2.1. Introduction. Consider the cosine and sine series

\[ \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \]

\[ \sum_{k=1}^{\infty} a_k \sin kx, \]

or together

\[ \sum_{k=1}^{\infty} a_k \phi_k(x), \]

where \( \phi_k(x) \) is \( \cos kx \) or \( \sin kx \) respectively. Let the partial sum of (2.1.3) be denoted by \( S_n(x) \) and \( f(x), g(x) = \lim_{n \to \infty} S_n(x) \).

Further, let \( \lim_{n \to \infty} S^{(r)}_n(x) = t^{(r)}(x), r \in \{0, 1, 2, \ldots \} \), where \( S^{(r)}_n(x) \) represents \( r \)-th derivative of \( S_n(x) \).

Generalizing the concept of quasi-convex null sequence, Sidon [29] introduced the following class of trigonometric series: Let \( a_k = o(1), k \to \infty \). If there exists a sequence \( < A_k > \) such that

\[ A_k \downarrow O, k \to \infty, \]

\[ \sum_{k=0}^{\infty} A_k < \infty, \]

\[ | \Delta a_k | \leq A_k, \text{ for all } k, \]
we say that (2.1.3) belongs to the class $S$. We observe that the trigonometric series (2.1.3) with quasi-convex null coefficients belongs to the class $S$ if we choose $A_k = \sum_{m=k}^{\infty} |\Delta^2 a_m|$. 

Sheng [28] generalized the notion of class $S$ in the following way: Let $a_k = o(1)$, $k \to \infty$. If there exists a sequence $\langle A_k \rangle$ such that 

\begin{equation}
A_k \not\to 0, \quad k \to \infty,
\end{equation}

\begin{equation}
\sum_{k=1}^{\infty} A_k < \infty, \quad \text{for some} \quad \lambda > 0,
\end{equation}

\begin{equation}
\frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1), \quad 1 < p \leq 2, \quad n \to \infty,
\end{equation}

we say that (2.1.3) belongs to the class $S_p$.

If $a_k = o(1)$, $k \to \infty$ and

\begin{equation}
\sum_{k=1}^{\infty} \frac{2}{\Delta^2 \left( \frac{a_k}{k} \right)} |^2 < \infty,
\end{equation}

we say that (2.1.3) belongs to the class $R$, introduced by Kano [14].

We introduce a new class $R_r$ in the following way: Let $a_k = o(1)$, $k \to \infty$ and

\begin{equation}
\sum_{k=1}^{\infty} k^{r+2} \left| \Delta^2 \left( \frac{a_k}{k} \right) \right| < \infty, \quad r \in \{0, 1, 2, \ldots\}
\end{equation}

we say that (2.1.3) belongs to the class $R_r$. Clearly, for $r = 0$,
Concerning the $L^1$ convergence of the cosine series, we have the following classical result of Kolmogorov \cite{15}:

\textbf{Theorem A.} \cite[Vol.II, p. 204]{2}. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (2.1.1) in the metric space $L$, it is necessary and sufficient that $\lim_{k \to \infty} a_k \log k = 0$.

The case, in which the sequence $\{a_k\}$ is convex, of this theorem was established by Young \cite{41}. That is why, some times the Theorem A is known as Young-Kolmogorov Theorem.

Garret and Stanojevic \cite{8} introduced a new cosine sum as

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{j=1}^{n} \Delta(a_j) \cos kx,$$

and proved the following:

\textbf{Theorem B.} Let $a_k = o(1)$, $k \to \infty$ and $\sum_{k=1}^{\infty} |\Delta a_k| < \infty$.

Then $g_n$ converges to $f$ in the $L^1$-metric if and only if given $\varepsilon > 0$, there exists $S(\varepsilon) > 0$ such that

$$\int_0^1 \left| \sum_{k=n+1}^{\infty} \Delta(a_k) D_k(x) \right| dx < \varepsilon, \text{ for all } n > S(\varepsilon).$$

Ram \cite{21} proved the following result regarding the $L^1$-convergence of $g_n(x)$:

\textbf{Theorem C.} If (2.1.1) belongs to the class $S$, then

$$\|f - g_n\|_1 = o(1), \; n \to \infty.$$
The following theorem of Teljakovskii [38] follows as a corollary of Theorem C.

**Theorem D.** If (2.1.1) belong to the class S, then a necessary and sufficient condition for \( L^1 \)-convergence of (2.1.1) is \( |a_n| \log n = o(1), n \to \infty \).

Singh and Sharma [30] generalised Theorem C by replacing monotonicity of the sequence \( \langle A_n \rangle \), in the definition of the class S, by quasi-monotonicity.

Ram and Kumari ([16],[24]) introduced new modified cosine and sine sums as

\[
f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{\infty} a_j \Delta(\frac{j}{n})k \cos kx
\]

and

\[
g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{\infty} a_j \Delta(\frac{j}{n})k \sin kx
\]

and studied their \( L^1 \)-convergence. They obtained the following results:

**Theorem E** [16]. Let (2.1.1) belongs to the class S. If

\[
\lim_{n \to \infty} |a_{n+1}| \log n = 0, \text{ then } \|f_n - g_n\|_1 = o(1), n \to \infty.
\]

**Theorem F** [24]. Let (2.1.1) belongs to the class R. If \( t_n(x) \) represents \( f_n(x) \) or \( g_n(x) \), then \( \|f_n - t_n\|_1 = o(1), n \to \infty \). Also they deduced as corollaries the \( L^1 \)-convergence of the trigonometric cosine and sine series from Theorems E and F.
The aim of this chapter is to study the $L^1$-convergence of
$t_n(x)$ under the conditions that (2.1.3) belongs to the class $R_r$ or $S_p^a$. The results of Ram and Kumari ([16], [24]) follow as
particular cases.

2.2. Lemmas. The following lemmas of Sheng [28] will be
required for the proof of our results:

Lemma 1. $||D_n^{(r)}(x)|| = 4/n(n^r \log n) + O(n^r)$,
$r \in \{0, 1, 2, \ldots\}$, where $D_n^{(r)}(x)$ represent the $r$-th derivative
of Dirichlet-kernel.

Lemma 2. $||\tilde{D}_n^{(r)}(x)|| = O(n \log n)$, $r \in \{0, 1, 2, \ldots\}$,
where $\tilde{D}_n^{(r)}(x)$ represents the $r$-th derivative of conjugate Dirichlet-kernel.

2.3. Results. We prove the following two theorems:

Theorem 1. Let (2.1.3) belong to the class $R_r$. Then

$$\lim_{n \to \infty} t_n^{(r)}(x) = t^{(r)}(x), \text{ for } x \in (0, \pi],$$

$$t^{(r)}(x) \in L^1[0, \pi].$$

$$||S_n^{(r)}(x) - t^{(r)}(x)|| = o(1), n \to \infty \text{ if and only if }$$

$$|a_n^{(r)}|/n \log n = o(1), n \to \infty.$$}

The case $r = 0$ yields Theorem F of Ram and Kumari [24].

Theorem 2. Let $\langle a_n \rangle_{p^a}$, $a > 0$, $r \in \{0, 1, \ldots [a]\}$. It
\[
\lim_{n \to \infty} \left| a_n \right| n \log n = 0, \text{ then } \left| f^{(r)}(x) - f^{(r)}(x) \right| = o(n^{-a}), n \to \infty.
\]

**Corollary 1.** If \(<a_n \in S, a > 0, r \in \{0, 1, \ldots \}[a]\),

\[
\lim_{n \to \infty} |a_n| n \log n = 0, \text{ then } \left| f^{(r)}(x) - S^{(r)}(x) \right| = o(n^{-a}), n \to \infty \text{ if and only if }
\]

Denoting \(S_p\) by \(S_0\) in the case of \(r = a = 0\), Theorem 2 and Corollary 1 reduce respectively to the following corollaries which are extensions of Theorem E and its corollary given in [24].

**Corollary 2.** If \(<a_n \in S_0\) and \(\lim_{n \to \infty} |a_n+1| \log n = 0\), then

\[
\left| f^{(r)}(x) - f(x) \right| = o(1), n \to \infty.
\]

**Corollary 2.** If \(<a_n \in S_0\), then

\[
\left| f^{(r)}(x) - S^{(r)}(x) \right| = o(1), n \to \infty,
\]

if and only if \(\lim_{n \to \infty} |a_{n+1}| \log n = 0\).

### 2.4. Proofs of the Theorems

**Proof of Theorem 1.** We shall consider only cosine sums as the proof for sine sums follows the same course. We have

\[
t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} a_j \Delta(j) k \cos kx
\]

\[
= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[ \Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \ldots + \Delta\left(\frac{a_n}{n}\right) \right]
\]

\[
= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[ \frac{a_k}{k} - \frac{a_{n+1}}{n+1} \right]
\]
\[
= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx
\]

\[
= S_n(x) - \frac{a_{n+1}}{n+1} D_n(x)
\]

We have then,

\[
(2.4.1) \quad t_n(x) = S_n(x) - \frac{a_{n+1}}{n+1} D_n(x)
\]

where \( D_n(x) \) represents the \((r+1)\)th derivative of the conjugate Dirichlet kernel. Since \(< a_n >\) is null, \( \lim_{n \to \infty} t_n(x) = t(x) \)

for \( x \in (0, \pi] \) and thus the proof of (2.3.1) follows.

For \( x \neq 0 \), it follows from (2.4.1) that

\[
t_n(x) - t_n(x) = \sum_{k=n+1}^{\infty} k a_k \cos (kx + \frac{r\pi}{2}) + \frac{a_{n+1}}{n+1} \tilde{D}(x)
\]

\[
= \lim_{m \to \infty} \left[ \sum_{k=n+1}^{m} \frac{a_k}{k} \cos (kx + \frac{r\pi}{2}) \right] + \frac{a_{n+1}}{n+1} \tilde{D}(x)
\]

Applying Abel's transformation twice, we have

\[
t_n(x) = \lim_{m \to \infty} \left[ \sum_{k=n+1}^{m-1} \frac{a_k}{k} \tilde{D}_k(x) + \frac{a}{m} \tilde{D}_m(x) \right] + \frac{a_{n+1}}{n+1} \tilde{D}(x)
\]

\[
= \frac{a_{n+1}}{n+1} \tilde{D}_n(x) + \frac{a_{n+1}}{n+1} \tilde{D}(x)
\]
\[ \lim_{m \to \infty} \left[ \sum_{k=n+1}^{m-2} (k+1) \Delta \left( \frac{a_k}{k} \right) K^{(r+1)}(x) + m \Delta \left( \frac{a_{m-1}}{m-1} \right) K^{(r+1)}(x) \right] \]

\[ = \sum_{k=n+1}^{\infty} (k+1) \Delta \left( \frac{a_k}{k} \right) K^{(r+1)}(x) - (n+1)^2 \Delta \left( \frac{a_{n+1}}{n+1} \right) K^{(r+1)}(x), \]

where \( K^{(r+1)}(x) \) denotes the \((r+1)\)th derivative of the conjugate Fejér kernel. Thus

\[ \| t^{(r)}(x) - t^{(r)}_n(x) \|_{L^1} \leq \sum_{k=n+1}^{\infty} \left( \frac{a_k}{k} \right) \| K^{(r+1)}(x) \|_{L^1} \]

\[ + (n+1)^2 \Delta \left( \frac{a_{n+1}}{n+1} \right) \| K^{(r+1)}_n(x) \|_{L^1} \]

But, by Zygmund Theorem \cite[Vol.II, p. 458]{Zygmund}, we have

\[ \int_0^\pi K^{(r+1)}_n(x) \, dx = O(1). \]

Moreover,

\[ \left| \Delta \left( \frac{a_{n+1}}{n+1} \right) \right| = \sum_{k=n+1}^{\infty} \left( \frac{a_k}{k} \right)^2 \leq \sum_{k=n+1}^{\infty} k^2 \left| \Delta \left( \frac{a_k}{k} \right) \right| \leq \frac{1}{(n+1)^2} \sum_{k=n+1}^{\infty} k^2 \left| \Delta \left( \frac{a_k}{k} \right) \right|. \]

Thus, it follows that
\[ \left| \frac{t^{(r)}_n(x)}{t^{(r)}_n(x)} - 1 \right| = O\left( \prod_{k=n+1}^{r+2} \Delta \left( \frac{a_k}{k} \right) \right) = o(1), \quad n \to \infty, \]

by using (2.1.11). Since \( t^{(r)}_n(x) \) is a polynomial, therefore
\[ x^{(r)} \in L^1. \]
This proves (2.3.2).

For the proof of (2.3.3), we note that
\[
\left| t^{(r)}_n(x) - S^{(r)}_n(x) \right| = \left| t^{(r)}_n(x) - t^{(r)}_n(x) + t^{(r)}_n(x) - S^{(r)}_n(x) \right| \\
\leq \left| t^{(r)}_n(x) - t^{(r)}_n(x) \right| + \left| t^{(r)}_n(x) - S^{(r)}_n(x) \right| \\
= \left| t^{(r)}_n(x) - t^{(r)}_n(x) \right| + \left| \frac{a_{n+1}}{n+1} D^{(r+1)}_n(x) \right|.
\]
and
\[
\left| \frac{a_{n+1}}{n+1} D^{(r+1)}_n(x) \right| = \left| t^{(r)}_n(x) - S^{(r)}_n(x) \right| \\
\leq \left| t^{(r)}_n(x) - S^{(r)}_n(x) \right| + \left| t^{(r)}_n(x) - t^{(r)}_n(x) \right| \\
= o(1), \quad n \to \infty.
\]

Since \( t^{(r)}_n(x) \leq o(1) \), by (2.3.2), by the use of Lemma 2, we have \( t^{(r)}_n(x) - S^{(r)}_n(x) = o(1), \quad n \to \infty \) if and only if \( |a_{n+1}| \log n = o(1) \), \( n \to \infty \).

Proof of Theorem 2. Proceeding as in the proof of Theorem 1, we have
\[
f^{(r)}_n(x) = S^{(r)}_n(x) - \frac{a_{n+1}}{n+1} D^{(r+1)}_n(x),
\]
where \( D^{(r+1)}_n(x) \) denotes the \((r+1)th\) derivative of the conjugate Dirichlet kernel.
Now, making use of Abel's transformation, we have

\((2.4.2)\)
\[
\| f^{(r)}_n(x) - f^{(r)}_n(x) \| < \int_0^{\infty} \sum_{k=n+1}^{\infty} \Delta a_k \Delta a_j (r) D_k(x) \, dx
\]
\[
+ \int_0^{\infty} \sum_{k=n+1}^{\infty} \Delta a_j (r) D_k(x) \, dx
\]
\[
\leq \int_0^{\infty} \sum_{k=n}^{\infty} \Delta a_k \sum_{j=1}^{\infty} \Delta a_j (r) D_k(x) \, dx
\]
\[
+ \int_0^{\infty} \sum_{k=n}^{\infty} \Delta a_j (r) D_k(x) \, dx
\]
\[
= I_1 + I_2 + I_2.
\]

Where,

\((2.4.3)\)
\[
I_1 = \sum_{k=n}^{\infty} \frac{\pi}{k} \sum_{j=1}^{\infty} \Delta a_j (r) D_j(x) \, dx
\]
\[
+ \sum_{k=n}^{\infty} \frac{\pi}{k} \sum_{j=1}^{\infty} \Delta a_j (r) D_j(x) \, dx = I_N + J_N, \text{say.}
\]
As in [26], we have \( I_N, J_N \) equal to \( o(n^{-\alpha}) \). Thus \( I_1 = o(n^{-\alpha}) \).

Similarly \( I_2 = o(n^{-\alpha}) \) holds. Also, by using Lemma 2, we have

\[
I = \int_0^{\pi} \frac{a}{n+1} \tilde{D}^{(r+1)}(x) \, dx
\]

\[
= \int_0^{\pi} \frac{a}{n+1} \tilde{D}^{(r+1)}(x) \, dx
\]

\[
= O(\frac{a}{n+1} n \log n) = o(1), \quad n \to \infty.
\]

The conclusion of the theorem now follows from (2.4.2).

Proof of Corollary 1. We notice that

\[
|f_n^{(r)}(x) - S_n^{(r)}(x)| = |f_n^{(r)}(x) - f_n^{(r)}(x) + f_n^{(r)}(x) - S_n^{(r)}(x)|
\]

\[
\leq |f_n^{(r)}(x) - f_n^{(r)}(x)| + |f_n^{(r)}(x) - S_n^{(r)}(x)|
\]

\[
= |f_n^{(r)}(x) - f_n^{(r)}(x)| + |\frac{a}{n+1} \tilde{D}^{(r+1)}(x)|
\]

and

\[
|\frac{a}{n+1} \tilde{D}^{(r+1)}(x)| = |f_n^{(r)}(x) - S_n^{(r)}(x)|
\]

\[
\leq |f_n^{(r)}(x) - S_n^{(r)}(x)| + |f_n^{(r)}(x) - f_n^{(r)}(x)|.
\]

Since \( |f_n^{(r)}(x) - f_n^{(r)}(x)| = o(n^{-\alpha}), \quad n \to \infty \) by Theorem 2, the
use of Lemma 2 yields \( \| f^{(r)}(x) - S_n^{(r)}(x) \|_q = o(n^{-a}) \), \( n \to \infty \), if

and only if \( |a_{n+1}| n^{r} \log n = o(1) \), as \( n \to \infty \).