), \lambda}(x) = (H(A, B) + \lambda M(f, g))^{-1}(x), \forall x \in E.
\]
Next, we prove that proximal mapping is Lipschitz continuous.

**Theorem 4.2.3.** Let $A, B, f, g : E \to E$ and let $H : E \times E \to E$ be $\alpha' \beta'$-symmetric accretive mapping with respect to $A$ and $B$. Suppose that $M : E \times E \to 2^E$ be a generalized $\alpha\beta$-symmetric accretive mapping with respect to the mappings $A, B, f$ and $g$. Then the proximal mapping $R_{M(\cdot, \lambda)}^{H(\cdot)} : E \to E$ is Lipschitz continuous with constant $L$, that is,

$$\| R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*) - R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*) \| \leq L\| x^* - y^* \|, \forall x^*, y^* \in E,$$

where $L := \frac{1}{\lambda(\alpha - \beta) + (\alpha' - \beta')}.$

**Proof.** Let $x^*, y^* \in E$. It follows that

$$R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*) = (H(A, B) + \lambda M(f, g))^{-1}(x^*),$$

and

$$R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*) = (H(A, B) + \lambda M(f, g))^{-1}(y^*),$$

and so

$$\frac{1}{\lambda}(x^* - H(A(R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*)), B(R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*)))) \in M(f(R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*)), g(R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*)))$$

and

$$\frac{1}{\lambda}(y^* - H(A(R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*)), B(R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*)))) \in M(f(R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*)), g(R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*))).$$

Since $M$ is $\alpha\beta$-symmetric accretive with respect to $f$ and $g$, we have

$$(\alpha - \beta)\| R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*) - R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*) \|^q \leq \frac{1}{\lambda}(x^* - H(A(R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*)), B(R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*))$$

$$- \frac{1}{\lambda}(y^* - H(A(R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*)), B(R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*)))) = \frac{1}{\lambda}(x^* - H(A(R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*)), B(R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*)$$

$$- \frac{1}{\lambda}(y^* + H(A(R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*)), B(R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*)), J_q(R_{M(\cdot, \lambda)}^{H(\cdot)}(x^*) - R_{M(\cdot, \lambda)}^{H(\cdot)}(y^*)).$$

Since $H$ is $\alpha' \beta'$-symmetric accretive with respect to $A$ and $B$, we have

56
\[ \|x^* - y^*\| \leq \|R_{M(i),\lambda}^{H(i)}(x^*) - R_{M(i),\lambda}^{H(i)}(y^*)\|^{q-1} \]
\[ \geq \langle x^* - y^*, R_{M(i),\lambda}^{H(i)}(x^*) - R_{M(i),\lambda}^{H(i)}(y^*) \rangle \]
\[ \geq \lambda(\alpha - \beta)\|R_{M(i),\lambda}^{H(i)}(x^*) - R_{M(i),\lambda}^{H(i)}(y^*)\|^q + \|H(A(R_{M(i),\lambda}^{H(i)}(x^*)), B(R_{M(i),\lambda}^{H(i)}(x^*))) - H(A(R_{M(i),\lambda}^{H(i)}(y^*)), B(R_{M(i),\lambda}^{H(i)}(y^*))) - J_q(R_{M(i),\lambda}^{H(i)}(x^*) - R_{M(i),\lambda}^{H(i)}(y^*)) \| \]
\[ \geq [\lambda(\alpha - \beta) + (\alpha' - \beta')]\|R_{M(i),\lambda}^{H(i)}(x^*) - R_{M(i),\lambda}^{H(i)}(y^*)\|^q \]
and so
\[ \|R_{M(i),\lambda}^{H(i)}(x^*) - R_{M(i),\lambda}^{H(i)}(y^*)\| \leq \frac{1}{[\lambda(\alpha - \beta) + (\alpha' - \beta')]^{q-1}}\|x^* - y^*\|. \]

This completes the proof.

**Remark 4.2.1.** The Definitions 4.2.3-4.2.4 and Theorems 4.2.1-4.2.3 generalize the corresponding concepts and results given in [42-44,83,122,137,139].

### 4.3. FORMULATION OF PROBLEM AND EXISTENCE THEOREM

Throughout the chapter unless otherwise stated, we assume that, for each \(i = 1, 2\), \(E_i\) is \(q_i\)-uniformly smooth Banach space with norm \(\|\cdot\|\).

Let \(A_1, B_1, f_1, g_1 : E_1 \to E_1\), \(A_2, B_2, f_2, g_2 : E_2 \to E_2\) be nonlinear mappings; let \(F_1, H_1 : E_1 \times E_2 \to E_1\), \(F_2, H_2 : E_1 \times E_2 \to E_2\) be nonlinear mappings, and let \(M_1 : E_1 \times E_1 \to 2^{E_1}\) and \(M_2 : E_2 \times E_2 \to 2^{E_2}\) be generalized \(\alpha_1\beta_1\)-accretive and generalized \(\alpha_2\beta_2\)-accretive mappings, respectively. We consider the following system of generalized variational inclusions (SGVI):

Find \((x, y) \in E_1 \times E_2\) such that
\[
\begin{cases}
F_1(x, y) + M_1(f_1(x), g_1(x)) \ni \theta_1; \\
F_2(x, y) + M_2(f_2(y), g_2(y)) \ni \theta_2,
\end{cases}
\]
where \(\theta_1\) and \(\theta_2\) are zero vectors of \(E_1\) and \(E_2\), respectively.

We remark that for suitable choices of the mappings \(A_1, A_2, B_1, B_2, f_1, f_2, F_1, F_2, g_1, g_2, H_1, H_2, M_1, M_2\) and the spaces \(E_1, E_2\), SVLI (4.3.1) reduces to various classes of the system of variational inclusions and system of variational inequalities, see for example [35,42-45,58,59,63,67,73,94,96,111,122,127,137] and the references therein.

**Definition 4.3.1.** Let \(A : E_1 \to E_1\). A mapping \(F : E_1 \times E_2 \to E_1\) is said to be:
(i) $\alpha$-strongly accretive with respect to $A$ in the first argument if there exists a constant $\alpha > 0$ such that

$$\langle F(x_1, y) - F(x_2, y), J_q(A(x_1) - A(x_2)) \rangle \geq \alpha \|x_1 - x_2\|_1^q, \quad \forall x_1, x_2 \in E_1, y \in E_2;$$

(ii) $(\beta, \gamma)$-mixed Lipschitz continuous if there exist constants $\beta > 0, \gamma > 0$ such that

$$\|F(x_1, y_1) - F(x_2, y_2)\|_1 \leq \beta \|x_1 - x_2\|_1 + \gamma \|y_1 - y_2\|_2, \quad \forall x_1, x_2 \in E_1, y_1, y_2 \in E_2.$$

**Remark 4.3.1.** If $A$ is $\xi$-Lipschitz continuous, it follows from Definition 4.3.1 and Definition 1.2.18 that $\alpha \leq \beta \xi^{q_1 - 1}$.

The following lemma, which will be used in the sequel, is an immediate consequence of the definitions of $R^{H_1(., .), \lambda_1}_{M_1(., .)}$, $R^{H_2(., .), \lambda_2}_{M_2(., .)}$.

**Lemma 4.3.1.** For any given $(x, y) \in E_1 \times E_2$, $(x, y)$ is a solution of SVLI (4.3.1) if and only if $(x, y)$ satisfies

$$x = R^{H_1(., .), \lambda_1}_{M_1(., .)}[H_1(A_1, B_1)(x) - \lambda_1 F_1(x, y)],$$

$$y = R^{H_2(., .), \lambda_2}_{M_2(., .)}[H_2(A_2, B_2)(y) - \lambda_2 F_2(x, y)],$$

where $\lambda_1, \lambda_2 > 0$ are constants; $R^{H_1(., .), \lambda_1}_{M_1(., .)}(x) \equiv (H_1(A_1, B_1) + \lambda_1 M(f_1, g_1))^{-1}(x)$;

$R^{H_2(., .), \lambda_2}_{M_2(., .)}(y) \equiv (H_2(A_2, B_2) + \lambda_2 M(f_2, g_2))^{-1}(y), \quad \forall x \in E_1, y \in E_2.$

Now, we prove the existence and uniqueness of solution for SGVI (4.3.1)

**Theorem 4.3.1.** For each $i = 1, 2$, let $E_i$ be $q_i$-uniformly smooth Banach spaces; let $A_i, B_i, f_i, g_i : E_i \to E_i$ be single valued mappings. Let $H_i : E_1 \times E_2 \to E_i$ be $\alpha_i \beta_i$-symmetric accretive mappings with respect to $A_i$ and $B_i$ and $(\nu_i, \delta_i)$-mixed Lipschitz continuous. Let $F_i : E_1 \times E_2 \to E_i$ be $\mu_i$-strongly accretive mapping in the $i^{th}$ argument and $(L_{F_i}, l_{F_i})$-mixed Lipschitz continuous; let $M_i : E_1 \times E_1 \to 2^{E_1}$ be generalized $\alpha_1 \beta_1 H_1(., .)$-accretive mappings with respect to $A_1, B_1, f_1$ and $g_1$ and $M_2 : E_2 \times E_2 \to 2^{E_2}$ be generalized $\alpha_2 \beta_2 H_2(., .)$-accretive mappings with respect to $A_2, B_2, f_2$ and $g_2$. Suppose that there are constants $\lambda_1, \lambda_2 > 0$ satisfy the following condition:

$$\begin{cases} k_1 := m_1 + \lambda_2 L_2 L_{F_2} < 1; \\ k_2 := m_2 + \lambda_1 L_1 l_{F_1} < 1, \end{cases}$$

(4.3.2)
Lipschitz continuous, respectively.

where

\[ m_1 := L_1 [(1 - 2q_1(\alpha'_1 - \beta'_1) + c_{q_1}(\nu_1 + \delta_1)^{q_1})^{\frac{1}{q_1}} + (1 - 2\lambda_1 q_1 \mu_1 + c_{q_1} \lambda_1^{q_1} L_{F_1} q_1) \frac{1}{q_1}]; \]

\[ m_2 := L_2 [(1 - 2q_2(\alpha'_2 - \beta'_2) + c_{q_2}(\nu_2 + \delta_2)^{q_2})^{\frac{1}{q_2}} + (1 - 2\lambda_2 q_2 \mu_2 + c_{q_2} \lambda_2^{q_2} L_{F_2} q_2) \frac{1}{q_2}]; \]

\[ L_1 := \frac{1}{[\lambda_1(\alpha_1 - \beta_1) + (\alpha'_1 - \beta'_1)]}; \]

\[ L_2 := \frac{1}{[\lambda_2(\alpha_2 - \beta_2) + (\alpha'_2 - \beta'_2)]}. \]

Then SGVI (4.3.1) has a unique solution.

**Proof.** For \( i = 1, 2 \), it follows that for \((x, y) \in E_1 \times E_2\) the proximal mappings \( R_{H_1,\lambda}^{M_1} \) and \( R_{H_2,\lambda}^{M_2} \) are \( \frac{1}{[\lambda_1(\alpha_1 - \beta_1) + (\alpha'_1 - \beta'_1)]} \)-Lipschitz continuous and \( \frac{1}{[\lambda_2(\alpha_2 - \beta_2) + (\alpha'_2 - \beta'_2)]} \)-Lipschitz continuous, respectively.

Next define a mapping \( Q : E_1 \times E_2 \to E_1 \times E_2 \) by

\[ Q(x, y) = (T(x, y), S(x, y)), \quad \forall (x, y) \in E_1 \times E_2, \quad (4.3.3) \]

where \( T : E_1 \times E_2 \to E_1 \) and \( S : E_1 \times E_2 \to E_2 \) are defined by

\[ T(x, y) = R_{H_1,\lambda}^{M_1}[H_1(A_1, B_1)(x) - \lambda_1 F_1(x, y)], \quad (4.3.4) \]

and

\[ S(x, y) = R_{H_2,\lambda}^{M_2}[H_2(A_2, B_2)(y) - \lambda_2 F_2(x, y)], \quad (4.3.5) \]

for \( \lambda_1, \lambda_2 > 0 \), respectively.

For any \((x_1, y_1), (x_2, y_2) \in E_1 \times E_2\), it follows from (4.3.4) and (4.3.5) and Lipschitz continuity of \( R_{H_1,\lambda}^{M_1} \) and \( R_{H_2,\lambda}^{M_2} \) that

\[ ||T(x_1, y_1) - T(x_2, y_2)||_1 \]

\[ \leq ||R_{H_1,\lambda}^{M_1}[H_1(A_1, B_1)(x_1) - \lambda_1 F_2(x_1, y_1)] - R_{H_1,\lambda}^{M_1}[H_1(A_1, B_1)(x_2) - \lambda_1 F_1(x_2, y_2)||_1 \]

\[ \leq L_1[||H_1(A_1, B_1)(x_1) - H_1(A_1, B_1)(x_2) - \lambda_1(F_1(x_1, y_1) - F_1(x_2, y_1)||_1 \]

\[ + \lambda_1||F_1(x_2, y_1) - F_1(x_2, y_2)||_1], \quad (4.3.6) \]

and

59
\[\|S(x_1, y_1) - S(x_2, y_2)\|_2 \]
\[\leq L_2 \|H_2(A_2, B_2)(y_1) - H_2(A_2, B_2)(y_2) - \lambda_2(F_2(x_1, y_1) - F_2(x_1, y_2))\|_2 + \lambda_2\|F_2(x_1, y_2) - F_2(x_2, y_2)\|_2. \quad (4.3.7)\]

Now,
\[\|H_1(A_1, B_1)(x_1) - H_1(A_1, B_1)(x_2) - \lambda_1(F_1(x_1, y_1) - F_1(x_1, y_1))\|_1 \]
\[\leq \|H_1(A_1, B_1)(x_1) - H_1(A_1, B_1)(x_2) - (x_1 - x_2)\|_1 + \|x_1 - x_2 - \lambda_1(F_2(x_1, y_1) - F_1(x_2, y_1))\|_1. \quad (4.3.8)\]

Also,
\[\|H_2(A_2, B_2)(y_1) - H_2(A_2, B_2)(y_2) - \lambda_2(F_2(x_1, y_1) - F_2(x_1, y_2))\|_2 \]
\[\leq \|H_2(A_2, B_2)(y_1) - H_2(A_2, B_2)(y_2) - (y_1 - y_2)\|_2 + \|y_1 - y_2 - \lambda_2(F_1(x_1, y_1) - F_2(x_1, y_2))\|_2. \quad (4.3.9)\]

Since, for \(i = 1, 2\), \(H_i\) is \(\alpha', \beta'_i\)-symmetric with respect to \(A_i\) and \(B_i\) and \((\nu_i, \delta_i)\)-mixed Lipschitz continuous, then using Lemma 1.2.4, we have
\[\|H_1(A_1, B_1)(x_1) - H_1(A_1, B_1)(x_2) - (x_1 - x_2)\|_1^{q_1} \]
\[\leq \|x_1 - x_2\|^{q_1} - 2q_1\langle H_1(A_1, B_1)(x_1) - H_1(A_1, B_1)(x_2), J_{q_1}(x_1 - x_2)\rangle_1 + c_{q_1}\|H_1(A_1, B_1)(x_1) - H_1(A_1, B_1)(x_2)\|_1^{q_1} \]
\[\leq (1 - 2q_1(\alpha'_1 - \beta'_1) + c_{q_1}(\nu_1 + \delta_1)^{q_1})\|x_1 - x_2\|^{q_1}, \]

which implies that
\[\|H_1(A_1, B_1)(x_1) - H_1(A_1, B_1)(x_2) - (x_1 - x_2)\|_1 \]
\[\leq (1 - 2q_1(\alpha'_1 - \beta'_1) + c_{q_1}(\nu_1 + \delta_1)^{q_1})^{\frac{1}{q_1}}\|x_1 - x_2\|_1 \quad (4.3.10)\]

Similarly, we have

60
\[ \| H_2(A_2, B_2)(y_1) - H_2(A_2, B_2)(y_2) - (y_1 - y_2) \|_2 \]

\[ \leq (1 - 2q_2(\alpha'_2 - \beta'_2) + c_{q_2}(\nu_2 + \delta_2)^{q_2}) \| \frac{1}{2} || y_1 - y_2 ||_2. \]  

(4.3.11)

Again, since \( F_i \) is \( \mu_i \)-strongly accretive in the first argument and \((L_F, l_F)\)-mixed Lipschitz continuous, then using Lemma 1.2.4, we have

\[ \| x_1 - x_2 - \lambda_1(F_1(x_1, y_1) - F_1(x_2, y_1)) \|_1 \]

\[ \leq \| x_1 - x_2 \|_1^q - 2\lambda_1 q_1 \| F_1(x_1, y_1) - F_1(x_2, y_1), J_{q_1}(x_1 - x_2) \|_1 + c_{q_1} \lambda_1^q \| F_1(x_1, y_1) - F_1(x_2, y_1) \|_1 \]

\[ \leq (1 - 2\lambda_1 q_1 \mu_1 + c_{q_1} \lambda_1^q L_{F_1}^q) \| x_1 - x_2 \|_1, \]

which implies that

\[ \| x_1 - x_2 - \lambda_1(F_1(x_1, y_1) - F_1(x_2, y_1)) \|_1 \leq (1 - 2\lambda_1 q_1 \mu_1 + c_{q_1} \lambda_1^q L_{F_1}^q) \frac{1}{2} \| x_1 - x_2 \|_1. \]  

(4.3.12)

Similarly, we have

\[ \| y_1 - y_2 - \lambda_2(F_2(x_1, y_1) - F_2(x_2, y_2)) \|_2 \leq (1 - 2\lambda_2 q_2 \mu_2 + c_{q_2} \lambda_2^q L_{F_2}^q) \frac{1}{2} \| y_1 - y_2 \|_2. \]  

(4.3.13)

From (4.3.6), (4.3.8), (4.3.10) and (4.3.12), we have

\[ \| T(x_1, y_1) - T(x_2, y_2) \|_1 \]

\[ \leq (L_1[(1 - 2q_1(\alpha'_1 - \beta'_1) + c_{q_1}(\nu_1 + \delta_1)^{q_1})^q + (1 - 2\lambda_1 q_1 \mu_1 + c_{q_1} \lambda_1^q L_{F_1}^q)\frac{1}{2}]) \| x_1 - x_2 \|_1 + L_1 \lambda_1 l_F \| y_1 - y_2 \|_2. \]  

(4.3.14)

From (4.3.7), (4.3.9), (4.3.11) and (4.3.13), we have

\[ \| S(x_1, y_1) - S(x_2, y_2) \|_2 \]

\[ \leq (L_2[(1 - 2q_2(\alpha'_2 - \beta'_2) + c_{q_2}(\nu_2 + \delta_2)^{q_2})^q + (1 - 2\lambda_2 q_2 \mu_2 + c_{q_2} \lambda_2^q L_{F_2}^q)\frac{1}{2}]) \| y_1 - y_2 \|_2 + L_2 \lambda_2 L_{F_2} \| x_1 - x_2 \|_1. \]  

(4.3.15)

From (4.3.14), and (4.3.15), we have
\[ \| T(x_1, y_1) - T(x_2, y_2) \|_1 + \| S(x_1, y_1) - S(x_2, y_2) \|_2 \]
\[ \leq k_1 \| x_1 - x_2 \|_1 + k_2 \| y_1 - y_2 \|_2 \]
\[ \leq \max\{k_1, k_2\}(\| x_1 - x_2 \|_1 + \| y_1 - y_2 \|_2), \quad (4.3.16) \]

where
\[
\begin{cases}
  k_1 := m_1 + \lambda_2 L_2 L_{F_2}; \\
  k_2 := m_2 + \lambda_1 L_1 l_{F_1}. 
\end{cases} \quad (4.3.17)
\]

and
\[
m_1 := L_1 [ (1 - 2q_1 (\alpha'_1 - \beta'_1) + c_{q_1} (\nu_1 + \delta_1)^{\eta_1})^{\frac{1}{\eta_1}} + (1 - 2\lambda_1 q_1 \mu_1 + c_{q_1} \lambda_1^{\eta_1} L_{F_1}^{\eta_1})^{\frac{1}{\eta_1}} ]; \\
m_2 := L_2 [ (1 - 2q_2 (\alpha'_2 - \beta'_2) + c_{q_2} (\nu_2 + \delta_2)^{\eta_2})^{\frac{1}{\eta_2}} + (1 - 2\lambda_2 q_2 \mu_2 + c_{q_2} \lambda_2^{\eta_2} L_{F_2}^{\eta_2})^{\frac{1}{\eta_2}} ]; \\
L_1 := \frac{1}{[\lambda_1 (\alpha_1 - \beta_1) + (\alpha'_1 - \beta'_1)]}; \\
L_2 := \frac{1}{[\lambda_2 (\alpha_2 - \beta_2) + (\alpha'_2 - \beta'_2)]}. 
\]

Now, define the norm \( \| . \|_1 \) on \( E_1 \times E_2 \) by
\[ \| (x, y) \|_1 = \| x \|_1 + \| y \|_2, \quad \forall (x, y) \in E_1 \times E_2. \quad (4.3.18) \]

We observe that \( (E_1 \times E_2, \| . \|_1) \) is a Banach space. Hence, it follows from (4.3.3), (4.3.16) and (4.3.18) that
\[ \| Q(x_1, y_1) - Q(x_2, y_2) \|_1 \leq \max\{k_1, k_2\}(\| (x_1, y_1) - (x_2, y_2) \|_1), \quad (4.3.19) \]

Since \( \max\{k_1, k_2\} < 1 \) by condition (4.3.2), it follows from (4.3.19) that \( Q \) is a contraction mapping. Hence, by Banach contraction principle (Theorem 1.2.16), there exists a unique point \( (x, y) \in E_1 \times E_2 \) such that
\[ Q(x, y) = (x, y), \]

which implies that
\[ x = R_{\eta_1, \eta_2} \left[ H_1(A_1, B_1)(x) - \lambda_1 F_1(x, y) \right], \]

62
It follows from Lemma 4.3.1, that \((x, y)\) is a unique solution of SGVI (4.3.1). This completes the proof.

4.4. ITERATIVE ALGORITHM AND CONVERGENCE ANALYSIS

Based on Lemma 4.3.1, we suggest and analyze the following iterative algorithm for finding an approximate solution for SGVI (4.3.1):

Iterative Algorithm 4.4.1. For any given \((x_0, y_0) \in E_1 \times E_2\), compute \((x_n, y_n) \in E_1 \times E_2\) by iterative scheme

\[
x_{n+1} = R_{M_1(\cdot, \cdot), H_1(\cdot, \cdot)}^{H_1(\cdot, \cdot)} [H_1(A_1, B_1)(x_n) - \lambda_1 F_1(x_n, y_n)],
\]

\[
y_{n+1} = R_{M_2(\cdot, \cdot), H_2(\cdot, \cdot)}^{H_2(\cdot, \cdot)} [H_2(A_2, B_2)(y_n) - \lambda_2 F_2(x_n, y_n)],
\]

where \(n = 0, 1, 2, \ldots\); \(\lambda_1, \lambda_2 > 0\) are constants.

Theorem 4.4.1. Let, for \(i = 1, 2\), \(A_i, B_i, E_i, f_i, g_i, F_i, H_i, M_i\) be same as in Theorem 4.3.1 and let condition (4.3.2) of Theorem 4.3.1 hold. Then approximate solution \((x_n, y_n)\) generated by Iterative Algorithm 4.4.1, converges strongly to the unique solution \((x, y)\) of SGVI (4.3.1).

Proof. By Theorem 4.3.1, there exists a unique solution \((x, y)\) of SGVI (4.3.1). It follows from Iterative Algorithm 4.4.1 and Theorem 4.2.3 that

\[
||x_{n+1} - x||_1 \leq ||R_{M_1(\cdot, \cdot), H_1(\cdot, \cdot)}^{H_1(\cdot, \cdot)} [H_1(A_1, B_1)(x_n) - \lambda_1 F_1(x_n, y_n)] - R_{M_1(\cdot, \cdot), H_1(\cdot, \cdot)}^{H_1(\cdot, \cdot)} [H_1(A_1, B_1)(x) - \lambda_1 F_1(x, y)]||_1
\]

\[
\leq L_1[||H_1(A_1, B_1)(x_n) - H_1(A_1, B_1)(x) - \lambda_1 (F_1(x_n, y_n) - F_1(x, y))||_1
+ \lambda_1 ||F_1(x, y_n) - F_1(x, y)||_1]
\]

\[
\leq (L_1[(1-2\lambda_1(\alpha_i - \beta_i') + c_n(\nu_1 + \delta_i)y_n)^{\frac{1}{\alpha_i}} + (1-2\lambda_1 q_1(\mu_1 + c_n(\nu_1 + \delta_i)y_n)^{\frac{1}{\alpha_i}}) ||x_n - x||_1
+ L_1 \lambda_1 \mu_1 ||y_n - y||_2).
\]
\[ \|y_{n+1} - y\|_2 \]
\[ \leq L_2 \|H_2(A_2, B_2)(y_n) - H_2(A_2, B_2)(y) - \lambda_2(F_2(x_n, y_n) - F_2(x, y))\|_2 \]
\[ + \lambda_2 \|F_2(x_n, y) - F_2(x, y)\|_2 \]
\[ \leq (L_2[(1 - 2q_2(\alpha' - \beta'_2) + c_q(\nu_2 + \delta_2)^q)\frac{1}{\lambda_2} + (1 - 2\lambda_2q_2\mu_2 + c_q^2\lambda_2^2\mu_2^2)\frac{1}{\lambda_2^2}])\|y_n - y\|_2 \]
\[ + L_2\lambda_2L_{F_2}\|x_n - x\|_1. \]  
(4.4.4)

From (4.4.3) and (4.4.4), we have
\[ \|x_{n+1} - x\|_1 + \|y_{n+1} - y\|_2 \leq k_1\|x_n - x\|_1 + k_2\|y_n - y\|_2 \]
\[ \leq \max\{k_1, k_2\}(\|x_n - x\|_1 + \|y_n - y\|_2). \]  
(4.4.5)

Since \((E_1 \times E_2, \|\cdot\|_*)\) is a Banach space with norm \(\|\cdot\|_*\) defined by (4.3.18). Hence, it follows from (4.4.1), (4.4.4) and (4.3.18) that
\[ \|(x_{n+1}, y_{n+1}) - (x, y)\|_* = \|x_{n+1} - x\|_1 + \|y_{n+1} - y\|_2 \]
\[ \leq \max\{k_1, k_2\}\|(x_n, y_n) - (x, y)\|_* \]
\[ \leq (\max\{k_1, k_2\})^n\|(x_0, y_0) - (x, y)\|_* \]  
(4.4.6)

By condition (4.3.1), it follows that \(\max\{k_1, k_2\} < 1\) and hence (4.4.6) implies that
\[ \|(x_{n+1}, y_{n+1}) - (x, y)\|_* \to \infty \text{ as } n \to \infty. \]

Thus \(\{(x_n, y_n)\}\) converges strongly to the unique solution \((x, y)\) of SGVI (4.3.1). This completes the proof.

**Remark 4.4.1.** The proximal mapping method presented in this chapter can be extended to the systems of variational-like inclusions in Banach spaces.