CHAPTER-IV
CHARACTERIZATIONS OF FUZZY MEASURES OF ENTROPY THROUGH CONCAVITY AND RECURSIVITY

4.1. INTRODUCTION

Shannon's [118] "a mathematical theory of communication" published in 1948 is the Magna Carta of the information age when he discovered the fundamentals laws of data compression and transmission and consequently marked the birth of Information Theory. A unifying theory with profound intersections with Probability, Statistics, Computer science, and other fields, Information Theory continues to set the stage for the development communications, data storage and processing, and other information technologies. The main achievements in information theory which confines itself to those disciplines directly spawned from Shannon's [118] paper are now commonly referred to as Shannon theory. In fact Shannon's measure of entropy has very nice properties and this entropy has been literally used in tens of thousands of papers in Physical, Mathematical, Biological and Social Sciences. Recently, Nanda and Paul [77] observed that Shannon's [118] entropy may be negative for some probability distribution and this fact led Khinchin to define residual entropy instead.

In the literature of information theory, the probabilistic measures of entropy due to Shannon [118] and Havrada-Charvat [39] are quiet familiar. There have been many characterizations of these measures which have been provided by Shannon [118], Havrada-Charvat [39], Mathai and Rathie [75] etc. Recently, Lavenda [68] has studied in-depth the analysis of mean entropies, particularly Shannon's [118] and Renyi's [114] entropies, which are expressed as negative logarithms of some means. The functional forms of these entropies follow from the multiplicative law of means. This is well-known standard result proved by Aczel and Daroczy [2]. Lavenda [68] has also shown that exponential entropy measures the extent of a probability...
distribution, that is, error occurring in using an estimated probability in place of the true probability. The analysis leads to some interesting results involving errors and entropies.

Zuripov [155] defined the abelian entropy group, described its most general properties, and derived a general law of composition of non-additive entropies with a quadratic nonlinearity. He introduced trigonometric entropies and, based on them, determined equilibrium distributions and also obtained thermodynamic relations for non-extensive systems. Pierre-Olivier and Charistophe [111] have studied the link between non-additivity of some entropies and their boundedness. The authors proposed an axiomatic construction of the entropy relying on the fact that entropy belongs to a group isomorphic to the usual additive group. This allows them to show that the entropies that are additive with respect to the addition of the group for independent random variables are nonlinear transforms of Renyi’s [114] entropies, including the particular case of the Shannon’s [118] entropy.

All these authors have discussed and derived different measures of probabilistic entropy but from characterization point of view, have not taken into consideration the concavity property of these measures which is very important because of its applications to maximum entropy principle. The importance arises due to the fact that the stationary value of a concave function, when it is obtained by Lagrange’s method, will give the globally maximum value. Thus, when we get the stationary value, we have not to worry whether it is maximum or minimum and we have not to worry whether it is the largest maximum value. The problem of checking these can be complicated problem, since entropy is a function of $n$ variables. However, because of the concavity of Shannon’s [118] and Havrada-Charvat’s [39] measures, these problems are automatically taken care of and the problem of entropy maximization becomes a relatively simple matter. It is therefore necessary that concavity property of these measures should be considered as a property of primary importance and should be emphasized in characterization theorems.
One of the main problems in the description and modeling of complex systems is the impossibility of defining their meaningful parameters in an unambiguous and crisp manner. As a consequence, a sort of vagueness is present from the beginning and often every effort of eliminating it at a subsequent stage induces an oversimplification of the model and so a loss of information of the real system under study.

A second problem is presented by the complexity of these systems which usually prevents the finding of solutions of the models without simplifications and forced approximations. A possible way of avoiding these two drawbacks may be found by a suitable change of the descriptive language of the system. This new language should be compatible with the presence of elements of "imprecision", "uncertainty", "fuzziness" and so on, and at the same time should be tractable those problems whose solution in the classical descriptive language would be unattainable without subsequent arbitrary simplifications. This is what fuzzy set theory aims to do.

A mathematical theory describing imprecise and vague notions is meaningful if the theory is also able to measure and control the indeterminacy which is introduced. A way of achieving this goal in the theory of fuzzy sets is provided by the theory of entropy measures.

The measures of fuzzy entropy corresponding to Shannon's [118] and Havrada-Charvat's [39] probabilistic measures have been derived by De Luca and Termini [18] and Kapur [58] respectively. Corresponding to Shannon's [118] measure of probabilistic entropy, De Luca and Termini [18] suggested the measure of fuzzy entropy as

$$H(A) = - \sum_{i=1}^{n} [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i))]$$

(4.1.1)

Corresponding to Renyi's [114] measure of probabilistic entropy, Bhandari and Pal [7] suggested the measure of fuzzy entropy as

$$H_\alpha(A) = \frac{1}{1 - \alpha} \sum_{i=1}^{n} \log [\mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^\alpha] ; \alpha > 0, \alpha \neq 1$$

(4.1.2)
Corresponding to Havrada-Charvat’s [39] probabilistic measure of entropy, Kapur [58] suggested the following measure of fuzzy entropy:

\[
H^\alpha_a(A) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \left[ \mu^\alpha_A(x_i) + (1-\mu_A(x_i))^\alpha \right] - 1; \alpha > 0, \alpha \neq 1 \tag{4.1.3}
\]

Parkash [85] introduced a new generalized measure of fuzzy entropy involving two real parameters. This measure is given by

\[
H_{\alpha, \beta}(A) = \left[ (1-\alpha) \beta \right]^{-1} \sum_{i=1}^{n} \left[ \left\{ \mu^\alpha_A(x_i) + (1-\mu_A(x_i))^\alpha \right\}^{\beta} - 1 \right]; \alpha > 0, \alpha \neq 1, \beta \neq 0 \tag{4.1.4}
\]

and called it \((\alpha - \beta)\) fuzzy entropy. The fuzzy entropy (4.1.4) includes some well known fuzzy entropies existing in the literature.

Corresponding to Kapur’s [48] probabilistic measure of entropy of degree \(\alpha, \beta\), Kapur [58] took the following expressions of fuzzy entropy:

\[
H^\alpha_{\alpha, \beta}(A) = \frac{1}{\alpha + \beta - 2} \left[ \sum_{i=1}^{n} \left[ \mu^\alpha_A(x_i) + (1-\mu_A(x_i))^\alpha + \mu^\beta_A(x_i) + (1-\mu_A(x_i))^\beta \right] - 2 \right] \tag{4.1.5}
\]

and

\[
H^\beta_{\alpha, \beta}(A) = \frac{1}{\beta - \alpha} \log \left[ \sum_{i=1}^{n} \left[ \mu^\alpha_A(x_i) + (1-\mu_A(x_i))^\alpha \right] \right] / \left[ \sum_{i=1}^{n} \left[ \mu^\beta_A(x_i) + (1-\mu_A(x_i))^\beta \right] \right]; \alpha \geq 1, \beta \leq 1 \text{ or } \alpha \leq 1, \beta \geq 1 \tag{4.1.6}
\]

Parkash and Sharma [97] developed and obtained some relationships among different measures of fuzzy entropy.

Pal and Pal [84] introduced their own exponential entropy, given by

\[
H_e(A) = \frac{1}{n \sqrt{e} - 1} \sum_{i=1}^{n} \log \left[ \mu_A(x_i) e^{1-\mu_A(x_i)} + (1 - \mu_A(x_i)) e^{\mu_A(x_i)} - 1 \right] \tag{4.1.7}
\]
Further, Pal and Pal [84] introduced the concept of hybrid entropy which takes into account both the probabilistic and possibilistic uncertainties and reduces to the probabilistic entropy in absence of fuzziness. They have also discussed the higher $r$th order entropy of a fuzzy set which gives average ambiguity associated with any sub collection of supports of size $r$. Many other parametric and non-parametric measures of fuzzy entropy have been discussed and derived by Zadeh [149], Kapur [58], Kandel [47], Klir and Folger [64], Kosko [65], Loo [72], Parkash [85], Parkash and Gandhi [87], Parkash and Sharma [97] etc. Again, all these authors have not taken into consideration the concavity property from the characterization point of view. Consequently, in the present chapter, our aim is to characterize measures of fuzzy entropy in terms of concavity and recursivity properties.

In section 4.2, two well-known measures of fuzzy entropy due to De Luca and Termini [18] and Kapur [58], which correspond to measures of entropy due to Shannon [118] and Havrada-Charvat [39], have been characterised by using sum, concavity and recursivity. In section 4.3, it is shown that the measures of fuzzy entropy due to De Luca and Termini [18] and Kapur [58] are the only measures which can be characterised in terms of sum property, concavity and recursivity. An interesting property known as strong recursivity of both these measures has been studied in section 4.4.

4.2. CHARACTERIZATION THEOREMS FOR MEASURES OF FUZZY ENTROPY

In this section, we derive measures of fuzzy entropy due to De Luca and Termini [18] and Kapur [58]. These fuzzy entropies correspond to Shannon’s [118] and Havrada-Charvat’s [39] probabilistic entropies.

(a) Derivation of De Luca and Termini’s [18] fuzzy entropy

**Theorem 4.1.** The only function $H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n))$ of the form

$$\sum_{i=1}^{n} [\phi(\mu_A(x_i)) + \phi(1 - \mu_A(x_i))]$$

where $\phi(x)$ is a continuous, differentiable,
concave function defined on \([0, 1]\) for which \(\phi(0) = \phi(1) = 0\) and which satisfies the recursivity property is an arbitrary positive multiple of fuzzy measure corresponding to Shannon's [118] measure.

Before proving the above theorem, we first of all prove the following Lemma:

**Lemma 4.1.** The only solution of the functional equation

\[
\phi(x) + \phi(y) = \phi(x + y) + (x + y)^\alpha \left[ \phi\left( \frac{x}{x+y} \right) + \phi\left( \frac{y}{x+y} \right) \right]
\]

is given by \(\phi(x) = -k x \log x\)

**Proof.** Putting \(x = y\), in the given functional equation, we get

\[
2 \phi(x) = \phi(2x) + 2^\alpha x^{\alpha} \cdot \left[ 2 \phi\left( \frac{1}{2} \right) \right]
\]

Taking \(\phi(x) = x^\alpha \psi(x)\) in equation (A), we get

\[
\psi(x) = 2^{\alpha-1} \psi(2x) + \psi\left( \frac{1}{2} \right)
\]

so that

\[
\psi'(x) = 2^\alpha \psi'(2x)
\]

The solution of the above equation is given by

\[
\psi(x) = \frac{A}{x^\alpha}, \text{ where } A \text{ is some constant}
\]

Integrating both sides, we get

\[
\psi(x) = \frac{Ax^{-\alpha+1}}{-\alpha + 1} - B, \text{ where } B \text{ is some other constant.}
\]

Thus, we have

\[
\phi(x) = \frac{Ax}{1-\alpha} - B x^\alpha
\]

Substituting in (A), we get

\[
\frac{2 A x}{1-\alpha} - 2 B x^\alpha = 2Ax - B 2^\alpha x^\alpha + 2^{\alpha+1} x^\alpha \left[ \frac{1}{2} \frac{A}{1-\alpha} - B \frac{1}{2^\alpha} \right]
\]
or \( B = \frac{A}{1-\alpha} \)

Thus, we have

\[
\phi(x) = A \frac{x^\alpha - x}{\alpha - 1}
\]

Since \( \phi(x) \) is to be a concave function, we choose

\[
A = -k, \quad \text{so that} \quad \phi(x) = k \frac{x^\alpha - x}{1-\alpha}, \quad k > 0
\]

As \( \alpha \to 1 \) \( \phi(x) = -k x \log x \), which proves the Lemma.

**Proof Theorem 4.1:** Let \( H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)) \)

\[
= \phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \ldots + \phi(\mu_A(x_n)) + \phi(1-\mu_A(x_1))
\]

\[
+ \phi(1-\mu_A(x_2)) + \ldots + \phi(1-\mu_A(x_n)) \quad (4.2.1)
\]

where \( \phi(x) \) is continuous, differentiable, concave function in \((0, 1)\) for which \( \phi(0) = \phi(1) = 0 \) and satisfies recursivity property.

We now study the properties of \( H_n(\mu_A(x_1), \ldots, \mu_A(x_n)) \):

(i) Since sum of any number of continuous, differentiable, concave functions is itself a continuous, differentiable, concave function. Thus \( H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)) \) is a continuous, differentiable and concave function.

(ii) \( H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)) \) is permutationally symmetric, that is, it does not change if \( \mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n) \) are permuted among themselves.

(iii) \( H_{n+1}(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n), 0) \)

\[
= \phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \ldots + \phi(\mu_A(x_n)) + \phi(0)
\]

\[
+ \phi(1-\mu_A(x_1)) + \phi(1-\mu_A(x_2)) + \ldots + \phi(1-\mu_A(x_n)) + \phi(1-0)
\]
\[= \phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \ldots + \phi(\mu_A(x_n)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)) + \ldots + \phi(1 - \mu_A(x_n))\]
\[= H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n))\]

Thus our function satisfies the property of expansibility.

Now we assume that the recursivity or the branching property holds so that
\[H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n))\]
\[= H_{n-1}(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3), \ldots, \mu_A(x_n))\]
\[+ (\mu_A(x_1) + \mu_A(x_2)) H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)}\right)\]
\[+ H_{n-1}(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3), \ldots, 1 - \mu_A(x_n))\]
\[+ (2 - \mu_A(x_1) - \mu_A(x_2)) H_2\left(\frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)}, \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)}\right)\]

(4.2.2.)

This gives the following result:
\[\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2))\]
\[= \phi(\mu_A(x_1) + \mu_A(x_2))\]
\[+ (\mu_A(x_1) + \mu_A(x_2)) \left(\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)}\right) + \phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)}\right)\right)\]
\[+ \phi(2 - \mu_A(x_1) - \mu_A(x_2)) + (2 - \mu_A(x_1) - \mu_A(x_2))\]
\[\left(\phi\left(\frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)}\right) + \phi\left(\frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)}\right)\right)\]

(4.2.3)

which is functional equation and using Lemma 4.1, the only solution of this equation (4.2.3) is given by the following equation:
\( \phi(x) = A x \log x \)

Since \( \phi(x) \) is to be concave function, \( A \) has to be negative. Let \( A = -K \), where \( K \) is positive. Thus, we have

\( \phi(x) = -K x \log x \)

where \( K \) is an arbitrary positive constant.

Consequently, we get the following fuzzy entropy:

\[
H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)) = \sum_{i=1}^{n} [\phi(\mu_A(x_i)) + \phi(1 - \mu_A(x_i))]
\]

\[
= -K \sum_{i=1}^{n} [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i))]
\]

which is arbitrary positive multiple of fuzzy measure of entropy corresponding to Shannon's [118] well known probabilistic measure.

(b) Derivation of Kapur's [58] fuzzy entropy

**Theorem 4.2.** The only function \( H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)) \) of the form

\[
\sum_{i=1}^{n} [\phi(\mu_A(x_i)) + \phi(1 - \mu_A(x_i))]
\]

where \( \phi(x) \) is continuous, differentiable and concave function defined on the closed interval \([0, 1]\) for which \( \phi(0) = \phi(1) = 0 \) and which satisfies the recursivity property given by

\[
H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n))
\]

\[
= H_{n-1}(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3), \ldots, \mu_A(x_n))
\]

\[
+ (\mu_A(x_1) + \mu_A(x_2))^{\alpha} H_2 \left( \frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)} \right)
\]

\[
+ H_{n-1} (2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3), \ldots, 1 - \mu_A(x_n))
\]

\[
+ (2 - \mu_A(x_1) - \mu_A(x_2))^{\alpha} H_2 \left( \frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)}, \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right)
\]

is a fuzzy measure of entropy corresponding to Havrada - Charvat's [39] probabilistic measure of entropy.
Proof. Let $H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n))$

$$= \phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \ldots + \phi(\mu_A(x_n)) + \phi(1 - \mu_A(x_1))$$

$$+ \phi(1 - \mu_A(x_2)) + \ldots + \phi(1 - \mu_A(x_n))$$

$$= \sum_{i=1}^{n} [\phi(\mu_A(x_i)) + \phi(1 - \mu_A(x_i))]$$

where $\phi(x)$ is a continuous, differentiable and concave function defined in the interval $[0, 1]$ for which $\phi(0) = \phi(1) = 0$.

Now by recursive property,

$$\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2))$$

$$= \phi(\mu_A(x_1) + \mu_A(x_2))$$

$$+ (\mu_A(x_1) + \mu_A(x_2))^{\alpha} \left[ \phi\left( \frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)} \right) + \phi\left( \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)} \right) \right]$$

$$+ \phi(2 - \mu_A(x_1) - \mu_A(x_2)) + (2 - \mu_A(x_1) - \mu_A(x_2))^{\alpha}$$

$$\left[ \phi\left( \frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) + \phi\left( \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) \right]$$

(4.2.4)

which is a functional equation for which the only continuous, differentiable solution provided by Lemma 4.1 is given by

$$\phi(x) = A(x^\alpha - x)$$

Since $\phi(x)$ has to be a concave function, to make it concave for all positive values of $\alpha \neq 1$, we take $A = \frac{1}{1 - \alpha}$

Thus, we have

$$\phi(x) = \frac{x^\alpha - x}{1 - \alpha}$$

This gives us the following mathematical result:
which is fuzzy measure of entropy corresponding to Havrada and Charvat’s [39] well known probabilistic measure of entropy.

4.3 MEASURES WHICH CAN BE CHARACTERIZED IN TERMS OF SUM-FUNCTION PROPERTY, CONCAVITY AND RECURSIVITY

We, now consider the more general recursivity relation, given by the following mathematical expression:

\[
\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2))
\]

\[
= \phi\{\mu_A(x_1) + \mu_A(x_2)\} + g\{\mu_A(x_1) + \mu_A(x_2)\}
\]

\[
+ \phi(2 - \mu_A(x_1) - \mu_A(x_2)) + g(2 - \mu_A(x_1) - \mu_A(x_2))
\]

\[
\left[ \phi\left( \frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) + \phi\left( \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) \right]
\]

This gives the following equation:

\[
\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(\mu_A(x_3)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)) + \phi(1 - \mu_A(x_3))
\]

\[
= \{\phi(\mu_A(x_1) + \mu_A(x_2)) + \phi(\mu_A(x_3))\}
+ g(\mu_A(x_1) + \mu_A(x_2)) \left[ \phi \left( \frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)} \right) + \phi \left( \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)} \right) \right] \\
+ \{\phi(2 - \mu_A(x_1) - \mu_A(x_2)) + \phi(1 - \mu_A(x_3))\} \\
+ g(2 - \mu_A(x_1) - \mu_A(x_2)) \\
\left[ \phi \left( \frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) + \phi \left( \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) \right] \\

(4.3.2)

Using (4.3.1) in (4.3.2), we get the following result:
\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(\mu_A(x_3)) + \phi(1 - \mu_A(x_1)) \\
\hspace{1cm} + \phi(1 - \mu_A(x_2)) + \phi(1 - \mu_A(x_3)) \\
= \phi(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)) + g(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)) \\
\left[ \phi \left( \frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)} \right) + \phi \left( \frac{\mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)} \right) \right] \\
+ g(\mu_A(x_1) + \mu_A(x_2)) \left[ \phi \left( \frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)} \right) + \phi \left( \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)} \right) \right] \\
+ \phi(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)) + g(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)) \\
\left[ \phi \left( \frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)} \right) + \phi \left( \frac{1 - \mu_A(x_3)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)} \right) \right] \\
+ g(2 - \mu_A(x_1) - \mu_A(x_2)) \\
\left[ \phi \left( \frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) + \phi \left( \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) \right] \\

(4.3.3.)

Again using equation (4.3.1), we get the following mathematical expression:
\[
\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) + \phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) \\
= \phi\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) + g\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right)
\]

or

\[
\phi\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) = \phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) \\
+ \phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) - g\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right)
\]

\[
\left[\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) + \phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right)\right]. \tag{4.3.4}
\]

Similarly, we have

\[
\phi\left(\frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) = \phi\left(\frac{1 - \mu_A(x_1)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right)
\]

\[
+ \phi\left(\frac{1 - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right) - g\left(\frac{2 - \mu_A(x_1) - \mu_A(x_2)}{3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)}\right)
\]

\[
\left[\phi\left(\frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)}\right) + \phi\left(\frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)}\right)\right]. \tag{4.3.5}
\]

Using equations (4.3.4) and (4.3.5) in (4.3.3.), we get

\[
\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(\mu_A(x_3)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2))
\]

\[
+ \phi(1 - \mu_A(x_3)) = \phi(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3))
\]

\[
+ g\{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)\}.
\]
Thus, we see that the recursivity property (4.3.1) can be extended to three fuzzy values if

\[
g\left(\frac{\mu_A(x_1) + \mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) = g\left(\frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) = g\left(\frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right) = g\left(\frac{\mu_A(x_3)}{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)}\right)
\]  

(4.3.7)
Equations (4.3.7) and (4.3.8) have the solution \( g(x) = x^\alpha \) so that equation (4.3.1) becomes

\[
\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)) = \phi(\mu_A(x_1) + \mu_A(x_2)) + (\mu_A(x_1) + \mu_A(x_2))^\alpha
\]

\[
\left[ \phi\left( \frac{\mu_A(x_1)}{\mu_A(x_1) + \mu_A(x_2)} \right) + \phi\left( \frac{\mu_A(x_2)}{\mu_A(x_1) + \mu_A(x_2)} \right) \right] + \phi(2 - \mu_A(x_1) - \mu_A(x_2)) + (2 - \mu_A(x_1) - \mu_A(x_2))^\alpha
\]

\[
\left[ \phi\left( \frac{1 - \mu_A(x_1)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) + \phi\left( \frac{1 - \mu_A(x_2)}{2 - \mu_A(x_1) - \mu_A(x_2)} \right) \right]
\]

which is same as equation (4.2.4).

Thus, the only fuzzy measures which can be characterized in terms of sum-property, concavity and recursivity are those measures which correspond to Shannon’s [118] and Havrada and Charvat’s [39] measures.

In the next section, we study an important property of these measures known as strong recursivity.

### 4.4. A STRONG RECURSIVITY PROPERTY

From equation (4.2.1), we have

\[
H(\mu(x), \mu(x), \ldots, \mu(x)) = \phi(\mu(x)) + \phi(\mu(x)) + \ldots + \phi(\mu(x)) + \phi(1 - \mu(x)) + \phi(1 - \mu(x)) + \ldots + \phi(1 - \mu(x))
\]

(4.4.1)

Also, we have the following mathematical expression:
\[ H(\mu(x), \mu(x), \ldots, \mu(x)) = H(\mu(x) + \mu(x), \ldots, \mu(x)) \]

\[ + g(\mu(x), \mu(x)) \frac{\mu(x)}{\mu(x) + \mu(x)} \frac{\mu(x)}{\mu(x) + \mu(x)} \]

\[ + H(2-\mu(x)-\mu(x), 1-\mu(x), \ldots, 1-\mu(x)) + g(1-\mu(x), 1-\mu(x)) \]

\[ H\left(\frac{1-\mu(x)}{2-\mu(x)-\mu(x)}, \frac{1-\mu(x)}{2-\mu(x)-\mu(x)}\right) \]

(4.4.2)

\[ = H(\mu(x) + \mu(x) + \mu(x), \mu(x), \ldots, \mu(x)) + g\{(\mu(x) + \mu(x)), \mu(x)\} \]

\[ H\left(\frac{\mu(x) + \mu(x)}{\mu(x) + \mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)}\right) \]

\[ + g\{\mu(x), \mu(x)\} H\left(\frac{\mu(x)}{\mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x)}\right) \]

\[ + H(3-\mu(x)-\mu(x)-\mu(x), 1-\mu(x), \ldots, 1-\mu(x)) \]

\[ + g\{2-\mu(x)-\mu(x), (1-\mu(x))\} \]

\[ H\left(\frac{2-\mu(x)-\mu(x)}{3-\mu(x)-\mu(x)-\mu(x)}, \frac{1-\mu(x)}{3-\mu(x)-\mu(x)-\mu(x)}\right) \]

\[ + g\{1-\mu(x), 1-\mu(x)\} H\left(\frac{1-\mu(x)}{2-\mu(x)-\mu(x)}, \frac{1-\mu(x)}{2-\mu(x)-\mu(x)}\right) \]

(4.4.3)

\[ = H(\mu(x) + \mu(x) + \mu(x) + \mu(x), \mu(x), \ldots, \mu(x)) \]

\[ 1-\mu(x), \ldots, 1-\mu(x) \]

\[ H\left(\frac{\mu(x) + \mu(x) + \mu(x)}{\mu(x) + \mu(x) + \mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x) + \mu(x)}\right) \]
\[ H \left( \frac{\mu(x) + \mu(x)}{\mu(x) + \mu(x) + \mu(x)} \right) \]

\[ + g\{\mu(x), \mu(x)\} \cdot H \left( \frac{\mu(x)}{\mu(x) + \mu(x)} \right) \]

\[ + \left\{ \left( 4 - \mu(x) - \mu(x) - \mu(x) - \mu(x) \right), 1 - \mu(x), \ldots, 1 - \mu(x) \right\} \]

\[ + g\{(3 - \mu(x) - \mu(x) - \mu(x)), 1 - \mu(x)\} \]

\[ H \left( \frac{3 - \mu(x) - \mu(x) - \mu(x)}{4 - \mu(x) - \mu(x) - \mu(x)} \right) \]

\[ + g\{(2 - \mu(x) - \mu(x)), 1 - \mu(x)\} \]

\[ H \left( \frac{2 - \mu(x) - \mu(x)}{3 - \mu(x) - \mu(x) - \mu(x)} \right) \]

\[ + g\{(1 - \mu(x)), (1 - \mu(x))\} \cdot H \left( \frac{1 - \mu(x)}{2 - \mu(x) - \mu(x)} \right) \]

\[ (4.4.4) \]

and so on.

In general, we have

\[ H(\mu(x), \mu(x), \ldots, \mu(x)) \]

\[ = H \left\{ \left( \mu(x) + \mu(x) + \ldots + \mu(x) \right), \mu(x), \ldots, \mu(x) \right\} \]

\[ + g\{\mu(x), \mu(x)\} \cdot H \left( \frac{\mu(x)}{\mu(x)} \right) \]

\[ + H \left\{ \left( k - \mu(x) - \mu(x) - \ldots - \mu(x), 1 - \mu(x), \ldots, 1 - \mu(x) \right) \right\} \]

\[ + g\{(1 - \mu(x)), (1 - \mu(x))\} \cdot H \left( \frac{(1 - \mu(x))}{(1 - \mu(x))} \right) \]

\[ (4.4.5) \]

Thus to calculate \( H_n \), we have first to calculate \( k \) fuzzy entropies of the type \( H_2 \) and one fuzzy entropy of the type \( H_{n-k} \).
A stronger type of recursivity enables us to express $H_n$ in terms of two entropies, one of the type $H_{n-k+1}$ and the other of the type $H_k$. In particular, we would like to express $H_n$ in terms of $H_{n-2}$ and $H_3$.

From equation (4.4.2), we have

$$H \left( \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)} \right)$$

$$= H \left( \frac{\mu(x) + \mu(x)}{\mu(x) + \mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)} \right)$$

$$+ g \left( \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)} \right)$$

$$H \left[ \frac{\mu(x)}{\mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x)} \right] \quad (4.4.6)$$

and

$$H \left( \frac{1 - \mu(x)}{3 - \mu(x) - \mu(x) - \mu(x)}, \frac{1 - \mu(x)}{3 - \mu(x) - \mu(x) - \mu(x)} \right)$$

$$= H \left( \frac{2 - \mu(x) - \mu(x)}{3 - \mu(x) - \mu(x) - \mu(x)}, \frac{1 - \mu(x)}{3 - \mu(x) - \mu(x) - \mu(x)} \right)$$

$$+ g \left( \frac{1 - \mu(x)}{3 - \mu(x) - \mu(x) - \mu(x)}, \frac{1 - \mu(x)}{3 - \mu(x) - \mu(x) - \mu(x)} \right)$$

$$H \left[ \frac{1 - \mu(x)}{2 - \mu(x) - \mu(x)}, \frac{1 - \mu(x)}{2 - \mu(x) - \mu(x)} \right] \quad (4.4.7)$$

Using (4.4.6) and (4.4.7), equation (4.4.3) gives

$$H \{ \mu(x), \mu(x), \ldots, \mu(x) \}$$

$$= H \{ (\mu(x) + \mu(x) + \mu(x)), \mu(x), \ldots, \mu(x) \}$$
This will be expressed in terms of $H_{n-2}$ and $H_3$ only if

$$-g\{(\mu(x) + \mu(x)), \mu(x)\}$$

$$g\left(\frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)}\right)$$
\[ + g\{\mu(x), \mu(x)\} + g\{1-\mu(x), 1-\mu(x)\} - g\{2-\mu(x)-\mu(x), 1-\mu(x)\} \]
\[ \cdot g\left(\frac{1-\mu(x)}{3-\mu(x)-\mu(x)-\mu(x)}, \frac{1-\mu(x)}{3-\mu(x)-\mu(x)-\mu(x)}\right) = 0 \]

or
\[ g\{\mu(x) + \mu(x), \mu(x)\} \cdot g\left(\frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)}, \frac{\mu(x)}{\mu(x) + \mu(x) + \mu(x)}\right) \]
\[ + g\{2-\mu(x)-\mu(x), 1-\mu(x)\} \cdot g\left(\frac{1-\mu(x)}{3-\mu(x)-\mu(x)-\mu(x)}, \frac{1-\mu(x)}{3-\mu(x)-\mu(x)-\mu(x)}\right) \]
\[ = g\{\mu(x), \mu(x)\} + g\{1-\mu(x), 1-\mu(x)\} \] (4.4.9)

Equation (4.4.9) will be satisfied if
\[ g\{\mu(x) + \mu(x), \mu(x)\} = f\{\mu(x) + \mu(x) + \mu(x)\} \]

or
\[ g\{\mu(x), \mu(x)\} = f(\mu(x) + \mu(x)) \] (4.4.10)

Using (4.4.10), equation (4.4.9) can be written as
\[ f(\mu(x) + \mu(x) + \mu(x)) \cdot f\left(\frac{\mu(x) + \mu(x)}{\mu(x) + \mu(x) + \mu(x)}\right) \]
\[ + f\{3-\mu(x)-\mu(x)-\mu(x)\} \cdot f\left(\frac{2-\mu(x)-\mu(x)}{3-\mu(x)-\mu(x)-\mu(x)}\right) \]
\[ = f(\mu(x) + \mu(x)) + f(2-\mu(x)-\mu(x)) \] (4.4.11)

The functional equation (4.4.11) has the solution
\[ f(x) = x^\alpha \] (4.4.12)

So that the equations (4.4.2) and (4.4.3) give
\[ H\{\mu(x), \mu(x), \ldots, \mu(x)\} \]
\[ = H\{\mu(x) + \mu(x), \mu(x), \ldots, \mu(x)\} + (\mu(x) + \mu(x)) \]
Thus in general, we have the following expression:

\[
H \{\mu(x), \mu(x), \mu(x), \ldots, \mu(x)\} = H_\mu(\{\mu(x) + \mu(x) + \ldots + \mu(x), \mu(x), \ldots, \mu(x)\} + \{\mu(x) + \mu(x) + \mu(x) + \ldots + \mu(x)\} H \left( \frac{\mu(x)}{\mu(x)} : \frac{\mu(x)}{\mu(x)} : \ldots : \frac{\mu(x)}{\mu(x)} \right) + \{k - \mu(x) - \mu(x) - \ldots - \mu(x), 1 - \mu(x), \ldots, 1 - \mu(x)\} + \{k - \mu(x) - \mu(x) - \ldots - \mu(x) \}
\]

Thus, we observe that the fuzzy measures corresponding to Havrada-Charvat’s [39] and Shannon’s [118] measures are the only fuzzy measures of entropy for which the strong recursivity condition applies, that is, for which \(H_n\) can be expressed in terms of \(H_{n-k+1}\) and \(H_k\) for all possible values of \(n\) and \(k (< n)\). Hence, we conclude that though many measures of fuzzy
entropy have been discussed, derived and characterized by various authors, the only fuzzy entropies which can be characterized in terms of sum property, concavity and recursivity are De Luca-Termini's [18] and Kapur's [58] measures. It is further shown that these are the only measures which satisfy strong recursivity property.

Concluding remarks: There have been many characterizations of the measures of entropy due to Shannon [118] and Havrada-Charvat [39]. Different authors have given different techniques for their characterizations but nobody has characterized these measures by using concavity and recursivity. For any measure of entropy, recursivity is not at all essential property but it is very nice property. The objective of the present chapter is to characterize fuzzy measures corresponding to Shannon's [118] and Havrada-Charvat's [39] measures by using concavity and recursivity properties.