1. Introduction. Let \( \{a_k\}, k = 0, 1, 2, \ldots \), be a quasi-convex sequence of real numbers tending to zero, that is, \( a_k \to 0 \) as \( k \to \infty \) and

\[
\sum_{k=1}^{\infty} k |\Delta^2 a_{k-1}| < \infty ,
\]

where \( \Delta^2 a_{k-1} = \Delta a_{k-1} - \Delta a_k = a_{k-1} - 2a_k + a_{k+1} \).

Then the series

\[
a_0 + \sum_{k=1}^{\infty} a_k \cos kx
\]

and

\[
\sum_{k=1}^{\infty} a_k \sin kx
\]

are uniformly convergent on every segment \([\varepsilon, 2\pi - \varepsilon] , \varepsilon > 0 \), and are therefore convergent for \( 0 < x < 2\pi \).

It is known (cf. [23], [11, vol. II, p. 202]) that if the null sequence satisfies (1.1), then (1.2) is a Fourier series and
Moore [25] generalized the notion of quasi-convex sequence in the following way:

A null sequence \( \langle a_k \rangle \) is said to be generalized quasi-convex if

\[
\sum_{k=1}^{\infty} k^{\alpha+1} |a_k| \triangleq a_{k-1} < \infty, \quad \text{for } \alpha > 0
\]

For this generalized notion, Moore [25] obtained the following more general result concerning the integrability of cosine series (1.2):

**Theorem A.** If \( \langle a_k \rangle \) is a null sequence satisfying (1.4) then the series (1.2) converges in the internal \( 0 < x \leq \pi \) and represents there an \( L \) - integrable function whose Fourier cosine development is given by (1.2).

Teljakovskii [43] considered the problem as to when the series (1.5) will be Fourier series, that is, when will the function to which the series (1.5) converges be integrable. Assuming \( a_0 = 0 \), he established the following theorem:
Theorem B. If \( \langle a \rangle \) is a quasi-convex null sequence, then (1.3) is a Fourier series if and only if

\[
\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty.
\]

Moreover, if the series in (1.5) converges, then

\[
\int_0^\pi \left| \sum_{k=1}^{\infty} a_k \sin kx \right| dx \leq C \sum_{k=1}^{\infty} \frac{|a_k|}{k} \sum_{k=1}^{\infty} k^2 |\Delta^2 a_{k-1}|,
\]

where \( C \) is an absolute constant.

Concerning the \( L^1 \)-convergence of cosine series, Singh and Sharma [38] established the following theorem:

Theorem C. Let \( k \) be a real number such that \( k > 0 \). If \( \langle a_k \rangle \) be a null sequence satisfying (1.4), then for \( L^1 \)-convergence of the series (1.2), it is necessary and sufficient that

\[
\lim_{k \to \infty} a_k \log k = 0.
\]

The object of this chapter is to generalize Theorem B for the sine series with generalized quasi-convex coefficients. Also we shall study \( L^1 \)-convergence of this series under the said condition on the coefficients.
In what follows, we shall use the following notations:

Given a sequence $S_0, S_1, S_2, \ldots$, we define for every $k = 0, 1, 2, \ldots$, the sequence $S^k_0, S^k_1, S^k_2, \ldots$, by the conditions

$$S^k_n = S^k_n + S^k_{n-1} + \ldots + S^k_0 \quad (n = 0, 1, 2, \ldots)$$

Similarly for $k = 0, 1, 2, \ldots$, we define the sequence of numbers $A^k_0, A^k_1, A^k_2, \ldots$ by the conditions

$$A^k_n = 1, A^k_n = A^k_0 + A^k_1 + \ldots + A^k_n \quad (n = 0, 1, 2, \ldots)$$

The $A^k_n$'s are called binomial coefficients and are given by the following relation:

$$\sum_{k=0}^{\infty} A^k_n x^k = (1-x)^{\alpha-1}$$

whereas $S^k_n$'s are given by

$$\sum_{k=0}^{\infty} S^k_n x^k = (1-x)^{\alpha} \sum_{k=0}^{\infty} S^k_n x^k$$

We shall use the following relations:

$$A^\alpha_n = \sum_{v=0}^{n} A^\alpha_v, \quad A^\alpha_n - A^\alpha_{n-1} = A^\alpha_n - A^\alpha_{n-1}.$$
The Cesaro means $T^\alpha_k$ of order $\alpha$ of $\Sigma a_n$ will be defined by $T^\alpha_k = \frac{S_k}{\alpha}$.

2. Lemma. The following lemma will be used for the proofs of our results.

**Lemma 1 [14].** If for $\alpha > 0$, $p > 0$,

(i) $\varepsilon_n = O(n^{-p})$

(ii) $\sum_{n=0}^{\infty} \frac{\alpha+p}{\alpha} \frac{\alpha+1}{\lambda^p \lambda+1} \varepsilon_n | < \infty$,

then

(iii) $\sum_{n=0}^{\infty} \frac{\lambda+p}{\lambda} \frac{\lambda+1}{\lambda^p} \varepsilon_n | < \infty$ for $-1 < \lambda < \alpha$, and

(iv) $\Delta^\lambda \varepsilon_n$ is of bounded variation for $0 < \lambda < \alpha$ and tends to zero as $n \to \infty$.

3. Results. We prove the following results:

**Theorem 1.** Let $\langle a_k \rangle$, $k = 0, 1, 2, \ldots$, $a_0 = 0$ be a generalized quasi-convex null sequence. Then (1.3) is a
Fourier series if and only if (1.5) is satisfied.

Moreover, if (1.5) holds, then

\[ \pi \int_{0}^{\infty} |\sum_{k=1}^{\infty} a_k \sin kx| \, dx - \sum_{k=1}^{\infty} \frac{|a_k|}{k} \leq C \sum_{k=1}^{\infty} k^\alpha \Delta^{\alpha+1} a_{k-1} \]

where \( C \) is an absolute constant not necessarily the same at each occurrence.

The assertions of Theorem 1 are immediate consequences of the following theorem:

Theorem 2. Let \( \langle a_k \rangle \), \( k = 0, 1, 2, \ldots \), \( a_0 = 0 \), be a generalized quasi-convex null sequence. Then for all \( s = 1, 2, \ldots \) we have

\[ \pi \int_{1}^{s+1} |\sum_{k=1}^{s} a_k \sin kx| \, dx - \sum_{k=1}^{s} \frac{|a_k|}{k} \leq C \sum_{k=1}^{s} k^\alpha \Delta^{\alpha+1} a_{k-1} \]

Theorem 3. Let \( \langle a_k \rangle \) be a generalized quasi-convex sequence. Then the series (1.3) converges in the metric space \( L \) if and only if (1.5) and

\( (3.1) \quad |a_k| \log k \to 0 \) as \( k \to \infty \),

hold.
Proof of Theorem 2. Let $0 < x < \pi$ and let

$$D_0(x) = -\frac{1}{2} \cot \frac{x}{2}$$

$$S_n(x) = D_0(x) + \sin x + \sin 2x + \ldots + \sin nx$$

$$S_n(x) = S_0(x) + S_1(x) + S_2(x) + \ldots + S_n(x)$$

$$S_n(x) = S_0(x) + \frac{1}{2} S_1(x) + \frac{1}{2} S_2(x) + \ldots + \frac{1}{2} S_n(x)$$

$$S_n(x) = S_0(x) + S_1(x) + S_2(x) + \ldots + S_n(x)$$

Now

$$\sum_{k=0}^{n-1} a_k \sin kx = \sum_{k=0}^{n-1} \Delta a_k \left( \sum_{v=0}^{k} \sin vx \right) + a_n \sum_{v=0}^{n-1} \sin vx$$

$$= \sum_{k=0}^{n-1} \Delta a_k \left( S_k(x) - S_0(x) \right) + a_n \left( S_n(x) - S_0(x) \right)$$

$$= \sum_{k=0}^{n-1} \Delta a_k S_k(x) + \sum_{k=0}^{n-1} \Delta a_k S_0(x) - a_n S_n(x)$$

$$= \sum_{k=0}^{n-1} \Delta a_k S_k(x) + a_n S_n(x) - a_0 S_0(x)$$
If we use Abel's transformation \((a + 1)\) times, we have similarly,

\[
\sum_{k=1}^{n} a_k \sin kx = \sum_{k=0}^{n-1} a_k S_k(x) + a_n S_n(x) + \Delta a_k S_k(x) + \Delta^2 a_{k-1} S_{k-1}(x) + a_n S_n(x) + \ldots
\]

Since \(S_n(x)\) and \(T_n(x)\) are uniformly bounded on every segment \([\epsilon, \pi]\), \(\epsilon > 0\), it follows, by Lemma 1, that

\[
(3.2) \quad \sum_{k=1}^{\infty} a_k \sin kx = \sum_{k=1}^{\infty} a_k \Delta^k \frac{\alpha}{t^{k-1}} T_k(x) = \sum_{k=1}^{\infty} a_k \Delta^k \frac{\alpha}{t^{k-1}} T_k(x)
\]
= \sum_{k=1}^{\infty} \Delta^{\alpha+1} a_{k-1} \tilde{S}_{k-1}^{\alpha}(x) \cdot

We introduce the auxiliary function

\[ \gamma(U) = \begin{cases} U & \text{for } 0 \leq U < 1 \\ 1 & \text{for } 1 \leq U < \infty \end{cases} \]

We have, by our notation,

\[
\tilde{S}_{k-1}^{\alpha}(x) = - \sum_{v=0}^{k-1} \frac{\alpha-1}{\Delta^{\alpha} A_{k-v-1}} \frac{\cos(v + \frac{1}{2})x}{2 \sin \frac{x}{2}}
\]

Applying Abel's transformation, we obtain

\[
(3.3) \quad \tilde{S}_{k-1}^{\alpha}(x) = - \sum_{v=0}^{k-2} \frac{\alpha-1}{\Delta^{\alpha} A_{k-v-1}} \frac{\sin(v+1)x}{4 \sin^2 \frac{x}{2}} - \frac{\alpha-1}{A_0} \frac{\sin kx}{4 \sin \frac{x}{2}}
\]

\[
= - \sum_{v=0}^{k-2} \frac{\alpha-2}{\Delta^{\alpha-2} A_{k-v-1}} \frac{1}{4 \sin^2 \frac{x}{2}} - \frac{\alpha-2}{A_0} \frac{1}{4 \sin \frac{x}{2}}
\]

\[
= \sum_{v=0}^{k-2} \frac{\alpha-2}{A_{k-v-1}} \tilde{S}_{v}^{\alpha}(x) + \frac{\alpha-2}{A_0} \tilde{S}_{k-1}^{\alpha}(x)
\]

\[
= \sum_{v=0}^{k-1} \frac{\alpha-2}{A_{k-v-1}} \tilde{S}_{v}^{\alpha}(x).
\]
Thus, from (3.2) and (3.3), it follows that

\[ (3.4) \sum_{k=1}^{\infty} a \sin kx = \sum_{k=1}^{\infty} \Delta a_{k-1} \sum_{v=0}^{k-1} \frac{1}{\Delta A_{v}} S_{v}(x) \]

\[ = \sum_{k=1}^{\infty} \Delta a_{k-1} \frac{1}{\Delta A_{v}} \left( \frac{1}{S_{v}(x)} + \frac{\gamma((k-1)x)}{x^2} \right) \]

\[ + \sum_{k=1}^{\infty} \Delta a_{k-1} \frac{1}{\Delta A_{v}} \left( \frac{1}{S_{v}(x)} + \frac{\gamma((k-1)x)}{x^2} \right) \]

But

\[ (3.5) \int_{0}^{\pi} \sum_{k=1}^{\infty} \Delta a_{k-1} \left( \frac{1}{\Delta A_{v}} \left( S_{v}(x) + \frac{S_{v}(x)}{x^2} \right) \right) \, dx \]

\[ \leq \sum_{k=1}^{\infty} \Delta a_{k-1} \int_{0}^{\pi} \sum_{v=0}^{k-1} \frac{\gamma((k-1)x)}{x^2} \, dx \]
Now, following Teljakovskii [43], we have

\[
\pi \sum_{v=0}^{k-1} A_{k-v-1}^{-1} (S_v(x) + \frac{\gamma((k-1)x)}{x^2}) \right| \, dx
\]

\[
= \sum_{v=0}^{k-1} A_{k-v-1}^{-1} \int S_v(x) + \frac{\gamma((k-1)x)}{x^2} \right| \, dx
\]

\[
< C_k \sum_{v=0}^{k-1} A_{k-v-1}^{-1}
\]

\[
= C_k^\alpha.
\]

The relation (3.5) then yields

\[
(3.6) \int \sum_{k=1}^{\alpha+1} \Delta a_{k-1} \left[ \sum_{v=0}^{k-1} A_{k-v-1}^{-1} (S_v(x) + \frac{\gamma((k-1)x)}{x^2}) \right] \, dx
\]

\[
\leq \sum_{k=1}^{k} k^\alpha \Delta a_{k-1}^\alpha
\]

Thus, by (3.4) and (3.6), we obtain

\[
(3.7) \int \sum_{k=1}^{\infty} a_k \sin kx \, dx - \int \sum_{k=1}^{\infty} \Delta a_{k-\alpha+1} \gamma((k-1)x) \, dx
\]

\[
\leq C \sum_{k=1}^{k} k^\alpha \Delta a_{k-1}^\alpha
\]
Moreover,

\[
\int_{s+1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2} \, dx = \int_{s+1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2} \, dx
\]

But

\[
(3.3) \quad \int_{s+1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2} \, dx \leq \sum_{k=1}^{\infty} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2}
\]

Now, let \( \frac{1}{p+1} \leq x \leq \frac{1}{p} \). Then

\[
\sum_{k=1}^{\infty} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2} = \sum_{k=1}^{p} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2} + \sum_{k=p+1}^{\infty} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2}
\]

\[
+ \sum_{k=p+1}^{\infty} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2}
\]

\[
= \sum_{k=1}^{p} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2} + \sum_{k=p+1}^{\infty} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2}
\]

\[
+ \sum_{k=p+1}^{\infty} \frac{a_{k-1}^{\alpha+1}(k-1)x}{x^2}
\]

\[
= \sum_{k=1}^{p} \frac{\alpha-1 \alpha+1 (k-1)}{x} + \sum_{k=p+1}^{\infty} \frac{\alpha-1 \alpha+1}{x}
\]

\[
+ \sum_{k=p+1}^{\infty} \frac{\alpha-1 \alpha+1}{x}
\]
But, a times application of Abel's transformation yield

$$ \sum_{k=p+1}^{\infty} a_{k-1} \Delta a_k = \sum_{k=p+1}^{\infty} a_{k-1} \Delta a_k - \sum_{r=1}^{\infty} a_p \Delta \frac{a_r}{p} $$

and

$$ \sum_{k=p+1}^{\infty} a_{k-1} \Delta a_k = \sum_{k=p+1}^{\infty} a_{k-1} \Delta a_k - \sum_{r=1}^{\infty} a_p \Delta \frac{a_r}{p} $$

Therefore

$$ \sum_{k=1}^{\infty} a_{k-1} \Delta a_k \frac{\gamma((k-1)x)}{x^2} < \frac{-a_p}{x} + \sum_{r=1}^{\infty} a_p \left( \frac{a_r}{p} - \frac{a_{r-1}}{p} \right) $$

Hence

$$ \left| \frac{1}{1+p} \sum_{k=1}^{\infty} a_{k-1} \Delta a_k \frac{\gamma((k-1)x)}{x^2} \right| \left| dx - \int_{1}^{1+p} \frac{|a_p|}{x} dx \right| $$
\[ \frac{\alpha}{r} \sum_{r=1}^{\infty} |\Delta a_p| \left( \frac{1}{p} \int_{x=1}^{\infty} \frac{1}{x^2} - \frac{1}{x^{1+p}} \right) dx \]

\[ = \sum_{r=1}^{\infty} |\Delta a_p| \left( \frac{1}{p} \int_{x=1}^{\infty} \frac{1}{x^2} - \frac{(p-1)}{rx} \right) dx \]

\[ \leq \sum_{r=1}^{\infty} |\Delta a_p| A_{p-1} . \]

Now, since

\[ \left| \frac{1}{p} - \frac{1}{p} \int_{x=1}^{\infty} \frac{1}{x} \right| \leq \frac{1}{2}, \]

we have

\[ \left( 3.9 \right) \left| \frac{1}{p} \int_{k=1}^{\infty} \sum_{k-1}^{\infty} \alpha_{k-1} \Delta a_{k-1} \frac{\Gamma((k-1)x)}{x^2} \right| \leq \left| \frac{a_p}{p} \right| \]

\[ \leq \frac{|a_p|}{p^2} + \sum_{r=1}^{\infty} \alpha_{r-1} |\Delta a_p| \]

\[ \leq \frac{|a_p|}{p^2} + C \sum_{r=1}^{\infty} \alpha_{r-1} |\Delta a_p| . \]

Since \( a_k \to 0 \) as \( k \to \infty \), we have
\[ \max_{k=0}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} \frac{\Delta a_k}{\Delta a_{k-1}} \leq \sum_{k=1}^{\infty} k^{\alpha + 1} \Delta a_{k-1}, \]

which yields
\[ \sum_{p=1}^{\infty} \frac{|a_p|^2}{p} \leq C \sum_{k=1}^{\infty} k^{\alpha + 1} \Delta a_{k-1}, \]

the last step being the consequence of the use of Abel's transformation \( \alpha \) times.

Moreover,
\[ \sum_{r=1}^{\alpha} \sum_{p}^{r-1} \Delta a_p \Delta a_p = \sum_{p=1}^{\infty} (p + 2) a_p a_p + \sum_{p=1}^{\infty} (p + 3) a_p a_p + \ldots + \sum_{p=1}^{\infty} (p + \alpha - 1) a_p a_p \]

and application of Abel's transformation for \( \alpha, \alpha - 1, \ldots, 2, 1 \) time, gives
\[ \sum_{p=1}^{\infty} \sum_{r=1}^{\alpha} \sum_{s=1}^{r-1} \Delta a_p \leq \sum_{p=1}^{\infty} \Delta a_p \alpha^2 \Delta a_p + \sum_{p=1}^{\infty} \Delta a_p \alpha^3 \Delta a_p + \ldots + \sum_{p=1}^{\infty} \Delta a_p \alpha^{\alpha-1} \Delta a_p \]

Thus
\[ \sum_{p=1}^{\infty} \sum_{r=1}^{\alpha} \sum_{s=1}^{r-1} \Delta a_p \leq \sum_{p=1}^{\infty} \Delta a_p \alpha^2 \Delta a_p + \sum_{p=1}^{\infty} \Delta a_p \alpha^3 \Delta a_p + \ldots + \sum_{p=1}^{\infty} \Delta a_p \alpha^{\alpha-1} \Delta a_p \]

The result now follows from the relations (3.7), (3.8), (3.9) and (3.10).
Proof of Theorem 3. Suppose that the conditions (1.5) and (3.1) are satisfied. By Theorem 1, the condition (1.5) together with the fact that $a_k \to 0$ as $k \to \infty$, implies that the sine series is a Fourier series of an $f \in L$. On the other hand, under the given conditions, the series (1.2) is the Fourier series of an $f \in L$, by Theorem A and it converges in $L$ by Theorem H. Hence the sine series also converges in $L$ (cf. [11], Vol.II, p.147).

Conversely, suppose that the sine series converges in $L$. Therefore it is a Fourier series (cf. [11], Vol.I, Chap.I, §12). Hence again by Theorem 1, condition (1.5) is satisfied. On the other hand, by Theorem A, the series (1.2) is a Fourier series of an $f \in L$ whence follows the convergence in $L$ of (1.2). Hence by Theorem C, condition (3.1) is satisfied.