CHAPTER-V
ON GENERALIZED MEASURES OF INFORMATION FOR DISCRETE AND CONTINUOUS DISTRIBUTIONS

5.1 INTRODUCTION

For any two discrete probability distributions \( P = (p_1, p_2, \ldots, p_n) \) and \( Q = (q_1, q_2, \ldots, q_n) \), it is noted that all the axioms of a measure of information for discrete distribution cannot be taken over to the continuous case. A general measure of divergence of a discrete probability distribution \( P \) from another probability distribution \( Q \) has been developed by Csiszer’s [17] and is given by

\[
D(P:Q) = \sum_{i=1}^{n} q_i \phi \left( \frac{p_i}{q_i} \right) \tag{5.1.1}
\]

where \( \phi(\cdot) \) is a twice differentiable convex function for which \( \phi(1) = 0 \).

Replacing \( p_i \) by \( f(x_i) \Delta x_i \) and \( q_i \) by \( g(x_i) \Delta (x_i) \) in (5.1.1), we get

\[
D(f : g) = \sum_{i=1}^{n} g(x_i) \phi \left( \frac{f(x_i)}{g(x_i)} \right) \Delta x_i \tag{5.1.2}
\]

For continuous–variate probability distribution in the interval \([a, b]\), this measure approaches to the following measure:

\[
D(f : g) = \int_{a}^{b} g(x) \phi \left( \frac{f(x)}{g(x)} \right) dx \tag{5.1.3}
\]

Taking different values for \( \phi(x) \), this measure gives many well known measures of divergence for the continuous probability distributions corresponding to the existing measures of divergence for discrete distributions as discussed below:

(i) When \( \phi(x) = x \log x \), we have the following expression:

\[
D(f : g) = \int_{a}^{b} f(x) \log \frac{f(x)}{g(x)} dx \tag{5.1.3}
\]

which is Kullback and Leibler’s [67] measure of divergence.
(ii) When \( \phi(x) = \frac{x - x}{r - 1} \), we have

\[
D(f : g) = \frac{1}{r - 1} \int_a^b g(x) \left[ \left( \frac{f(x)}{g(x)} \right)^r - f(x) \right] \frac{dx}{g(x)}
\]

\[
= \frac{1}{r - 1} \left[ \int_a^b f''(x) g^{1 - r}(x) \, dx - 1 \right], \quad r \neq 1
\]

which is a measure given by Havrada and Charvat [39].

(iii) When \( \phi(x) = x \log x - \frac{1}{c} (1 + c \, x) \log (1 + c \, x) + \frac{x}{c} (1 + c) \log (1 + c) \), we get

\[
D(f : g) = \int_a^b g(x) \left[ \log f(x) g(x) - \frac{1}{c} \left( 1 + c f(x) \right) \log \left( 1 + c \frac{f(x)}{g(x)} \right) \right] \frac{dx}{g(x)}
\]

\[
+ \frac{1}{c} \frac{f(x)}{g(x)} (1 + c) \log (1 + c)
\]

\[
= \int_a^b f(x) \log \frac{f(x)}{g(x)} \, dx - \frac{1}{c} \int_a^b [g(x) + c \, f(x)] \log \frac{g(x) + c \, f(x)}{g(x)} \, dx
\]

\[
+ \frac{1}{c} (1 + c) \log (1 + c)
\]

which is a measure of divergence given by Kapur [57].

Thus, we can derive the measures of directed divergence axiomatically and rigorously for discrete variate probability distributions and then obtain the corresponding measures for the continuous variate probability distributions by considering a limiting process. There exist many well-known measures of directed divergence due to Kullback and Leibler [67], Havrada and Charvat [39], Renyi [114], Ferreri [26], Kapur [51,52], etc. for the discrete probability distributions. Similar expressions of directed divergence or distance measures for continuous variate distributions have been obtained by Salicru and Taneja [117] and Ferentinos and Papaioannou [25].
Since, we have also to deal with fuzzy measures of information; we take into consideration the notion of fuzzy sets introduced by Zadeh [148]. "The notion of fuzzy sets provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary set, but is more applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variable."

Taking into consideration the idea of fuzzy sets, many fuzzy measures of divergence have been discussed and derived by Kapur [58], Lowen [73], Parkash [86], Zadeh [149], Pal and Bezdek [82], Bhandari, Pal and Majumder [8] etc. Motivated by the existing probabilistic measures, many other measures for continuous fuzzy distributions can be developed.

Thus a measure of fuzzy directed divergence for a continuous fuzzy distribution corresponding to Kullback and Leibler's [67] probabilistic measure for continuous probability distribution can be taken as:

\[ I(A, B) = \int_a^b \left( \mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1 - \mu_A(x_i)) \log \frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)} \right) dx_i \]  

(5.1.6)

Similar expressions for the measure of fuzzy directed divergence for continuous cases corresponding to probabilistic measures due to Renyi [114] and Havrada and Charvat [39] can be taken as:

\[ I_r(A, B) = (r - 1)^{-1} \int_a^b \log \left[ \mu_A^r(x_i) \mu_B^{1-r}(x_i) + (1 - \mu_A(x_i))^r (1 - \mu_B(x_i))^{1-r} \right] dx_i \]

; \( r \neq 1, r > 0 \)  

(5.1.7)

\[ I^r(A, B) = (r - 1)^{-1} \int_a^b \left[ \mu_A^r(x_i) \mu_B^{1-r}(x_i) + (1 - \mu_A(x_i))^r (1 - \mu_B(x_i))^{1-r} - 1 \right] dx_i \]

; \( r \neq 1, r > 0 \)  

(5.1.8)
In this chapter, we have introduced some new measures of fuzzy divergence for continuous fuzzy distributions which correspond to some well-known existing probabilistic measures. The essential properties of the proposed measures have been studied for their validity. A new generalized measure of fuzzy divergence has been introduced which contains many existing measures of fuzzy divergence.

In section 5.2, we have introduced a new measure of divergence, known as information radius and studied its important properties. In section 5.3, we have proposed some new generalized measures of divergence for discrete fuzzy distributions which correspond to well known probabilistic measures of divergence already existing in the literature of information theory. In section 5.4, we have introduced new measures of fuzzy directed divergence for the continuous fuzzy distributions and proved their validity. Also, a new generalized measure of fuzzy directed divergence has been developed and many more well known measures have been derived from it. In section 5.5, the unified expressions of entropy, inaccuracy and directed divergence have been presented. A more generalized unified measure of information depending upon two real parameters, which includes these unified expressions, has also been computed.

5.2 INFORMATION RADIUS BASED ON BURG’S [13] MEASURE OF ENTROPY

Let \( P = (p_1, p_2, \ldots, p_n) \) and \( Q = (q_1, q_2, \ldots, q_n) \) be any two discrete probability distributions. It is well-known in the literature of information theory that Shannon’s [118] measure of entropy satisfies the following inequalities:

\[
H(P) \leq H(P \Box Q) \tag{5.2.1}
\]

and

\[
\left( \frac{H(P) + H(Q)}{2} \right) \leq H\left( \frac{P + Q}{2} \right) \tag{5.2.2}
\]

where \( H(P) \) is Shannon’s [118] entropy and \( H(P \Box Q) \) is a measure of inaccuracy due to Kerridge’s [62].
The inequality (5.2.1) is known as Shannon-Gibbs inequality and the inequality (5.2.2) arises due to concavity property of Shannon entropy. The difference in two inequalities (5.2.1) and (5.2.2) given by
\[ D(P; Q) = H(P \oplus Q) - H(P) \]
\[ = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \]  
(5.2.3)
is known as Kullback and Leibler’s [67] directed divergence. And the difference
\[ R(P; Q) = H\left(\frac{P + Q}{2}\right) - \frac{1}{2}(H(P) + H(Q)) \]
(5.2.4)
is known as information radius given by Sibson [124] or Jensen difference divergence measure introduced by Burbea and Rao [12]. By simple calculations, we can write
\[ R(P; Q) = \frac{1}{2} \left\{ D\left(P, \frac{P + Q}{2}\right) + D\left(Q, \frac{P + Q}{2}\right) \right\} \]
(5.2.5)
The expression (5.2.5) is known as R-divergence. Now, we introduce Burg’s [13] entropy instead of Shannon’s [118] entropy in (5.2.4) to obtain a new measure of divergence. By Burg’s [13] entropy, we have
\[ B(P) = \sum_{i=1}^{n} \log p_i, \quad B(Q) = \sum_{i=1}^{n} \log q_i \]
(5.2.6)
Thus, equation (5.2.4) becomes
\[ R(P; Q) = B\left(\frac{P + Q}{2}\right) - \frac{1}{2}(B(P) + B(Q)) \]
\[ = \sum_{i=1}^{n} \log \left(\frac{p_i + q_i}{2}\right) - \frac{1}{2} \sum_{i=1}^{n} \log p_i - \frac{1}{2} \sum_{i=1}^{n} \log q_i \]
\[ = \frac{1}{2} \left\{ \sum_{i=1}^{n} \log \left(\frac{p_i + q_i}{2p_i}\right) + \sum_{i=1}^{n} \log \left(\frac{p_i + q_i}{2q_i}\right) \right\} \]
where

where

\[ R(P;Q) = D(P;Q) + D(Q;P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i q_i} \]  

is directed divergence between P and Q corresponding to Burg's [13] measure of entropy with following properties:

(1) \( D(P;Q) \geq 0 \)

(2) \( D(P;Q) = 0 \) iff \( P = Q \)

(3) \( D(P;Q) \) is a convex function of \( p_1, p_2, \ldots, p_n \) and \( q_1, q_2, \ldots, q_n \)

Under these three properties (5.2.8) is a valid measure of directed divergence.

Next, we discuss properties and generalizations of \( R(P;Q) \):

Properties:

It is obvious that

(1) \( R(P;Q) \geq 0 \)

(2) \( R(P;Q) = 0 \) \( \Rightarrow \) \( P = Q = \frac{P + Q}{2} \)

(3) \( R(P;Q) \) being sum of directed divergences is a convex function of \( p_1, p_2, \ldots, p_n \) and \( q_1, q_2, \ldots, q_n \).

(4) \( R(P;Q) \) is symmetric in \( P \) and \( Q \) that is \( R(P;Q) = R(Q;P) \)

Of course \( R(P;Q) \neq D(P;Q) + D(Q;P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i q_i} \)

Thus, we see that \( R(P;Q) \) defined above satisfies all the essential properties of a measure of directed divergence. Hence, it is a valid measure of R-divergence.

Next, we generalize the R-divergence \( R(P;Q) \) by introducing non-negative \( \lambda \) such that \( 0 \leq \lambda \leq 1 \) as follows:
\( R_\lambda (P;Q) = B \left( \lambda P + (1 - \lambda) Q \right) - \lambda B(P) - (1 - \lambda) B(Q) \) 

(5.2.9)

\[
\begin{align*}
\sum_{i=1}^{n} \log (\lambda p_i + 1 - \lambda q_i) - \lambda \sum_{i=1}^{n} \log p_i - (1 - \lambda) \sum_{i=1}^{n} \log q_i \\
= \lambda D(P; \lambda P + (1 - \lambda) Q) + (1 - \lambda) D(Q; \lambda P + (1 - \lambda) Q)
\end{align*}
\]

(5.2.10)

Now, we study properties of \( R_\lambda (P;Q) \):

(1) \( R_\lambda (P;Q) \geq 0 \) being sum of directed divergence with +ve constants

(2) \( R_\lambda (P;Q) = R_\lambda (P;Q) = 0 \)

(3) \( R_{\lambda+\lambda} (P;Q) = R_{\lambda} (P;Q) \)

(4) \( R_{\lambda-\lambda} (P;Q) = R_{\lambda} (Q;P) \)

(5) \( R_{\lambda+\lambda} (P;Q) = R_{\lambda-\lambda} (Q;P) \)

This indicates that about \( \lambda = \frac{1}{2} \), \( R_{\lambda} (P;Q) \) and \( R_{\lambda-\lambda} (Q;P) \) are symmetrical as shown in Fig.-5.2.1.

Again, we have the following expression:

\[
\frac{dR}{d\lambda} = \sum_{i=1}^{n} \frac{p_i - q_i}{\lambda p_i + (1 - \lambda) q_i} + \ln \sum_{i=1}^{n} \frac{q_i}{p_i}
\]

\[
\frac{d^2R}{d\lambda^2} = -\sum_{i=1}^{n} \frac{(p_i - q_i)^2}{(\lambda p_i + (1 - \lambda) q_i)^2} \leq 0
\]
So \( R_\lambda(P;Q) \) is concave function of \( \lambda \).

We can further generalize \( R_\lambda(P;Q) \) as follows:

Let \( \lambda_j \geq 0 \) and \( \sum_{j=1}^{m} \lambda_j = 1 \)

Then, we have

\[
R(P_1, P_2, \ldots, P_m) = \sum_{j=1}^{m} \lambda_j D\left(P_j; \sum_{j=1}^{m} \lambda_j P_j\right)
\]

is another generalization of information radius given by \( R_\lambda(P;Q) \).

\( R(P_1, P_2, \ldots, P_m) \) is called information radius and \( \sum_{j=1}^{m} \lambda_j P_j \) is called centre where \( 0 \leq \lambda_j \leq 1 \) and \( \sum_{j=1}^{m} \lambda_j = 1 \).

5.3 NEW GENERALIZED MEASURES OF FUZZY DIRECTED DIVERGENCE FOR DISCRETE FUZZY DISTRIBUTIONS

In this section, we have proposed the following new measures of fuzzy directed divergence for the discrete probability distributions:

(1) Generalized Fuzzy Divergence Measure Corresponding to Burg's [13]

Divergence Measure:

We propose the following generalized expression for fuzzy directed divergence measure corresponding is Burg's [13] probabilistic measure of divergence:

\[
D(A : B) = \sum_{i=1}^{n} \left[ \frac{1 + a \mu_A(x_i)}{1 + a \mu_B(x_i)} - \log \frac{1 + a \mu_A(x_i)}{1 + a \mu_B(x_i)} + \frac{1 + a (1 - \mu_A(x_i))}{1 + a (1 - \mu_B(x_i))} \right. \\
\left. - \log \frac{1 + a (1 - \mu_A(x_i))}{1 + a (1 - \mu_B(x_i))} - 2 \right] ; a > 0 \tag{5.3.1}
\]

The measure (5.3.1) is a more generalized form of the measure of fuzzy divergence developed by Parkash and Sharma [99].

To check the validity of the proposed measure, we study its following properties:
I. Non-negativity:

Equation (5.3.1) can be rewritten as

\[ D_1(A:B) = Y + Z \]  \hspace{1cm} (5.3.2)

where

\[ Y = \sum_{i=1}^{n} \left[ \frac{1 + a \mu_A(x_i)}{1 + a \mu_B(x_i)} - \log \frac{1 + a \mu_A(x_i)}{1 + a \mu_B(x_i)} \right] - 1 \]  \hspace{1cm} (5.3.3)

and

\[ Z = \sum_{i=1}^{n} \left[ \frac{1 + a(1 - \mu_A(x_i))}{1 + a(1 - \mu_B(x_i))} - \log \frac{1 + a(1 - \mu_A(x_i))}{1 + a(1 - \mu_B(x_i))} \right] - 1 \]  \hspace{1cm} (5.3.4)

Next, we take

\[ Y = x - \log x - 1 \]  \hspace{1cm} (5.3.5)

where

\[ x = \frac{1 + a \mu_A(x_i)}{1 + a \mu_B(x_i)} \]  \hspace{1cm} (5.3.6)

Since \( x - \log x - 1 \) is positive for \( x > 0 \) and equal to zero for \( x = 1 \), we see that \( Y \geq 0 \) and vanishes iff \( \mu_A(x_i) = \mu_B(x_i) \) for all \( i \).

Similarly, by taking

\[ Z = x' - \log x' - 1 \]  \hspace{1cm} (5.3.7)

where

\[ x' = \frac{1 + a(1 - \mu_A(x_i))}{1 + a(1 - \mu_B(x_i))} \]  \hspace{1cm} (5.3.8)

We can easily show that \( Z \geq 0 \) and vanishes iff \( \mu_A(x_i) = \mu_B(x_i) \) for all \( i \).

Hence, equation (5.3.2) gives that

\[ D_1(A:B) = Y + Z \geq 0 \]

Thus \( D_1(A:B) \geq 0 \) and the equality holds only when \( A = B \).
II. Convexity: We have
\[
\frac{dY}{dx} = 1 - \frac{1}{x} \quad \text{and} \quad \frac{d^2Y}{dx^2} = -\frac{1}{x^2} > 0
\]
which shows that \( Y \) is a convex function of \( \mu_A(x_i) \) and \( \mu_B(x_i) \) \( \forall i \). Similarly, it can be shown that \( Z \) is a convex function of \( \mu_A(x_i) \) and \( \mu_B(x_i) \). Thus, we have:

1. \( D_1(A:B) \geq 0 \)
2. \( D_1(A:B) \) is a convex function of both \( \mu_A(x_i) \) and \( \mu_B(x_i) \) \( \forall i \).
3. \( D_1(A:B) \) does not change when \( \mu_A(x_i) \) is replaced by \( 1 - \mu_A(x_i) \) and \( \mu_B(x_i) \) is replaced by \( 1 - \mu_B(x_i) \).
4. \( D_1(A:B) = 0 \) when \( A = B \).

Under the above four conditions, the proposed measure \( D_1(A:B) \) is a valid measure of weighted fuzzy divergence.

Note: We have \( D(A,B) = \)
\[
\lim_{\alpha \to \infty} D_1(A:B) = \sum_{i=1}^{n} \left[ \left( \frac{\mu_A(x_i)}{\mu_B(x_i)} - \log \frac{\mu_A(x_i)}{\mu_B(x_i)} - 1 \right) + \left( \frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)} - \log \frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)} - 1 \right) \right]
\]
\[(5.3.9)\]
which is another measure of divergence introduced by Parkash and Sharma [99].

(II) Two Generalized Fuzzy Divergence Measures Corresponding to Parkash and Sharma's [99] Divergence Measure:

(a) Consider, \( D_2(A:B) = \)
\[
\sum_{i=1}^{n} \left[ \frac{\mu_A(x_i)}{\lambda \mu_A(x_i) + (1 - \lambda) \mu_B(x_i)} - \log \frac{\mu_A(x_i)}{\lambda \mu_A(x_i) + (1 - \lambda) \mu_B(x_i)} - \frac{1 - \mu_A(x_i)}{1 - \lambda \mu_A(x_i) + (1 - \lambda) (1 - \mu_B(x_i))} - \log \frac{1 - \mu_A(x_i)}{1 - \lambda \mu_A(x_i) + (1 - \lambda) (1 - \mu_B(x_i))} - 2 \right]
\]
\[(5.3.10)\]
If we put $\lambda = 0$ in (5.3.10), we get $D_0(A : B) = D(A : B)$.

(b) Consider $D_2(A : B) = \sum_{i=1}^{n} \left[ \frac{1 + a \mu_A(x_i)}{\lambda(1 + a \mu_A(x_i)) + (1 - \lambda)(1 + a \mu_B(x_i))} - \log \frac{(1 + a \mu_A(x_i))}{\lambda(1 + a \mu_A(x_i)) + (1 - \lambda)(1 + a \mu_B(x_i))} \right]

+ \sum_{i=1}^{n} \left[ \frac{1 + a(1 - \mu_A(x_i))}{\lambda(1 + a(1 - \mu_A(x_i))) + (1 - \lambda)(1 + a(1 - \mu_B(x_i)))} \right]

- \sum_{i=1}^{n} \left[ \log \frac{1 + a(1 - \mu_A(x_i))}{\lambda(1 + a(1 - \mu_A(x_i))) + (1 - \lambda)(1 + a(1 - \mu_B(x_i)))} - 2 \right]

(5.3.11)

Again, if we put $\lambda = 0$ in (5.3.11), we get the following result:

$D_0(A : B) = D(A : B)$.

It can easily be verified that the generalized measures introduced in (5.3.10) and (5.3.11) are valid measures of fuzzy directed divergence.

**Note:** The monotonic character of the above parametric divergence measures can be discussed as below:

(I) Differentiating (5.3.10) with respect to $\lambda$, we get

$$\frac{d}{d\lambda} D_\lambda(A : B) = \sum_{i=1}^{n} \frac{(\lambda - 1)(\mu_A(x_i) - \mu_B(x_i))^2}{\lambda(1 + (\lambda - 1)\mu_A(x_i)) + (1 - \lambda)(1 + (1 - \lambda)\mu_B(x_i))}$$

Thus, we see that the weighted fuzzy measure $D_\lambda(A, B)$ is monotonically decreasing function of $\lambda$, $0 < \lambda < 1$.

To check the decreasing behavior of the divergence measure $D_\lambda(A, B)$, we have computed different values of $D_\lambda(A, B)$ with the help of different parameters and different fuzzy values. The results so obtained have been presented in Table-5.3.1. The different values of the weighted measure of fuzzy entropy have been presented graphically as shown in Fig-5.3.1:
Table 5.3.1

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</table>
Differentiating (5.3.11) with respect to $x$, we again see that
\[
\frac{d}{d\lambda} D_\lambda (A \cup B) < 0
\]
Thus, we see that $D_\lambda (A \cup B)$ is monotonically decreasing function of $\lambda$, $0 < \lambda < 1$.

Next, to check the decreasing behavior of $D_\lambda (A \cup B)$, we have computed different values of $D_\lambda (A \cup B)$ with the help of different parameters and different fuzzy values. The results so obtained have been presented in the Table-5.3.2.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu_\lambda (x_i)$</th>
<th>$\mu_\beta (x_i)$</th>
<th>$D_\lambda (A \cup B)$</th>
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<td>0.9</td>
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The different values of the weighted fuzzy measure have been presented graphically as shown in following Fig-5.3.2:

![Graph with 2D(A, B) vs lambda](image)

**Fig-5.3.2**

(III) Measure Of Fuzzy Divergence Corresponding To Jensen-Shannon Measure And Its Generalization:

We take the following expression for this measure:

\[
\hat{D}_1(A : B) = \\
\sum_{i=1}^{n} \left[ \mu_a(x_i) \log \frac{\mu_a(x_i)}{\lambda \mu_a(x_i) + (1-\lambda) \mu_b(x_i)} + (1- \mu_a(x_i)) \log \frac{1-\mu_a(x_i)}{\lambda (1-\mu_a(x_i)) + (1-\lambda)(1-\mu_b(x_i))} \right]
\]

(5.3.12)

We, prove that the weighted fuzzy measure (5.3.12) is a convex function of \(\mu_a(x_1), \mu_a(x_2), ..., \mu_a(x_n)\) and \(\mu_b(x_1), \mu_b(x_2), ..., \mu_b(x_n)\). The other properties for being a divergence measure are obvious. Next, rewriting equation (5.3.12), we have the following expression for the divergence measure:
\[ D_1(A : B) = \]
\[ -\sum_{i=1}^{n} \left[ \mu_a(x_i) \log \{ \lambda \mu_a(x_i) + (1 - \lambda) \mu_b(x_i) \} + (1 - \mu_a(x_i)) \log \{ \lambda (1 - \mu_a(x_i)) + (1 - \lambda)(1 - \mu_b(x_i)) \} \right] 
+ \sum_{i=1}^{n} \left[ \mu_a(x_i) \log \mu_a(x_i) + (1 - \mu_a(x_i)) \log(1 - \mu_a(x_i)) \right] \]

Differentiating (5.3.13) w.r.t. \( \mu_b(x_i) \), we get
\[ \frac{d}{d \mu_b(x_i)} \, D_1(A : B) = \]
\[ \frac{(1 - \lambda) \mu_a(x_i)}{\lambda \mu_a(x_i) + (1 - \lambda) \mu_b(x_i)} + \frac{(1 - \lambda)(1 - \mu_a(x_i))}{\lambda(1 - \mu_a(x_i)) + (1 - \lambda)(1 - \mu_b(x_i))} \]

Also, we have
\[ \frac{\partial^2 D_1(A : B)}{\partial \mu_b^2(x_i)} = \frac{(1 - \lambda)^2 \mu_a(x_i)}{[\lambda \mu_a(x_i) + (1 - \lambda) \mu_b(x_i)]^2} + \frac{(1 - \lambda)^2(1 - \mu_a(x_i))}{[\lambda(1 - \mu_a(x_i)) + (1 - \lambda)(1 - \mu_b(x_i))]^2} \]
\[ > 0 \]
Thus \( D_1(A : B) \) is a convex function of \( \mu_b(x_i) \). Similarly, we can prove that it is convex function of \( \mu_a(x_i) \).

The above directed divergence can be generalized as follows:
\[ D_2(A : B) = \]
\[ \lambda \sum_{i=1}^{n} \left[ \mu_a(x_i) \log \frac{\mu_a(x_i)}{\lambda \mu_a(x_i) + (1 - \lambda) \mu_b(x_i)} + (1 - \mu_a(x_i)) \log \frac{1 - \mu_a(x_i)}{\lambda(1 - \mu_a(x_i)) + (1 - \lambda)(1 - \mu_b(x_i))} \right] 
+ (1 - \lambda) \sum_{i=1}^{n} \left[ \mu_b(x_i) \log \frac{\mu_b(x_i)}{\lambda \mu_a(x_i) + (1 - \lambda) \mu_b(x_i)} + (1 - \mu_b(x_i)) \log \frac{1 - \mu_b(x_i)}{\lambda(1 - \mu_a(x_i)) + (1 - \lambda)(1 - \mu_b(x_i))} \right] \]

(5.3.14)
\[(1-\lambda)\sum_{i=1}^{n} \left[ \mu_a(x_i) \log \mu_b(x_i) + (1-\mu_a(x_i)) \log(1-\mu_b(x_i)) \right] \]

\[-\sum_{i=1}^{n} \left[ (\lambda \mu_a(x_i) + (1-\lambda)\mu_b(x_i)) \log (\lambda \mu_a(x_i) + (1-\lambda)\mu_b(x_i)) \right] \]

\[-\sum_{i=1}^{n} \{ \lambda (1-\mu_a(x_i)) + (1-\lambda)(1-\mu_b(x_i)) \} \log \{ \lambda (1-\mu_a(x_i)) + (1-\lambda)(1-\mu_b(x_i)) \} \]

\[= H(\lambda A + (1-\lambda)B) - \lambda H(A) - (1-\lambda)H(B) \]

where

\[H(A) = -\sum_{i=1}^{n} \left[ \mu_a(x_i) \log \mu_a(x_i) + (1-\mu_a(x_i)) \log(1-\mu_a(x_i)) \right] \]

and it represents a measure of fuzzy entropy corresponding to Shannon's [118] entropy. The measure (5.3.14) is a generalization of Parkash and Sharma's [99] measure of fuzzy divergence which arises when \(\lambda = \frac{1}{2}\).

5.4 NEW GENERALIZED MEASURES OF FUZZY DIRECTED DIVERGENCE FOR CONTINUOUS FUZZY DISTRIBUTIONS

In this section, we have proposed the following generalized measures of fuzzy directed divergence for continuous fuzzy distributions:

(I) \(K(A: B) = \int_{a}^{b} \left[ \mu_A(x_i) \log \left( \frac{\mu_A(x_i)}{(\mu_A(x_i) + \mu_B(x_i))/2} \right) \right] dx_i \) \hspace{1cm} (5.4.1)

(II) \(S(A: B) = \frac{1}{\alpha - \beta} \int_{a}^{b} \left[ \mu_A^\alpha(x_i) \mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha} - \mu_A^\beta(x_i) \mu_B^{1-\beta}(x_i) - (1-\mu_A(x_i))^\beta(1-\mu_B(x_i))^{1-\beta} \right] dx_i \) \hspace{1cm} (5.4.2)

;\(\alpha < 1, \beta > 1\) or \(\alpha > 1, \beta < 1\)
The fuzzy measure (5.4.1) corresponds to Lin's [69] probabilistic measure of directed divergence whereas the measure (5.4.2) corresponds to Sharma and Taneja's [123] probabilistic measure of directed divergence.

**Validity of New Measures of Fuzzy Divergence**

Taking \( \int_a^b \mu_A(x_i) \, dx_i = \alpha_0 \); \( \int_a^b \mu_B(x_i) \, dx_i = \beta_0 \),

where \( \alpha_0, \beta_0 \) may be different from unity.

We know that

\[
\int_a^b \mu_A(x_i) \log \left( \frac{\mu_A(x_i)}{(\mu_A(x_i)+\mu_B(x_i))/2} \right) \, dx_i \geq 0
\]

that is,

\[
\int_a^b \mu_A(x_i) \log \frac{\mu_A(x_i)}{(\mu_A(x_i)+\mu_B(x_i))/2} \, dx_i \geq \alpha_0 \log \frac{\alpha_0}{(\alpha_0+\beta_0)/2}
\]

(5.4.3)

Similarly,

\[
\int_a^b (1-\mu_A(x_i)) \log \frac{1-\mu_A(x_i)}{(1-\mu_A(x_i)+1-\mu_B(x_i))/2} \, dx_i \geq (n-\alpha_0) \log \frac{n-\alpha_0}{(2n-\alpha_0-\beta_0)/2}
\]

(5.4.4)

Adding (5.4.3) and (5.4.4), we get

\[ K(A: B) \geq f(\alpha_0) \]

where

\[ f(\alpha_0) = \alpha_0 \log \frac{\alpha_0}{(\alpha_0+\beta_0)/2} + (n-\alpha_0) \log \frac{n-\alpha_0}{(2n-\alpha_0-\beta_0)/2} \]

Now

\[ f'(\alpha_0) = \log \alpha_0 - \frac{\alpha_0}{\alpha_0+\beta_0} - \log (\alpha_0+\beta_0) - \log (n-\alpha_0) + \frac{n-\alpha_0}{2n-\alpha_0-\beta_0} + \log (2n-\alpha_0-\beta_0) \]
and

\[
f''(\alpha_0) = \frac{1}{\alpha_0} - \frac{1}{\alpha_0 + \beta_0} \cdot \frac{\beta_0}{(\alpha_0 + \beta_0)^2} + \frac{1}{n - \alpha_0} - \frac{1}{2n - \alpha_0 - \beta_0} \cdot \frac{n - \beta_0}{(2n - \alpha_0 - \beta_0)^2}
\]

\[
= \frac{\beta_0^2}{\alpha_0 (\alpha_0 + \beta_0)^2} + \frac{(n - \beta_0)^2}{(n - \alpha_0)(2n - \alpha_0 - \beta_0)^2} > 0
\]

The above expression shows that \( f(\alpha_0) \) is a convex function of \( \alpha_0 \) which has its minimum value when \( \alpha_0 = \beta_0 \) and the minimum value is 0 so that \( f(\alpha_0) > 0 \) and vanishes when \( \alpha_0 = \beta_0 \). Consequently, \( K(A; B) \) is a convex function of \( \mu_A(x_i) \).

Similarly, we can show that \( K(A; B) \) is a convex function of \( \mu_B(x_i) \). Thus, for all values of \( \alpha_0 \) and \( \beta_0 \), we have

(i) \( K(A; B) \geq 0 \)

(ii) \( K(A; B) = 0 \) iff \( A = B \)

(iii) \( K(A; B) \) is a convex function

(iv) \( K(A; B) \) doesn't change when \( \mu_A(x_i) \) is changed to \( 1 - \mu_A(x_i) \) and \( \mu_B(x_i) \) is changed to \( 1 - \mu_B(x_i) \).

Hence \( K(A; B) \) is a valid measure of fuzzy directed divergence and

Again, we know that

\[
b \int_a^b \left[ \frac{\mu_A(x_i)}{\alpha_0} \left( \frac{\mu_B(x_i)}{\beta_0} \right)^{-\alpha} - 1 \right] dx_i \geq 0
\]

that is,

\[
b \int_a^b \mu_A^{\alpha}(x_i) \mu_B^{1-\alpha}(x_i) dx_i \geq \alpha_0^{\alpha} \beta_0^{1-\alpha}
\]

\[ (5.4.5) \]

Similarly

\[
b \int_a^b (1-\mu_A(x_i))^{\alpha} (1-\mu_B(x_i))^{1-\alpha} dx_i \geq (n-\alpha_0)^{\alpha} (n-\beta_0)^{1-\alpha}
\]

\[ (5.4.6) \]

Adding (5.4.5) and (5.4.6), we get

\[
b \int_a^b \left[ \mu_A^{\alpha}(x_i) \mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} (1-\mu_B(x_i))^{1-\alpha} \right] dx_i
\]
Similarly,
\[
\int_{a}^{b} \left[ \mu_{A}^{\beta} (x_i) \mu_{B}^{1-\beta} (x_i) + (1-\mu_{A}(x_i))^{\beta} (1-\mu_{B}(x_i))^{1-\beta} \right] dx_i \\
\geq \alpha_0 \alpha \beta^0_{1-\alpha} + (n-\alpha_0)^{\alpha} (n-\beta_0)^{1-\alpha} \tag{5.4.7}
\]
Subtracting (5.4.8) from (5.4.7) and dividing by (\alpha-\beta), we get
\[
S (A:B) \geq F (\alpha_0)
\]
where
\[
F (\alpha_0) = \frac{1}{\alpha - \beta} \left[ \alpha_0 \alpha \beta^0_{1-\alpha} + (n-\alpha_0)^{\alpha} (n-\beta_0)^{1-\alpha} \right]
\]
Now
\[
F'(\alpha_0) = \frac{1}{\alpha - \beta} \left[ \alpha \left( \frac{\alpha_0}{\beta_0} \right)^{\alpha-1} - \alpha \left( \frac{n-\alpha_0}{n-\beta_0} \right)^{\alpha-1} \right]
\]
and
\[
F''(\alpha_0) = \frac{1}{\alpha - \beta} \left[ \alpha (\alpha - 1) \left( \frac{\alpha_0}{\beta_0} \right)^{\alpha-2} + \alpha (\alpha - 1) \left( \frac{n-\alpha_0}{n-\beta_0} \right)^{\alpha-2} \right]
\]
\[
= \frac{\alpha (\alpha - 1)}{\alpha - \beta} \left[ \frac{1}{\beta_0} \left( \frac{\alpha_0}{\beta_0} \right)^{\alpha-2} + \frac{1}{n-\beta_0} \left( \frac{n-\alpha_0}{n-\beta_0} \right)^{\alpha-2} \right]
\]
\[
> 0 \quad \text{for} \quad \alpha > 1, \beta < 1 \quad \text{or} \quad \alpha < 1, \beta > 1
\]
so that \(F(\alpha_0)\) is a convex function of \(\alpha_0\) whose minimum value arises when
\[
\frac{\alpha_0}{\beta_0} = \frac{n-\alpha_0}{n-\beta_0} = 1 \quad \text{that is,} \quad \alpha_0 = \beta_0
\]
and the minimum value is 0 so that \(F(\alpha_0) > 0\) and vanishes only when \(\alpha_0 = \beta_0\),
that is, when \(A = B\). Consequently, \(S(A:B)\) is a convex function of \(\mu_{A}(x_i)\).
Similarly, \(S(A:B)\) is a convex function of \(\mu_{B} (x_i)\). Thus for all values of \(\alpha_0 \) and \(\beta_0\),
we have the following desirable properties:
(i) \( S(A: B) \geq 0 \)
(ii) \( S(A: B) = 0 \) iff \( A = B \)
(iii) \( S(A: B) \) is a convex function
(iv) \( S(A: B) \) doesn’t change when \( \mu_A(x_i) \) is replaced by \( 1-\mu_A(x_i) \) and \( \mu_B(x_i) \) by \( 1-\mu_B(x_i) \).

Hence, \( S(A: B) \) is a valid measure of fuzzy directed divergence and if we take \( \beta = 1 \), \( S(A: B) \) becomes a measure of fuzzy directed divergence corresponding to Havrada and Charvat’s [39] measure and if we take \( \beta = 1 \) and \( \alpha \rightarrow 1 \), \( S(A: B) \), becomes Bhandari and Pal’s [7] measure of fuzzy directed divergence.

### III. Generalized Fuzzy Directed Divergence

We consider

\[
I_k(A:B) = \int_a^b \left( \lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i) \right) \phi \left( \frac{\mu_A(x_i)}{\lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i)} \right) + \frac{(1-\lambda) \mu_B(x_i)}{\lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i)} \right] dx_i \quad (5.4.9)
\]

where \( \phi(\cdot) \) is twice differentiable convex function for which \( \phi(1) = 0 \)

Now

\[
\frac{\partial I_k(A:B)}{\partial \mu_A(x_i)} = \lambda \phi \left( \frac{\mu_A(x_i)}{\lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i)} \right) + \frac{(1-\lambda) \mu_B(x_i)}{\lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i)}
\]

\[
\phi' \left( \frac{\mu_A(x_i)}{\lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i)} \right) - \lambda \phi \left( \frac{1-\mu_A(x_i)}{\lambda (1-\mu_A(x_i)) + (1-\lambda) (1-\mu_B(x_i))} \right)
\]

\[
- \frac{(1-\lambda) \mu_B(x_i)}{\lambda (1-\mu_A(x_i)) + (1-\lambda) (1-\mu_B(x_i))} \phi' \left( \frac{1-\mu_A(x_i)}{\lambda (1-\mu_A(x_i)) + (1-\lambda) (1-\mu_B(x_i))} \right)
\]

and

\[
\frac{\partial^2 I_k(A:B)}{\partial \mu_A^2(x_i)} = \lambda^2 \frac{(1-\lambda)^2 \left(1-\mu_B(x_i)\right)^2}{\left(\lambda (1-\mu_A(x_i)) + (1-\lambda) (1-\mu_B(x_i))\right)^3} + \frac{(1-\lambda)^2 \mu_B^2(x_i)}{\left[\lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i)\right]^3} > 0
\]
so that $I_\lambda (A : B)$ is a convex function of $\mu_A(x_i)$ which has its minimum value when $\mu_A(x_i) = \mu_B(x_i)$ and the minimum value is 0 so that $I_\lambda (A : B) > 0$ and vanishes when $\mu_A(x_i) = \mu_B(x_i)$. Similarly, $I_\lambda (A : B)$ is a convex function of $\mu_B(x_i)$.

Thus for all values of $\mu_A(x_i)$ and $\mu_B(x_i)$, we have

(i) $I_\lambda (A : B) \geq 0$

(ii) $I_\lambda (A : B) = 0 \iff A = B$

(iii) $I_\lambda (A : B)$ is a convex function

(iv) $I_\lambda (A : B)$ doesn’t change when $\mu_A(x_i)$ is replaced by $1 - \mu_A(x_i)$ and $\mu_B(x_i)$ by $1 - \mu_B(x_i)$.

Hence $I_\lambda (A : B)$ is a valid generalized measure of fuzzy directed divergence.

**Special Cases:**

(i) Taking $\phi(x) = x \log x$ in (5.4.9), we get

$$I_{1,\lambda} (A : B) = \int_a^b \mu_A(x_i) \log \left( \frac{\mu_A(x_i)}{\lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i)} \right)$$

$$+ \left( 1 - \mu_A(x_i) \right) \log \left( \frac{1 - \mu_A(x_i)}{\lambda (1 - \mu_A(x_i)) + (1 - \lambda) (1 - \mu_B(x_i))} \right) dx_i \quad (5.4.10)$$

The expression (5.4.10) is a generalization of (5.4.1).

(a) If we take $\lambda = 0$ in (5.4.10), we get

$$I_{1,0} (A : B) = \int_a^b \mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1 - \mu_A(x_i)) \log \frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)} dx_i \quad (5.4.11)$$

which is a measure of fuzzy directed divergence corresponding to probabilistic measure of divergence introduced by Kullback and Leibler [67].

(b) If we take $\lambda = \frac{1}{2}$ in (5.4.10), we get

$$I_{1,\frac{1}{2}} (A : B) = \int_a^b \mu_A(x_i) \log \left( \frac{\mu_A(x_i)}{(\mu_A(x_i) + \mu_B(x_i))/2} \right)$$
which is equation (5.4.1).

(ii) Let \( \phi(x) = \frac{x^\alpha - x}{\alpha (\alpha - 1)} \), \( \alpha \neq 0, \alpha \neq 1 \), then equation (5.4.9) gives

\[
I_{2,\lambda} (A: B) = \frac{1}{\alpha (\alpha - 1)} \int_a^b \left[ \mu_A^\alpha(x_i) \{ \lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i) \}^{1-\alpha} + (1-\mu_A(x_i)) \{ \lambda (1-\mu_A(x_i)) + (1-\lambda) (1-\mu_B(x_i)) \}^{1-\alpha} \right] dx_i \tag{5.4.12}
\]

(a) If we take \( \lambda = 0 \) in equation (5.4.12), we get

\[
I_{2,0}(A:B) = \frac{1}{\alpha (\alpha - 1)} \int_a^b \left[ \mu_A^\alpha(x_i) \mu_B^{-\alpha}(x_i) + (1-\mu_A(x_i))^{1-\alpha} \right] dx_i
\]

which is a measure of fuzzy directed divergence corresponding to Havrada and Charvat's \([39]\) probabilistic divergence.

(b) From equation (5.4.12), we have \( I_{2,\lambda} (A: B) = I_{1,\lambda} (A: B) \).

(c) When \( \lambda = \frac{1}{2} \) and \( \alpha \to 1 \), equation (5.4.12) becomes \( K (A: B) \).

(d) If we take \( \lambda = 0, \alpha \to 1 \) in (5.4.12), we get Bhandari and Pal's \([7]\) measure.

(iii) If we take \( \phi(x) = \frac{x^\alpha - x^\beta}{\alpha - \beta} \) in (5.4.9), we get

\[
I_{3,\lambda} (A: B) = \frac{1}{\alpha - \beta} \int_a^b \left[ \mu_A^\alpha(x_i) \{ \lambda \mu_A(x_i) + (1-\lambda) \mu_B(x_i) \}^{1-\alpha} + (1-\mu_A(x_i)) \{ \lambda (1-\mu_A(x_i)) + (1-\lambda) (1-\mu_B(x_i)) \}^{1-\alpha} \right] dx_i \tag{5.4.13}
\]

which is a generalization of the measure (5.4.2).
(a) If we take \( \lambda = 0 \) in equation (5.4.13), we get
\[
I_{3,0}^*(A: B) = \frac{1}{\alpha - \beta} \sum_a \left[ \mu_A^\alpha(x_i) \mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha (1-\mu_B(x_i))^{1-\alpha} \mu_A^\beta(x_i) \mu_B^{1-\beta}(x_i) - (1-\mu_A(x_i))^\beta (1-\mu_B(x_i))^{1-\beta} \right] \alpha_i
\]
which is (5.4.2)

(b) If we take \( \lambda = 0 \) and \( \beta = 1 \) in equation (5.4.13), we get
\[
I_{3,0}^*(A: B) = \frac{1}{\alpha - 1} \sum_a \left[ \mu_A^\alpha(x_i) \mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha (1-\mu_B(x_i))^{1-\alpha} \right] \alpha_i
\]
which is a measure of fuzzy directed divergence corresponding to Havrada and Charvat's [39] probabilistic directed divergence.

5.5 GENERALIZED UNIFIED INFORMATION MEASURE FOR PROBABILITY DISTRIBUTIONS

In this section, we have been presented the unified expressions of entropy, inaccuracy and directed divergence for discrete probability distributions. A generalized unified measure of information, depending upon two real parameters, which includes these unified expressions, has been computed. For this purpose, let
\[
\Delta_n = \{p = p_1, p_2, \ldots, p_n\}, p_i \geq 0, \sum_{i=1}^{n} p_i = 1, n \geq 2, \text{be a set of all probability distributions, associated with a discrete finite random variable. Then, we know that the following measures are well known:}
\]
\[
H(P) = -\sum_{i=1}^{n} p_i \log p_i \quad (5.5.1)
\]
\[
H(P\|Q) = -\sum_{i=1}^{n} p_i \log q_i \quad (5.5.2)
\]
\[
D(P\|Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \quad (5.5.3)
\]
The above results are respectively called Shannon’s [118] entropy, Kerridge’s [62] inaccuracy and Kullback and Leibler’s [67] relative information. Rathie and Taneja [113] obtained the following $(r,s)$-entropy involving two real parameters $r$ and $s$:

$$H_{sr}^s(P) = [(1-r)s]^{-1} \left\{ \sum_{i=1}^{n} p_i^r \right\}^s - 1; s \neq 0, r \neq 1, r > 0$$  \hspace{1cm} (5.5.4)

This $(r,s)$-entropy (5.5.4) includes Shannon’s [118] entropy and Renyi’s [114] entropy as limiting cases whereas Havrda and Charvat’s [39] entropy is a particular case of it. Next, we have developed the unified expressions of entropy, inaccuracy and directed divergence. A generalized unified measure which includes these unified expressions has also been provided.

**Generalized Unified Information Measures**

(a) **Unified $(r,s)$-entropy**

$$H_{r,s}^s(P) = \begin{cases} 
H_{sr}^s(P) = [(1-r)s]^{-1} \left\{ \sum_{i=1}^{n} p_i^r \right\}^s - 1; s \neq 0, r \neq 1, r > 0 \\
H_{rl}^1(P) = (1-r)^{-1} \left[ \sum_{i=1}^{n} p_i^r - 1 \right]; s = 1, r \neq 1 \\
H_{r0}^0(P) = (1-r)^{-1} \log \left( \sum_{i=1}^{n} p_i^r \right); s = 0, r \neq 1 \\
H(P) = -\sum_{i=1}^{n} p_i \log p_i; s = 0, r = 1 
\end{cases}$$  \hspace{1cm} (5.5.5)

Parkash and Singh [106] have obtained the following expression for $(r,s)$-directed divergence measure:

$$D_{r,s}^s(P \parallel Q) = [(r-1)s]^{-1} \left\{ \sum_{i=1}^{n} p_i^{r^*} q_i^{s^*} \right\}^s - 1; s \neq 0, r \neq 1, r > 0$$  \hspace{1cm} (5.5.6)
The measure (5.5.6) includes Renyi’s [114] directed divergence and Kullback and Leibler’s [67] divergence measure as limiting cases whereas Havrada and Charvat’s [39] divergence measure is a particular case of it.

The unified expression of these measures of relative information can be drawn as:

**(b) Unified (r,s)-relative information**

\[
D^r_s(P \| Q) = \begin{cases} 
D^1_r(P \| Q) = (r-1)^{-1} \left[ \sum_{i=1}^{n} p_i^{r-r} q_i^{1-r} - 1 \right] ; s \neq 0, r \neq 1, r > 0 \\
D^0_s(P \| Q) = (r-1)^{-1} \left[ \sum_{i=1}^{n} p_i^{r-r} - 1 \right] ; s = 1, r \neq 1 \\
D^s_0(P \| Q) = (r-1)^{-1} \log \left[ \sum_{i=1}^{n} p_i^{r-r} \right] ; s = 0, r \neq 1 \\
D(P \| Q) = \sum_{i=1}^{n} p_i \log \frac{P_i}{Q_i} ; s = 0, r = 1 
\end{cases} 
\]  

(5.5.7)

Kapur [53] has provided following measures of inaccuracy:

\[
H_r(P \| Q) = (r-1)^{-1} \left[ \sum_{i=1}^{n} p_i^{r-r} q_i^{1-r} - \sum_{i=1}^{n} p_i^{r} \right] ; r \neq 1 
\]  

(5.5.8)

and

\[
H_r(P \| Q) = (r-1)^{-1} \log \left[ \frac{\sum_{i=1}^{n} p_i^{r-r}}{\sum_{i=1}^{n} p_i^{r}} \right] ; r \neq 1 
\]  

(5.5.9)

We propose the following measure of inaccuracy depending upon two real parameter’s r and s:

\[
H^s_r(P \| Q) = [(r-1)s]^{-1} \left[ \left( \sum_{i=1}^{n} p_i^{r-r} q_i^{1-r} \right)^s - \left( \sum_{i=1}^{n} p_i^{r} \right)^s \right] ; s \neq 0, r \neq 1, r > 0 
\]  

(5.5.10)
We call the measure (5.5.10) as (r,s)- inaccuracy. It may be noted that the inaccuracy (5.5.10) contains (5.5.2) and (5.5.9) in the limiting cases and (5.5.8) in particular case. The unified expression of these measures of inaccuracy can be obtained as given below:

(c) Unified (r,s) inaccuracy

\[
H^r_s(P \| Q) = \left( (r-1)s \right)^{-1} \left\{ \left( \sum_{i=1}^{n} p_i^r q_i^{1-r} \right) - \left( \sum_{i=1}^{n} p_i^r \right)^s \right\}; s \neq 0, r \neq 1
\]

\[
H^0_r(P \| Q) = (r-1)^{-1} \log \left( \sum_{i=1}^{n} p_i^r q_i^{1-r} \right); s = 0, r \neq 1
\]

\[
H(P \| Q) = -\sum_{i=1}^{n} p_i \log q_i; s = 0, r = 1
\]

(5.5.11)

It is customary to study the generalized measures of entropy, directed divergence and inaccuracy mentioned in equations (5.5.5), (5.5.7) and (5.5.11) for positive values of r and s. In this chapter, our aim is to study these generalized measures for \((r, s) \in (-\infty, \infty)\). By considering \(r < 0\), the problem which arises is that, we can find probability distributions such that the generalized measures (5.5.5), (5.5.7) and (5.5.11) become infinite. To overcome this drawback, Taneja [136] defined the set of probability \(\Delta_s\) in the following way:
Thus any result holding for \( r \neq 1, \ s \neq 0 \) extends for any \( r \) and \( s \).

Now for \( (V,W) \in \Delta_n \times \Gamma_n \), let us consider the following generalized measure:

**Generalized Unified \((r,s)\) Measure**

\[
\Phi^t_r(V \parallel W) = \begin{cases} 
(1-t)s^{-1} & \text{for } s \neq 0, t \neq 1 \\
(1-t)^{-1} \left[ \sum_{i=1}^{n} v_i w_i^{t-1} \right] & \text{for } s = 1, t \neq 1 \\
\log \left( \sum_{i=1}^{n} v_i w_i^{t-1} \right) & \text{for } s = 0, t \neq 1 \\
-\sum_{i=1}^{n} v_i \log w_i & \text{for } s = 0, t = 1 
\end{cases}
\]

for all \( t, s \in (-\infty, \infty) \), where

\[
\Gamma_n = \{ W = (w_1, w_2, \ldots, w_n), w_i \geq 0, i = 1, 2, \ldots, n \} \subset R^n 
\]

for \( t < 0, w_i > 0, v_i > 0 \) for \( i = 1, 2, \ldots, n \).

The measures (5.5.5), (5.5.7) and (5.5.11) can be obtained as the particular cases of the measure (5.5.12) in the following way:

(i) Take \( v_i = w_i = p_i, \forall i = 1, 2, \ldots, n \) and \( t = r \), in (5.5.12), we get

\[
\Phi^t_r(V \parallel W) = H^t_r(P).
\]

(ii) Take \( v_i = p_i', w_i = q_i, \forall i = 1, 2, \ldots, n \) and \( t = (2-r) \), in (5.5.12), we get

\[
\Phi^t_r(V \parallel W) = H^t_r(P \parallel Q).
\]

(iii) Take \( v_i = p_i, w_i = \frac{q_i}{p_i}, \forall i = 1, 2, \ldots, n \) and \( t = 2-r \), in (5.5.12), we get

\[
\Phi^t_r(V \parallel W) = D^r(P \parallel Q).
\]

We call the measure (5.5.12) as generalized unified measure of information.
Concluding Remarks: In real life situation, we know that there are non-exponential growth models, a variety of models in Economics, Social Sciences, Biology and even in Physical Sciences, thus we need a variety of information measures for each field to extend the scope of their applications. Hence the development of new generalized parametric measures is necessary. Thus, it is concluded that taking into consideration the existing fuzzy as well as probabilistic measures of divergence, some new generalized measures of directed divergence for discrete as well as continuous probability have been developed. With similar motive some new fuzzy divergence measures for discrete as well as continuous fuzzy distributions have been developed in this chapter.