CHAPTER I

INTRODUCTION

1.1. The present thesis comprises certain investigations carried out by the author on convergence and integral modulus of continuity of trigonometric series with special coefficients. The work on the integral modulus of continuity was initiated by S. Aljančić and M. Tomic [6] while the work on $L^1$-convergence was initiated by A.N. Kolmogoroff [23]. Further researches on these topics were carried out by S. Aljančić, M. Tomic, M. Izumi and S. Izumi, C.S. Rees, S.A. Teljakovskii, B. Ram, C.V. Stanojević, W. Garrett, T. Kano etc. A number of theorems of different characters, most of which are directly associated with the works of the above mentioned authors, have been proved in the present thesis.

Before giving a résumé of the earlier researches in the light of which the various results have been investigated by the author, it seems desirable to present here those definitions and notations which will be used in the sequel. However, some of the definitions and notations will be repeated occasionally in various chapters for the sake of convenience. In this introductory chapter, a brief chapterwise résumé of the results contained in the thesis is also included.
1.2. Definitions and Notations

**Convex sequence.** A sequence \(< a_n >\) is said to be convex if \( \Delta^2 a_n > 0 \), where \( \Delta a_n = a_n - a_{n+1} \) and

\[
\Delta^2 a_n = \Delta (a_n - a_{n+1}) = a_n - 2a_{n+1} + a_{n+2}.
\]

**Quasi-convex sequence.** [11, Vol. II, p. 202]. A sequence \(< a_n >\) is said to be quasi-convex if

\[
\sum_{n=1}^{\infty} |\Delta^2 a_n| < \infty.
\]

**Generalized quasi-convexity** [25]. A sequence \(< a_n >\) is said to be generalized quasi-convex if

\[
\sum_{n=1}^{\infty} n^\alpha |\Delta^{\alpha+1} a_n| < \infty, \quad \text{for } \alpha > 0.
\]

**Hyper convex sequence** [26]. A sequence \(< a_n >\) is said to be hyper-convex of order \( h \) if

\[
\Delta^{h+2} a_n > 0 \quad (h = 0, 1, 2, \ldots)
\]

Obviously, hyper-convexity of order zero is the same as convexity.

**Bounded variation.** A sequence \(< a_n >\) is said to be of bounded variation if

\[
\sum_{n=0}^{\infty} |\Delta a_n| < \infty.
\]
Quasi-monotone sequence ([35], [39]). A sequence $\langle a_n \rangle$ of non-negative numbers is said to be quasi-monotone if

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n}\right)$$

for some constant $\alpha > 0$ and all $n > n_0(\alpha)$. An equivalent definition is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

Fourier series. A trigonometric series

$$a_0 \sum_{n=-\infty}^{\infty} \frac{1}{2} \left(a_n \cos nx + b_n \sin nx\right),$$

the coefficients $a_n$ and $b_n$ of which are determined by the Fourier formulae

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx,$$

derived from the function $f(x)$, is called the Fourier series of the function $f$. We then write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

The series

$$\tilde{f}(x) \sim \sum_{n=1}^{\infty} \left(-b_n \cos nx + a_n \sin nx\right)$$

is called the conjugate Fourier series of $f(x)$. 
Dirichlet kernel \([11]\). Let

\[ D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \ldots + \cos nx, \]

then

\[ 2\sin \frac{x}{2} D_n(x) = \sin \frac{x}{2} \cos x + \ldots + 2\sin \frac{x}{2} \cos nx \]

\[ = \sin(n + \frac{1}{2})x \]

whence

\[ D_n(x) = \frac{\sin(n + \frac{1}{2})x}{2\sin \frac{x}{2}}. \]

This expression is known as Dirichlet's kernel and

\[ \bar{D}_n(x) = \sin x + \sin 2x + \ldots + \sin nx = \frac{\cos \frac{x}{2} - \cos(n + \frac{1}{2})x}{2\sin \frac{x}{2}} \]

is called the kernel conjugate to the Dirichlet kernel.

If \( x \not\equiv 0 \pmod{2\pi} \), then

\[ |D_n(x)| \leq \frac{1}{2\left[ \sin \frac{x}{2} \right]} \quad \text{and} \]

\[ |\bar{D}_n(x)| \leq \frac{1}{\left[ \sin \frac{x}{2} \right]} . \]

Also

\[ |\mu_n(x)| \leq \frac{\pi}{2x} \quad \text{for} \ 0 < |x| \leq \pi \]

and

\[ |\bar{\mu}_n(x)| \leq \frac{\pi}{x} \quad \text{for} \ 0 < |x| \leq \pi . \]

Oftenly we use the estimates
$D_n(x) = O\left(\frac{1}{x}\right)$ and $\bar{D}_n(x) = O\left(\frac{1}{x}\right)$ as $x \to 0$.

**Fejér kernel** [11]. The Fejér kernel $K_n(x)$ is defined as

$$K_n(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} D_{\nu}(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} \frac{\sin(\nu + \frac{1}{2})x}{2 \sin \frac{x}{2}}$$

Using $|\bar{D}_n(x)| < n+1$, it follows that $K_n(x) < n+1$.

Moreover, we note that

$$K_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{\cos kx - \cos(k+1)x}{4\sin^2 \frac{x}{2}}$$

$$= \frac{1 - \cos(n+1)x}{(n+1)4\sin^2 \frac{x}{2}} = \frac{1}{2(n+1)} \left[\frac{\sin(n+1)x}{\sin \frac{x}{2}}\right]^2.$$

From this expression, we immediately derive the following properties of Fejér kernel.

(i) $K_n(x) \geq 0$

(ii) $K_n(x) \leq \frac{1}{2(n+1)\sin^2 \frac{x}{2}} \leq \frac{\pi^2}{2(n+1)x^2}$, for $0 < |x| \leq \pi$,

and therefore

$$K_n(x) = 0 \left(\frac{1}{nx^2}\right) \text{ for } 0 < |x| \leq \pi$$

and

$$K_n(x) \leq \frac{\pi^2}{2(n+1)\delta^2} \text{ for } 0 < \delta \leq |x| \leq \pi.$$

(iii) $\frac{1}{\pi} \int_{-\pi}^{\pi} K_n(x)dx = 1$. 
The class $S$ ([36], [42]). Let
\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \quad \text{and} \quad \sum_{k=1}^{\infty} a_k \sin kx
\]
be the cosine series and sine series respectively. If $a_k = o(1)$, $k \to \infty$ and there exists a sequence $\{A_k\}$ such that

(i) $A_k \downarrow 0$, $k \to \infty$

(ii) $\sum_{k=0}^{\infty} A_k < \infty$

(iii) $|\Delta a_k| \leq A_k$ for all $k$,

we say that the above mentioned cosine and sine series belong to the class $S$, introduced by S. Sidon.

Letting $A_k = \sum_{n=k}^{\infty} |\Delta^2 a_n|$, we observe that every quasi-convex null sequence satisfies the class $S$.

The class $R$ (cf. [21]). If $a_k = o(1)$, $k \to \infty$ and
\[
\sum_{k=1}^{\infty} k^2 |\Delta^2 \left( \frac{a_k}{k} \right)| < \infty,
\]
we say that the above said cosine and sine series belong to the class $R$.

Sequence of convergence factors [11, Vol. I, p. 143]. Let $\{\lambda_n\}$ be a sequence of real numbers. Then it is said to be sequence of convergence factors for some series
\[
u_0(x) + u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots
\]
in the interval \([a, b]\), if the series \(\sum \lambda_n u_n(x)\) converges almost everywhere in \([a, b]\).

Thus for the Fourier series, the sequence \(\{\lambda_n\}\) will be a sequence of convergence factors if

\[
\frac{1}{2} a_0 \lambda_0 + \sum_{n=1}^{\infty} \lambda_n (a_n \cos nx + b_n \sin nx)
\]

converges almost everywhere in \([0, 2\pi]\).

Differences of order \(k\). The differences of order \(k\) of the sequence \(\{a_n\}\) are defined by the equations

\[
\Delta a_n = a_n - a_{n+1}, \quad \Delta^2 a_n = \Delta a_n - \Delta a_{n+1}, \ldots, \quad \Delta^k a_n = \Delta(\Delta^{k-1} a_n),
\]

\(n = 0, 1, 2, \ldots\).

**Abel's transformation.** Let \(u_0, u_1, \ldots, v_0, v_1, \ldots, v_n\) be any real numbers and let \(v_n = v_0 + v_1 + \ldots + v_n\). Then for any values of \(m\) and \(n\) we have

\[
\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} \Delta u_k V_k + u_n V_n - u_m V_{m-1},
\]

under the condition that if \(m = 0\), \(V_{-1} = 0\).

**The broken differences** [10]. The broken differences \(\Delta^k a_n\) are defined by

\[
\Delta^k a_n = \sum_{m=0}^{n-p} \Delta^{k-1} a_{m}, \quad a_{n-p+m}.
\]

By means of broken differences we can condense the Abel formula of order \(k\) as
The binomial coefficients \([50]\). For \(k = 0, 1, 2, \ldots\) we define the sequence of numbers \(A_k, A_1, A_2, \ldots\) by the conditions

\[
A_0 = 1, \quad A_k = A_{k-1} + A_{k-2} + \ldots + A_n \quad (n = 0, 1, 2, \ldots)
\]

The \(A_n\)'s are called binomial coefficients and are given by the following relation:

\[
\sum_{k=0}^{\infty} A_k x^k = \frac{1}{(1-x)^{\alpha+1}}
\]

The following relations hold good for binomial coefficients

\[
A_n = \sum_{v=0}^{\alpha} A_v, \quad A_n - A_{n-1} = A_n
\]

\[
A_n = \binom{n+\alpha}{\alpha} \frac{n^\alpha}{\sqrt{\alpha+1}} \quad (\alpha \neq -1, -2, \ldots)
\]

The Cesáro sums and the Cesáro means \([50]\). Given a sequence \(S_0, S_1, S_2, \ldots\), we define for every \(k = 0, 1, 2, \ldots\), the sequence \(S^k_0, S^k_1, S^k_2, \ldots\), by the conditions

\[
S^0_n = S_n, \quad S^k_n = S^k_{n-1} + S^k_{n-2} + \ldots + S^k_0 \quad (n = 0, 1, 2, \ldots)
\]

The Cesáro means \(T^\alpha_k\) of order \(\alpha\) of \(S_n\) are defined by

\[
T^\alpha_k = \frac{S^\alpha_k}{A^\alpha_k}
\]

The integral modulus of continuity of various orders \([47, pp. 106-10]\). Let \(F\) be a function of period \(2\pi\) in \(L_p(1 \leq p < \infty)\).
Then the integral modulus of continuity of order \( k > 1 \) of \( F \) in \( L_p \) is defined by

\[
\omega^k_p(h;F) = \sup_{0 < |t| < h} |\Delta^k_F(x)|_{L_p},
\]

where

\[
\Delta^k_F(x) = \sum_{\alpha=0}^{k} \binom{k}{\alpha} (-1)^{k-\alpha} F(x + \alpha t)
\]

and \( ||.||_{L_p} \) denotes the norm in \( L_p \).

Thus, in particular the integral modulus of continuity of order 1 of \( F \), denoted by \( \omega^1_p(h;F) \) is defined by

\[
\omega^1_p(h;F) = \sup_{0 < |t| < h} \frac{1}{2\pi} \int_0^{2\pi} |F(x+t) - F(x)|^p dx^{1/p}.
\]

We call it simply the \( L^p \)-modulus of continuity of \( F \).

The following properties of \( \omega^k_p(h;F) \) follow directly from the above definition.

(i) \( \omega^k_p(0;F) = 0 \)

(ii) The function \( \omega^k_p(h;F) \) does not decrease with \( h \).

(iii) If the function \( F \) is continuous, then \( \omega^k_p(h;F) \) is also continuous in the interval \( 0 \leq h \leq \frac{2\pi}{k} \).

(iv) If \( F \) has a bounded derivative of \( r \)th order (\( r \) being an integer) on \( [0, 2\pi] \), then for any integer \( k \geq 0 \).
Similarly, if \( F \) is absolutely continuous and
\( F \in L^p (p > 1) \), then
\[
\omega^{k+r}_p (h; F) \leq h^r \omega^k_p (h; F). \]

(v) If \( n \geq 0 \) is an integer, then
\[
\omega^k_p (nh; F) \leq n^k \omega^k_p (h; F). \]

1.3. Concerning the integral modulus of continuity of
first order Lebesgue \([24]\) has proved that if \( F \in L \), then the
Fourier coefficients \( a_n \) and \( b_n \) of \( F \) satisfy the relation
\[
|a_n|, |b_n| \leq \frac{1}{2\pi} \omega_1 (F; \frac{\pi}{n}). \tag{1.3.1} \]

Conversely, some authors discussed the problem : How to
obtain estimates for the integral modulus of continuity of a
function given by its Fourier coefficients, that is, how to
obtain some kind of converse of the Riemann-Lebesgue Lemma?

In 1964, Aljančić and Tomic \([6]\) proved that under the
appropriate conditions the integral modulus of continuity
\( \omega_1 (F; \frac{\pi}{n}) \) is majorized by the coefficients of order \( n \). They have
established the following theorems :

Theorem 1. If the function \( a(x) (x \geq 0) \) satisfies the
following conditions
(i) \(0 < a(x) \downarrow 0\) as \(x \to +\infty\),

(ii) \(a(x)\) is convex,

(iii) \(\int_0^1 a(t)dt = 0[xa(x)]\); 

then \(f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \left[ a_n = a(n) \right]\) is a function in \(L(0,\pi)\) and

\[\omega_1(f; \frac{\pi}{n}) = O(a_n).\]  

If the conditions of Theorem 1 are satisfied, then (1.3.1) and (1.3.2) imply

\[0 < c_1 a_n < \omega_1(f; \frac{\pi}{n}) < c_2 a_n.\]

It may also be remarked that the condition (iii) of Theorem 1 will be satisfied if there exists a number \(\beta(0 < \beta < 1)\) such that \(x^\beta a(x) \uparrow\). If \(x a(x) \uparrow\), then \(\omega_1(f; \frac{\pi}{n}) = O(a_n \log n)\). This is the best possible evaluation because according to Zygmund \([49]\) if \(f_1(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n}\), then \(\omega_1(f_1; \frac{\pi}{n}) > \frac{c}{n} \log n\).

Next, the result of Theorem 1 also holds for the integral modulus of continuity of higher order. However, the analogous theorem for the sine series requires an additional condition on the function \(a(x)\).

**Theorem II.** If the function \(a(x)\) satisfies the conditions (i) - (iii) of Theorem 1 and if in addition
(iv) \[ \int_{x}^{t} a(t) \, dt = o\left( a(x) \right), \]
then
\[ g(x) = \sum_{n=1}^{\infty} a \sin nx \]
is in \( L(0,\pi) \) and
\[ \omega_i(g; \frac{n}{x}) = o(a_n). \]

Condition (iv) is satisfied if there exists a number \( \beta \) \((0 < \beta < 1)\) such that \( x^\beta a(x) \downarrow \).

Later on, the following similar results were also obtained by Aljančić and Tomic [6].

**Theorem III.** Let \( \{a_n\} \) be a convex sequence such that
\[ a_n \downarrow 0 \text{ and} \]
(i) \[ \sum_{v=1}^{n} a_v = o(n a_v) . \]

Then
\[ \omega_i(f; \frac{n}{2n}) = o(a_n) . \]

**Theorem IV.** Let \( \{b_n\} \) be a convex sequence such that
(i) \( b_n \downarrow 0 , \)
(ii) \[ \sum_{v=1}^{n} b_v = o(n b_v) , \]
(iii) \[ \sum_{v=n+1}^{\infty} \frac{b_v}{v} = o(b_v) . \]

Then
\[ \omega_i(g; \frac{n}{n}) = o(b_n) . \]
In 1967, Aljancić [5] extended Theorem III for the integral modulus of continuity of order $k$ in the following form:

**Theorem V.** Let $\{a_n\}$ be a convex sequence such that $a_n \downarrow 0$. Then

$$\omega_k^1 \left( \frac{1}{n}; f \right) \leq C_k n^{-k} \sum_{v=1}^{n} a_v^{k-1}.$$ 

Rem [26] generalized Theorem V by proving the following result:

**Theorem VI.** Let $\{a_n\}$ be a hyper-convex sequence of order $h$ such that $n^h \Delta^n a_n \to 0 \ (r = 0, 1, 2, \ldots, h+1)$. Then

$$\omega_k^1 \left( \frac{1}{n}; f \right) \leq C_k n^{-k} \sum_{v=1}^{n} a_v^{h+k-1} \Delta^h a_v,$$

where $0 \leq h \leq h+1$.

**Theorem V** is a particular case for $h = 0$ of **Theorem VI**, since $\Delta^2 a_n \geq 0$ and $a_n \downarrow 0$ imply $n \Delta a_n \to 0$.

Omitting the condition (i) in **Theorem III** and (ii) and (iii) in **Theorem IV**, Izumi and Izumi [20] obtained the following result:

**Theorem VII.** Let $\{a_n\}$ be a convex sequence tending to zero. Then

$$\omega_k^1 \left( \frac{1}{m}; f \right) \leq C m^{-1} \sum_{n=1}^{m} a_n.$$

The result holds for $g$ also.
In 1966, Aljanić [4] established the following result concerning the integral modulus of continuity in \( L(1 < p < \infty) \).

**Theorem VIII.** Let \( \langle a_n \rangle \) be a sequence which is monotonically decreasing to zero and such that for a fixed \( p(1 < p < \infty) \),

\[
\sum_{n=1}^{\infty} n^{2p-2} a_n < \infty.
\]

Then

\[
\omega_p \left( \frac{1}{n}; f \right) \leq C_n \left[ \sum_{v=1}^{n-1} v^{2p-2} a_v \right]^{1/p} + C \left[ \sum_{v=n}^{\infty} v^{p-2} a_v \right]^{1/p}.
\]

The result also holds for \( g \).

Rees [32] generalized this theorem in the sense that he replaced \( a_n \downarrow 0 \) by a null sequence \( \langle a_n \rangle \) satisfying \( a_n > 0 \) and

\[
\sum_{k=n}^{\infty} |a_k - a_{k+1}| \leq C a_n.
\]

The conditions imposed are certainly weaker than monotonically decreasing null sequence.

Theorem VIII was further extended to the integral modulus of continuity of order \( k \) by Aljanić [5]. He proved the following result:

**Theorem IX.** If the sequence \( \langle a_n \rangle \) decreases monotonically to zero and satisfies

\[
\sum_{v=1}^{\infty} v^{p-2} a_v < \infty \quad (1 < p < \infty),
\]

then

\[
\omega_{p}^{k} \left( \frac{1}{n}; f \right) \leq C_{k,p} \left[ n \left[ \sum_{v=1}^{n} v^{(k+1)p-2} a_v \right]^{1/p} + \left[ \sum_{v=n+1}^{\infty} v^{p-2} a_v \right]^{1/p} \right].
\]

The theorem holds for \( g \) also.
Ram [27] extended Theorem IX to the case \( p = 1 \). His theorem reads as follows:

**Theorem X.** Let \( \{a_n\} \) be a monotonically decreasing sequence of positive terms such that

\[
\sum_{v=1}^{\infty} \frac{a_v}{v} < \infty
\]

Then

\[
\omega_1\left(\frac{1}{n}, \phi\right) \leq C_k n^{-k} \log n \sum_{v=1}^{n-1} a_v + C \sum_{v=k}^{\infty} \frac{a_v}{v},
\]

where \( \phi \) is sum of either of the series \( \sum_{n=1}^{\infty} a_n \cos nx \) or \( \sum_{n=1}^{\infty} a_n \sin nx \).

The case \( k = 1 \) of this theorem yields also the result of Rees [33].

Izumi and Izumi [20] obtained the following result for the integral modulus of continuity of functions defined by trigonometric series.

**Theorem XI.** If the sequence \( \{b_v\} \) is quasi-convex, then

\[
\omega_1\left(\frac{1}{n}, \delta\right) \leq \frac{A}{n} \sum_{v=1}^{n} v^2 |\Delta^2 b_{v-1}| + \frac{A}{n} \sum_{v=n+1}^{\infty} v(1 + \log \frac{v}{n}) |\Delta^2 b_{v-1}|.
\]

In 1970, Teljakovskii [40] further improved Theorem XI by establishing the following theorem:

**Theorem XII.** Let \( \{b_v\} \) be a quasi-convex null sequence satisfying \( \sum_{v=1}^{\infty} \frac{|b_v|}{v} < \infty \). Then the integral modulus of continuity of \( g \) satisfies the relation
The above theorem was further generalized by Ram [29] for the integral modulus of continuity of higher order.

Some more results concerning integral modulus of continuity of Fourier series of summable (continuous) functions can be found in Aljancić ([1], [2], [3]), Aljancić and Tomic ([8], [9]), and Teljakovskii [41].

In Chapter II, we obtain an estimate for the integral modulus of continuity of order $k$ of the sine series belonging to the class $S$ of Sidon.

Chapter III of the present thesis is devoted to find an estimate for the integral modulus of continuity of sine series whose coefficients satisfy the conditions

$$a_v - 0, \ v \to \infty \quad \text{and} \quad \sum_{v=1}^{\infty} v^2 |\Delta^2 \left( \frac{a_v}{v} \right)| < \infty.$$ 

In Chapter IV, we generalize the results of Chapter III for integral modulus of continuity of higher order.

1.4. The following results about the behaviour of cosine and sine series are known:

**Theorem XIII** ([43], [23], [11], Vol. II, p. 202). If $\langle a_k \rangle$ is a quasi-convex null sequence, then
(1.4.1) \[ f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L[0, \pi] . \]

Theorem XIV ([12], [43]). If \( \{a_k\} \) is a quasi-convex null sequence, then

\[ \sum_{k=1}^{\infty} |a_k| \sin kx \]

is a Fourier series if and only if \( \sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty \).

Kano [21] generalized Theorem XIII and XIV in the following form:

Theorem XV. If \( \{a_k\} \) is a null sequence such that

\[ \sum_{k=1}^{\infty} \frac{k^2 |\Delta^2 \left( \frac{a_k}{k} \right)|}{k} < \infty , \]

then (1.4.1) and (1.4.2) are Fourier series, or equivalently, they represent integrable functions.

It is clear that Theorem XIII and XV provide us only sufficient conditions for the integrability of the cosine series. Rees and Stanojevic [34] showed that \( \sum_{k=1}^{\infty} \frac{a_k}{k} < \infty \) is a necessary and sufficient condition for \( L_1[0,\pi] \) integrability but for a different type of cosine-sums. They obtained the following results:

Theorem XVI. Let \( b_k = \frac{a_k}{k} \downarrow 0 \). Then
\[ g(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \frac{b_k}{2} + \left( \sum_{j=k}^{n} b_j \right) \cos kx \right] \]

exists for \( x \in (0, \pi) \) and \( g \in L^1(0, \pi) \) if and only if

\[ \sum_{k=1}^{\infty} b_k < \infty. \]

**Theorem XVII.** Let \( b_k = \frac{a_k}{k} \to 0. \) Then

\[ \frac{1}{x} \sum_{k=1}^{\infty} b_k \sin (k + \frac{1}{2})x = \frac{h(x)}{x} \]

converges for \( x \neq 0 \) and \( \frac{h(x)}{x} \in L^1(0, \pi) \) if and only if

\[ \sum_{k=1}^{\infty} b_k < \infty. \]

**Theorem XVIII.** Let \( (k+1)! \Delta^2 a_k \to 0. \) Then

\[ h(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \frac{1}{2} (k+1)! \Delta^2 a_k \right] + \sum_{j=k}^{n} \left( j+1 \right) \Delta^2 a_j \cos kx \]

exists for \( x \in (0, \pi) \) and \( h \in L^1(0, \pi) \) if and only if \( \langle a_k \rangle \) is quasi-convex.

Ram [30] showed that the condition \( S \) introduced already
in §1.2) is sufficient for the integrability of Rees-Stanojević sums [16]

\[ g_n(x) = \frac{1}{2^n} \left( \sum_{k=0}^{n} \Delta a_k + \sum_{j=1}^{n} \sum_{j=k}^{n} a_j \cos kx \right). \]

He proved the following theorems:

**Theorem XIX.** Let the sequence \( \langle a_k \rangle \) satisfy the condition \( S \). Then

\[ g(x) = \lim_{n \to \infty} g_n(x) \text{ exists for } x \in (0, \pi] \text{ and } \]

\[ \int_0^\pi |g(x)| \, dx < C \sum_{k=0}^{\infty} a_k. \]

**Theorem XX.** Let \( \langle a_k \rangle \) be a sequence satisfying the condition \( S \). Then

\[ \frac{1}{2} \sum_{k=1}^{\infty} \Delta a_k \sin \left( k + \frac{1}{2} \right) x = \frac{h(x)}{x} \]

converges for \( x \in (0, \pi] \) and \( \frac{h(x)}{x} \in L(0, \pi] \).
The above theorems were further generalized by Ram [31] under a weaker condition where monotonicity of the sequence $< A_n >$ in the definition of class $S$ is replaced by quasi-monotonicity.

The aim of Chapter V of the present thesis is to obtain necessary and sufficient conditions for convergence and $L(0, \pi)$ integrability of certain new trigonometric sums.

1.5. Concerning the $L^1$-convergence of the cosine series, we have the following classical result of Kolmogorov [23]:

**Theorem XXI.** If $a_k \downarrow 0$ and $< a_k >$ is convex or even quasi-convex, then for the convergence of the series $\sum_{k=1}^{\infty} a_k \cos k$ in the metric space $L$ it is necessary and sufficient that

$$\lim_{k \to \infty} a_k \log k = 0.$$ 

Generalizing the above classical result Teljakovskii [42] proved the following result:

**Theorem XXII.** If

$$a_0 + \sum_{k=2}^{\infty} a_k \cos kx$$

(1.5.1)
belongs to the class $S$, then a necessary and sufficient condition for $L^1$-convergence of (1.5.1) is $a_n \log n = o(1)$, $n \to \infty$.

Garrett and Stanojević [16] obtained an analogue of Theorem XXI for the cosine sums (1.4.4). These modified cosine sums approximate their limits better than the classical cosine series as they converge in $L^1$-metric to their limit whereas the classical cosine series may not. They proved the following interesting results:

**Theorem XXIII.** Let $f$ be the sum of the cosine series (1.5.1). Then $g_n$ converges to $f$ in the $L^1$-metric if and only if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$
\int_0^\infty \sum_{k=n+1}^\infty \Delta a_k D_k(x) \, dx < \varepsilon \quad \text{for all } n \geq 0.
$$

In 1977, Ram [28] proved the following result on $L^1$-convergence of Rees-Stanojević sums $g_n(x)$.

**Theorem XXIV.** If (1.5.1) belongs to the class $S$, then

$$
||f - g_n||_{L^1} = o(1), \quad n \to \infty.
$$

Theorem XXII of Teljakovskii follows as a corollary of Theorem XXIV.

Singh and Sharma [37] generalized Theorem XXIV replacing the monotonicity of the sequence $\langle A_n \rangle$ in the definition of class $S$ by quasi-monotonicity.
In Chapter VI, we introduce new modified cosine and sine sums as

\[
\begin{align*}
    f_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=1}^{n} \Delta(\frac{a_j}{j})k \cos kx \\
    g_n(x) &= \sum_{k=1}^{n} \sum_{j=1}^{n} \Delta(\frac{a_j}{j})k \sin kx
\end{align*}
\]

and study their \(L^1\)-convergence under the condition that the cosine series and the sine series belong to the classes \(R\) and \(S\). We also deduce there the results concerning \(L^1\)-convergence of cosine and sine series.

1.6. As we have pointed out earlier, if \(\langle a_n \rangle\) is a quasi-convex null sequence, then the cosine series is a Fourier series and

\[
\int_0^\pi a_0 + \sum_{k=1}^{\infty} k \cos kx \, dx \leq \frac{\pi}{2} \sum_{k=1}^{\infty} 2 k |\Delta a_{k-1}|.
\]

Moore [25] obtained the following more general result:

**Theorem XXV.** If \(\langle a_k \rangle\) is a null sequence satisfying

\[
\sum_{k=1}^{\infty} k^\alpha |\Delta^{\alpha+1} a_k| < \infty, \quad \text{for } \alpha > 0,
\]

then the series

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx
\]

converges in the interval \(0 < x \leq \pi\) and represents there an \(L\)-integrable function whose Fourier cosine development is given by (1.6.2).
Teljakovskii [43] considered the integrability of sine series

\[\sum_{k=1}^{\infty} a_k \sin kx\]

and established the following theorem:

**Theorem XXVI.** If \(\langle a_k \rangle\) is a quasi-convex null sequence, then (1.6.3) is a Fourier series if and only if

\[\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty.\]

Moreover, if (1.6.4) holds, then

\[\int_0^{\pi} \left| \sum_{k=1}^{\infty} a_k \sin kx \right| dx - \sum_{k=1}^{\infty} \frac{|a_k|}{k} \leq C \sum_{k=1}^{\infty} \Delta^2 a_{k-1}.\]

Assuming the sequence \(\langle a_n \rangle\) to be of bounded variation, Garrett and Stanojević [17] obtained the following result:

**Theorem XXVII.** Let \(a_n = o(1), \sum_{n=1}^{\infty} |\Delta a_n| < \infty\) and \(a_n \log n = o(1)\). Then \(S_n\) converges to \(f\) in \(L^1\) metric if and only if

\[\int_0^{\infty} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| < \epsilon.\]

Teljakovskii [44] studied the \(L^1\) convergence of the sine and cosine series under a different set of conditions. His theorems read as follows:

**Theorem XXVIII.** Let the coefficients of the series (1.6.2) satisfy the conditions
(1.6.5) \( a_k = o(1), \ k \to \infty \)

(1.6.6) \( \sum_{k=0}^{\infty} \Delta a_k < \infty \)

(1.6.7) \[ \sum_{m=2}^{\infty} \left| \sum_{k=1}^{m} \frac{\Delta a_{m-k} - \Delta a_{m+k}}{k} \right| < \infty. \]

If

(1.6.8) \( \lim_{n \to \infty} a_n \log n = 0 \), then the cosine series in question converges in the metric \( L \).

Theorem XXIX. Let the coefficients of the sine series (1.6.3) satisfy the conditions (1.6.5), (1.6.6), (1.6.7) and \( a_k < \infty \). Then the condition (1.6.8) is sufficient for \( L^1 \)-convergence of the sine series in question.

Later on, Teljakovskii and Fomin [46] considered the \( L^1 \)-convergence of Fourier series

(1.6.9) \[ f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \]

(1.6.10) \[ g(x) \sim \sum_{k=1}^{\infty} a_k \sin kx \]

by establishing the following results:

Theorem XXX. If the series (1.6.9) is a Fourier series and if its coefficients are quasi-monotone, then it converges in metric \( L \) if and only if \( a_n \log n = o(1), \ n \to \infty \).
Theorem XXXI. If the series (1.6.10) is a Fourier series and if its coefficients are quasi-monotone, then it converges in $L$ if and only if $a_n \log n = o(1), n \to \infty$.

The proof of Theorem XXX involves certain results on summability and interpolation processes. Garrett, Rees and Stanoević [18] gave a simple proof of this result. They deduced it as a corollary from the following theorem established by them in the above said paper.

Theorem XXXII. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

be a Fourier series with quasi-monotone coefficients. Then $||S_n - \sigma_n|| = o(1), n \to \infty$, if and only if $a_n \log n = o(1), n \to \infty$, where $\sigma_n$ is the Fejér sum.

Singh and Sharma [38] further added the following result of $L^1$-convergence of cosine series:

Theorem XXXIII. Let $k$ be a real number such that $k > 0$. If $<a_k>$ be a null sequence satisfying (1.6.1), then for $L^1$-convergence of the series (1.6.9), it is necessary and sufficient that

$$\lim_{k \to \infty} a_k \log k = 0.$$

The aim of Chapter VII is to generalize Theorem XXVI to the integrability of sine series with generalized quasi-convex coefficients. Also we study $L^1$-convergence of this series under the said condition on the coefficients.
1.7. Let $S_n(x)$ denote the partial sum of the series $\sum u_n(x)$. The following theorem about the convergence factors is known:

Theorem XXXIV [11, Vol. II, p. 454]. If the series $\sum u_n$ is summable (C, 1) and $S_n = o\left(\frac{1}{\lambda_n}\right)$, where $\lambda_n > 0$ is a convex sequence tending to zero, then $\sum \lambda_n u_n$ converges.

If we take $\lambda_n = \log n$, $n = 2, 3, \ldots$, then Theorem XXXIV along with Fejér-Lebesgue Theorem [11, Vol. I, p. 139] and the fact that $S_n(x) = o(\log n)$ almost everywhere [11, Vol. I, p. 141] yields

Theorem XXXV [11, Vol. I, p. 143]. If $a_n$ and $b_n$ are Fourier coefficients ($n = 1, 2, \ldots$), then the series

$$\sum_{n=2}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{\log n}$$

converges almost everywhere in $[-\pi, \pi]$.

In 1973, Husain [19] obtained the following results for the convergence factors of the Fourier series of summable functions.

Theorem XXXVI. Let $\lambda_n > 0$ be a quasi-convex and convergent sequence. Then a necessary and sufficient condition
for \( \lambda_n \) to be a sequence of convergence factors of the Fourier series of all \( f \in L[0, 2\pi] \) is that

\[
\lambda_n \log n = O(1) \quad (n \to \infty).
\]

**Theorem XXXVII.** Let \( \lambda_n \) be a quasi-convex and convergent sequence. Then a necessary and sufficient condition for \( \lambda_n \) to be a sequence of convergence factors of the conjugate Fourier series \( \sum_{n=1}^{\infty} (-b_n \cos nx + a_n \sin nx) \) of all \( f \in L[0, 2\pi] \) is that

\[
\lambda_n \log n = O(1) \quad (n \to \infty).
\]

In the last chapter of the present thesis, we have obtained necessary and sufficient condition for generalized quasi-convex null sequence \( \lambda_n \) to be a sequence of convergence factors of the Fourier series and the conjugate Fourier series of \( f \in L[0, 2\pi] \).