CHAPTER – 1

INTRODUCTION
1.1 Inventory System in Operations Research:

Operations Research (OR) is the collection of modern methods on the problems which have been arising in the management of large systems of men, machines, materials and money in industry, business, defence etc. According to Churchman et. al. OR is defined as the application of scientific methods, techniques and tools for decision making problems (DMP) involving the operations of systems so as to provide these in the control of the operations with optimum solution to the problems. The main origin of OR was introduced during the second world war. At that time, the military management in England called upon a team of scientist to study the strategic and tactical problems related to air and land defence of the country with limited military resources, it was necessary to decide the most effective utilization of them, e.g. the efficient ocean transport, effective bombing, etc. As the team was dealing with research on military operations, the work of this team of scientist was named as 'Operations Research'.

After the end of the war, the success of military team attracted the 'industrial managers', who were seeking solutions to their complex executive type problems. The first mathematical technique in this field (called the Simplex Method of linear programming) was developed in 1947 by a American Mathematician George B. Dantzing. Since then, new techniques and applications have been developed through the efforts and co-operation of interested individuals in academic institutions and industry. One of them is Inventory Control. Today, the impact of OR
can be felt in many areas. Apart from military and business applications, the OR activities include transportation system, libraries, hospitals, city planning, financial institution etc.

In the big production firms as well as in departmental stores or shops, the stocking of items depends upon various factors e.g. demand, time of ordering, time lag between orders and actual receipts, deterioration, amelioration, time value of money, inflation etc. and the impression of these factors. So the problem for the managers and retailers is to have a compromise between overstocking and understocking. The study of such type of problems is known by the terms “material management” or “Inventory control”.

Inventory control is concerned with the flow of raw materials from supplier to producer and the subsequent flow of products through distribution centers to the customers. It is responsible for the planning, acquisition, storage, movement, and control of raw materials and final products. It attempts to get the right goods at the right price at the right time to maintain desired service level at minimum cost.

The Inventory control may be defined as “the function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finishing goods orderly mannered to meet the objectives of maximum customer service with minimum investment and efficient (low-cost) plant operation. Inventory can be defined as the stock of goods, commodities or other economic recourses that are stored or reserved in order to ensure smooth and efficient running of business affairs. Stock of goods may be kept in any one of the following forms:
raw materials;
- Partly finished goods (work in process inventory);
- finished (or produced) goods;
- spare parts, maintenance, repair etc.

The control and maintenance is a problem Common to all organizations in any sector of an economy. Inventories of physical goods are maintained in government and non-government establishments, e.g. inventories are maintained in Agriculture, Industry, Military, Business etc. The some reasons for maintaining inventories are as follows:

1. Inventory helps in smooth and efficient running of business.
2. Inventory provides service to the customers immediately or at a short notice.
3. Due to absence of stock, the company may have to pay high prices because of piecewise purchasing (maintaining of inventory may earn price discount because of bulk-purchasing).
4. Inventory also acts as a buffer stock when raw materials are received late and also so many sale orders are likely to be rejected without it.
5. Without inventories, customers will have to wait till their orders are supplied from a source or the items are manufactured. But customers will not or cannot be forced to wait for long period of time though sometimes some particular type of goods is not available.
6. A retailer can influence customers to buy more and more by displaying large number of goods in showroom.
7. The prices of some raw materials used by a manufacturer or the prices of some commonly used goods (such as paddy, wheat etc.) may exhibit seasonal fluctuations. When the price is low it is profitable to procure a sufficient quantity of these raw materials to be used later during the high priced season or when the need arises.

8. To take the financial advantage of transporting and shipping economics.

In any country the total investment on inventories amounts to a sizable proportion of the General Net Production (G.N.P.). This being the situation the control and management of inventories are problems of great concern in any sector of a given economy. Some costs like holding, set-up, shortage, purchasing, material costs etc. are involved in an inventory system. Although inventories constitute an idle resource, which incurs holding costs. They are justified by the results occurred in saving in shortage, set-up and procurement costs. The two fundamental questions in controlling the inventory of a physical goods are:

- When should the inventory of the physical goods be replenished?
- How much physical goods should be purchased or produced at the beginning of each time interval?

An inventory problem is a problem of making optimal decisions regarding the above questions. In other words, an inventory problem deals with decision that minimizes the total average cost or maximizes the total average profit gained while meeting the customer's demands.

Our aim is to develop the operating doctrines that should be used to control the inventory system with the help of mathematical analysis.
For this purpose the task is to construct a mathematical model of the real-life inventory system. However, such a mathematical model is based on various assumptions and approximations. It is difficult to devise and operate with an exact (real world) model, as normally these are not amenable to mathematical treatment. For this reason, some approximations and simplifications must be made during the model-building process.

Here the solution of an inventory problem is a set of specific values of variables that minimize the total average cost or maximize the total average profit of the system. In some cases specific values cannot be obtained because of the general nature of the problem investigated or due to the occurrence of some other reasons. In these cases the solution is given by a set of decision rules (known as algorithm), which evaluates the decision variables involved in the system.

1.2 Basic Concepts And Terminologies In Inventory

The inventory system depends on several parameters such as demand, replenishment, resources and various types of costs etc.

❖ Demand

Demand refers to the quantity of a commodity required at a given time. Generally it cannot be controlled directly and in many cases even indirectly also. It usually depends on the decision of people outside the organizations. Demand can be categorized according to its size and pattern. Demand size refers to the magnitude of demand as well as the dimension of quantity. When the demand size is the same from period to period, we say that it is constant or crisp. Otherwise, we refer to it as variable. Also, for unknown demand size, it is possible in some cases to
ascertain its probability distribution, and then the demand is called probabilistic. In some situations, it partly depends upon the stock level (initial or on hand), time, selling price of the item and advertisement, etc. In some cases, demand may be represented by vague, imprecise and uncertain data. This type of demand is termed as fuzzy demand. The demand rate is simply the demand size per unit time.

❖ **Replenishment**

Replenishment refers to the amount of quantities that are scheduled to be put into inventories, at the time when decisions are made about ordering these quantities or to the time when they are actually added to stock. It can be categorized according to size, pattern and lead-time. Replenishment size may be constant or variable, depending upon the type of the inventory system. It may depend on time, demand and / or on-hand inventory level. The replenishment pattern are usually instantaneous, uniform or in batch. The replenishment data again may be probabilistic or fuzzy in nature.

❖ **Constraints**

Constraints are the limitations imposed on the inventory system. Constraints may be imposed on the amount of investment, available space, the amount of inventory held, average inventory expenditure, number of orders, etc. These constraints can also be fuzzy in nature i.e. data for constraints goals may be imprecise and vague. There may be also some chance constraints in inventory system i.e. the minimum probability of satisfying a constraint is specified or some parameters in the constraint(s) may be probabilistic.
Inventory Costs:

Inventory costs are the costs associated with the operation of an inventory system. They are basic economic parameters to any inventory decision model.

- **The holding or carrying cost** is the cost associated with the storage of the inventory until its use or sale. It is directly proportional to the quantity in inventory and the time for which the stocks are held. This cost generally includes the costs such as rent for storage space, interest on the money locked-up, insurance, taxes, etc.

- **The ordering or set up cost** is the cost associated with the expense of issuing a purchase order to an out-side supplier or setting up machines before internal production. These costs also include clerical and administrative costs, telephone charges, telegram, transportation costs, loading and unloading costs etc. Generally this cost is assumed to be independent of the quantity ordered for or produced. In the costs like transportation cost, etc. some part of it may be quantity dependent.

- **The purchase or unit cost** of an item is the unit purchase price to obtain the item from an external source or the unit replenishment cost for internal production. It may also depend upon the demand when production is done in large quantities as it results in reduction of production cost per unit. Also when quantity discounts are allowed for bulk orders, unit price is reduced and depends on the quantity purchased or ordered.

- **The shortage or stock out cost or penalty cost** is the penalty incurred when the stock proves inadequate to meet the demand of
the customers. This cost parameter does not depend upon the source of replenishment of stock but upon the amount of inventory not supplied to the customer.

- **Lead Time**

  The time gap between the time of placing an order or starting of the production and the time of actual arrival or delivery of goods to the inventory is called lead time. It can be crisp (i.e., a finite number) probabilistic or imprecise.

- **Deterioration / Damageability / Perishability**
  
  - **Deterioration** is defined as decay, evaporation, obsolescence, and loss of unity or marginal value of a commodity that results in the decreasing usefulness from the original condition. Vegetables, food grains, gasoline and semiconductor chips, etc. are examples of such products.
  
  - **Damageability** is defined by the damage when the items are broken or loose their utility due to the accumulated stress, bad handling, etc. The amount of damage by the stress varies with the size of stock and duration for which the stress is applied. Items made of glass; china-clay, ceramic, mud, etc. are examples of such product.
  
  - **Perishable** items are those, which have finite life-time (fixed or random). Fixed life-time product (e.g. human blood, etc.) has a deterministic shelf-life while the random life-time scenario assumes that the useful life of each unit is a random variable. The random lifetime scenario is closely related to the case of an
inventory which experiences continuous physical depletion due to deterioration or decay.

- **Fully back-logged / Partially back-logged shortages**
  
  During stock-out period, the sales and/or good-will may be lost either by a delay or complete refusal in meeting the demand. In the case where the unfulfilled demand for the goods can be satisfied completely at a later date, then it is a case of fully back-logged shortages i.e. it is assumed that no customer back away during this period and the demand of all these waiting customers is met at the beginning of the next period. Again it is normally observed that during the stock out period, some of the customers wait for the product and others back away. When this happens, the phenomenon is called partially backlogged shortages.

- **Salvage**
  
  During shortage, some items partially spoiled or damaged, i.e. some items lose their utility. But in a developing country, it is normally observed that some of these are sold in a lower price (less than the purchasing price) to a section of customers and this gives some revenue to the management. This revenue is called salvage value.

- **Environment**
  
  - **Crisp Environment**
    
    In inventory management when the values of all inventory parameters and resources are constant, then the model is called crisp model.
1.3 Fuzzy Set Theory (Preliminary Idea)

The fuzzy set theory was developed to define and solve the complex system with sources of uncertainty or imprecision which are non-statistical in nature. Fuzzy set theory is a theory of graded concept (a matter of degree) but not a theory of chance or probability. Prof. L. A. Zadeh proposed the term "FUZZY" in 1965. Prof. L. A. Zadeh formally discussed the concept "Fuzzy sets" in [109]. A short description of the fuzzy set theory is given below.

**Crisp Set**

A classical set is defined by crisp boundaries i.e. there is no uncertainty in the prescription of the elements of the set.

Def.:- “A classical or crisp set is normally defined as a well defined collection of elements or objects \( x \in X \) which can be finite, countable or over countable.”

**Fuzzy Set**

A fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do that.

Def.:- “Let \( X \) be a collection of objects and \( x \) be an element of \( X \), then a fuzzy set \( \tilde{A} \) in \( X \) is a set of ordered pairs
\( \tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / x \in X\} \), Where \( \mu_{\tilde{A}}(x) \) is called the membership function or grade of membership of \( x \) in \( \tilde{A} \) which maps \( X \) to the membership space \( M \) which is considered as the closed interval \([0, 1]\\)".

Note: When \( M \) consists of only two points 0 and 1, \( A \) becomes a non-fuzzy set (or Crisp set) and \( \mu_{\tilde{A}}(x) \) reduces to the characteristic function of the non-fuzzy set (or crisp set).

- **Equality**
  Two fuzzy sets \( \tilde{A} \) and \( \tilde{B} \) are said to be equal if and only if \( \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x) \ \forall \ x \in X \).

- **Containment**
  A fuzzy set \( \tilde{A} \) is contained or is a subset of a fuzzy set \( \tilde{B} \), written as \( \tilde{A} \subseteq \tilde{B} \) if and only if \( \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x) \ \forall \ x \in X \).

- **Support**
  The support of a fuzzy set \( \tilde{A} \) is a crisp set \( S(\tilde{A}) \) such that \( x \in S(\tilde{A}) \Leftrightarrow \mu_{\tilde{A}}(x) > 0 \).

- **Normality**
  A fuzzy set \( \tilde{A} \) is called normal if and only if \( \max_{x \in X} \mu_{\tilde{A}}(x) = 1 \).

- **\( \alpha \)-Level Set**
  The set of elements which belongs to the fuzzy set \( \tilde{A} \) at least to the degree \( \alpha \) is called the \( \alpha \)-Level Set:
  \( A_{\alpha} = \{ x \in X / \mu_{\tilde{A}}(x) \geq \alpha \} \) and \( A_{\alpha}^c = \{ x \in X / \mu_{\tilde{A}}(x) > \alpha \} \) is called 'strong \( \alpha \)-Level Set'.
Convexity
Let $\tilde{A}$ be a fuzzy set in $X$. Then $\tilde{A}$ is convex if and only if for any $x_1, x_2 \in X$, the membership function of $\tilde{A}$ satisfies the inequality
$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \quad \text{for } 0 \leq \lambda \leq 1$$

1.4 Fuzzy number and membership functions
A fuzzy number is a special class of a fuzzy set. Different definitions and properties of fuzzy numbers are encountered in the literature but they all agree on that a fuzzy number represents the conception of "a set of real numbers close to a "where A is the number being fuzzified.

A fuzzy number is a fuzzy set in the universe of discourse $X$ that is both convex and normal. Fig-1.1 shows a fuzzy number $\tilde{A}$ of the universe of discourse $X$ that is both convex and normal. The term fuzzy number is used to handle imprecise numerical quantities. For example shortage cost of a commodity is about $5.
A general definition of a fuzzy number according to Dubois and Prade [42] is a real fuzzy number $\tilde{A}$ described as a fuzzy subset on the real line $R$ whose membership function $\mu_{\tilde{A}}(x)$ is

1. a continuous mapping for $R$ to the closed interval $[0, 1]$, 
2. constant on $(-\infty, a_1] : \mu_{\tilde{A}}(x) = 0, \forall x \in (-\infty, a_1]$,
3. strictly increasing on $[a_1, a_2] : \mu_{\tilde{A}}(x) = f(x) \forall x \in [a_1, a_2]$
where $f(X)$ is a strictly increasing function of $x$,
4. constant on $(a_2, a_3] : \mu_{\tilde{A}}(x) = 1, \forall x \in (a_2, a_3]$,
5. strictly decreasing on \([a_3, a_4]\) : e.g., \(\mu_{\tilde{A}}(x) = g(x) \quad \forall X \in [a_3, a_4]\)
where \(g(x)\) is a strictly decreasing function of \(X\).

6. constant on \([a_4, \infty)\) : \(\mu_{\tilde{A}}(x) = 0, \quad \forall X \in [a_4, \infty)\),

![Membership function of a general fuzzy number](image)

Fig-1.1: Membership function of a general fuzzy number \(\tilde{A} = (a_1, a_2, a_3, a_4)\)

A general shape of a fuzzy number following the above definition may be shown pictorially as in fig.1.1 Here \(a_1, a_2, a_3\) and \(a_4\) are real numbers. A fuzzy number \(\tilde{A}\) in \(X\) is said to be discrete or continuous according as its membership function \(\mu_{\tilde{A}}(x)\) is discrete or continuous. Triangular, Trapezoidal, Parabolic flat, Hyperbolic and Inverse-Hyperbolic are some special class of fuzzy numbers.

- **Linear Fuzzy Number (LFN):**

A LFN of \(\tilde{A}\) is specified by two parameters \((a_1, a_2)\) and is defined by its continuous membership function \(\mu_{\tilde{A}}(x) : X \rightarrow [0, 1]\) as follows:
Triangular Fuzzy Number (TFN):

A TFN of $\tilde{A}$ is specified by the triplet $(a_1, a_2, a_3)$ and is defined by its continuous membership function $\mu_{\tilde{A}}(x): X \rightarrow [0,1]$ as follows:
\[
\mu_\tilde{A}(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\
\frac{a_3-x}{a_3-a_2} & \text{if } a_2 \leq x \leq a_3 \\
0 & \text{otherwise}
\end{cases}
\]

➤ Parabolic Fuzzy Number (PFN):

A PFN \( \tilde{A} \) is also specified by the triplet \((a_1, a_2, a_3)\) and is defined by its continuous membership function \( \mu_\tilde{A}(x): X \rightarrow [0, 1] \) as follows:

\[
\mu_\tilde{A}(x) = \begin{cases} 
1 - \left( \frac{a_2-x}{a_2-a_1} \right)^2 & \text{if } a_1 \leq x \leq a_2 \\
1 - \left( \frac{x-a_2}{a_3-a_2} \right)^2 & \text{if } a_2 \leq x \leq a_3 \\
0 & \text{otherwise}
\end{cases}
\]

Fig-1.4: Membership function of a PFN
- **Trapezoidal Fuzzy Number**:

  A TrFN $\tilde{A}$ is specified by four parameters $(a_1, a_2, a_3, a_4)$ and is defined by its continuous membership function $\mu_{\tilde{A}}(x): X \rightarrow [0, 1]$ as follows

  $$
  \mu_{\tilde{A}}(x) = \begin{cases} 
  \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\
  1 & \text{if } a_2 \leq x \leq a_3 \\
  \frac{a_4-x}{a_4-a_3} & \text{if } a_3 \leq x \leq a_4 \\
  0 & \text{otherwise}
  \end{cases}
  $$

  ![Fig-1.5: Membership function of a TrFN](image)

- **$\alpha$-Cut of a Fuzzy Number**:

  A cut of fuzzy number $\tilde{A}$ in $X$ is denoted by $A_\alpha$ and is defined as the following crisp set

  $$
  A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X\} \text{ where } \alpha \in [0, 1]
  $$
A_\alpha is a non-empty bounded closed interval contained in X and it can be denoted \( A_\alpha = [A_L(\alpha), A_R(\alpha)] \). \( A_L(\alpha) \) and \( A_R(\alpha) \) are the lower and upper bounds of the closed interval respectively. Fig 1.6 shows a fuzzy number \( \tilde{A} \) with \( \alpha \)-cuts \( A_{\alpha_1} = [A_L(\alpha_1), A_R(\alpha_1)] \), \( A_{\alpha_2} = [A_L(\alpha_2), A_R(\alpha_2)] \) it is seen that if \( \alpha_2 \geq \alpha_1 \) then \( A_L(\alpha_2) \geq A_L(\alpha_1) \) and \( A_R(\alpha_1) \geq A_R(\alpha_2) \)

![Graph showing \( \alpha \)-cut of a generalized fuzzy number \( \tilde{A} = (a_1, a_2, a_3, a_4) \)]

**1.5 Difference between Randomness and Fuzziness:**

Randomness and Fuzziness differ conceptually and mathematically though both systems use the unit intervals as their measure. Fuzziness describes ambiguous events. It measures the certainty of event occurrence. An event may occur at random, but to what degree it occurs, is fuzzy probability of a fuzzy event involves in the measure of the occurrence of ambiguous events. As for example when we say “there is 80% chance of lower price of products for next month”.
<table>
<thead>
<tr>
<th>Randomness</th>
<th>Fuzziness</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) It is represented as probability function i.e. $X \in A \rightarrow P(A) = a$ , It implies that probability of $X$ belonging to crisp set is $a$ . Here $a$ is a real number in $[0, 1]$ .</td>
<td>i) It is represented as possibility function . i.e. $\mu_A(x)=a$. It implies that membership grade of an element $X$ in a fuzzy set $A$ is $a$ . Here $a$ is a real number in $[0,1]$ .</td>
</tr>
<tr>
<td>ii) It is statistical inexactness due to random events. Example : There is a 50% chance that the selling price of an item is $20$</td>
<td>ii) It is imprecise and inexact due to the human perception process. Example : The selling price of an item may be about $20$ .</td>
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1.6 METHODOLOGY:

1.6.1 Single Objective Mathematical Programming Problem :

The problem of optimization (maximizing or minimizing) of an algebraic or a transcendental function of one or more variables subject to constraint optimization problem or precisely a single objective mathematical programming problem (SOMPP). The SOMPP can be formulated as :
Determine the values of variable $X=(x_1, x_2, x_3, \ldots, x_n)$ that optimize the function (called the objective function or criterion function)

**Optimize $f(x)$**

subject to

$g_i(x) \leq 0 \quad i=1,2,3,\ldots,m$

$h_j(x) = 0 \quad j=1,2,3,\ldots,l$

$x \geq 0$

Where $f(x), g_1(x), g_2(x), g_3(x), \ldots, g_m(x), h_1(x), h_2(x), \ldots, h_l(x)$ are functions defined on an $n$-dimensional set and $x$ is a vector of $n$ components $x_1, x_2, x_3, \ldots, x_n$.

**Note**: When both the objective function and the constraints are linear, the above SOMPP becomes single objective linear programming problem (SOLPP) otherwise it becomes single objective non-linear programming problem (SONLPP).

In illustrating SONLPP, if the problem is of minimization type, it can be converted in to maximization type by taking $f(x)=-F(x)$. Similarly $g_i(x) \geq 0$ type constraints can be easily converted into $G_i(x) \leq 0$ type where $G_i(x) = -g_i(x)$

An Ideal SONLPP:

**Minimize $g_o(x)$**

subject to the constraints

$g_k(x) \leq a_k \quad k=1,2,3,\ldots,m$

$h_j(x) = b_j \quad j=1,2,3,\ldots,l$

$x \geq 0$
with non-negativity restrictions of decision variables i.e. $x_i \geq 0$, $i=1,2,3,...,n$. Here, $x=(x_1,x_2,x_3,...,x_n)^T$ is a decision variable vector.

A vector $x$ satisfying all the constraints is called a feasible solution to the problem. The collection of all such solutions forms the feasible region. The SONLPP is to find a feasible solution such that $f(x) \geq f(\bar{x})$ for each feasible point $x$. Here $\bar{x}$ is called an optimum solution or solution to the problem.

For the solution of SONLPP, lot of mathematical techniques based on linearization, use of gradient, etc. available in the literature. Here, we illustrate few methods in crisp and fuzzy environment $s$, which have been used in this thesis to solve the inventory problems which are highly non-linear in nature.

- **Fuzzy Non-linear Programming (FNLP) Method.**

  Ever since Zadeh [109] developed the concept of fuzzy set theory, some authors exhibited their interests in the topics of fuzzy mathematical programming. Zimmermann [113] first apply fuzzy set theory to linear programming. Datta, et.al. [36] pursued the idea with a discussion of sensitivity analysis in fuzzy linear programming. Others, such as Tanaka and Asai [95] considered the fuzzy linear programming model with fuzzy coefficients in the objective function and in the constraints. In all cases, the authors considered ambiguous situations, vague parameters, loose restrictions or non-exact objectives in the linear programming domain. Detailed literature on fuzzy linear and Non-linear Programming with applications is available in the book of Lai and Hwang [61].
Let us now draw an outline regarding fuzzy non-linear programming and its solutions.

- **Fuzzy non-linear programming with fuzzy objective and fuzzy constraints**

  A crisp non-linear programming problem may be defined as follows:

  Minimize \( g_o(x) \)

  subject to the constraints

  \[ g_i(x) \leq b_i \quad i=1,2,3,\ldots,m \]

  \( x \geq 0 \) where \( x=(x_1,x_2,x_3,\ldots,x_n)^T \)

  Introducing fuzziness in the objective and constraint goals the non-linear programming problem i.e., a fuzzy non-linear programming problem with fuzzy resources and objective is:

  Minimize \( g_o(x) \)

  subject to the constraints

  \[ g_i(x) \leq \bar{b}_i \quad i=1,2,3,\ldots,m \]

  \( x \geq 0 \) where \( x=(x_1,x_2,x_3,\ldots,x_n)^T \)

  The impreciseness for objective and constraints described by the membership functions \( \mu_o(x) \) and \( \mu_i(x) \) respectively. These functions may be linear or non-linear.

  To solve the above problem, we use the max-min operator of Bellman and Zadeh [8] and the approach of Zimmermann [110]. The membership function of the decision set, \( \mu_D(x) \), is

  \[ \mu_d(x)=\min\{\mu_o(x),\mu_1(x),\ldots,\mu_m(x)\} \quad \text{for all } x \in X \]
The min operator is used here to model the intersection of the fuzzy sets of objective and constraints. Since the decision maker wants to have a crisp decision proposal, the maximizing decision will correspond to the value of x i.e., \( x^* \) which has the highest degree of membership in the decision set.

\[
\mu_b(x^*) = \max_{x \geq 0} \left[ \min \{ \mu_0(x), \mu_1(x), \ldots, \mu_m(x) \} \right]
\]

It is equivalent to solving the following crisp non-linear programming problem

Maximize \( \alpha \)
subject to
\[
\begin{align*}
\mu_0(x) & \geq \alpha \\
\mu_i(x) & \leq \alpha, \quad (i=1,2,3,\ldots,m) \\
0 & \leq \alpha \leq 1, \quad x \geq 0
\end{align*}
\]

**Fuzzy Non-linear Programming with Fuzzy Objective, Constraints and Coefficients:**

A crisp non-linear programming problem may be defined as follows:

Minimize \( g_0(x,c_0) \)
subject to the constraints
\[
\begin{align*}
g_i(x,c_i) & \leq b_i \quad (i=1,2,3,\ldots,m) \\
x & \geq 0
\end{align*}
\]

where \( x=(x_1,x_2,x_3,\ldots,x_n)^T \) is a variable vector. \( g_0,g_i \)'s are algebraic expressions in \( x \) with coefficients \( c_0 = (c_{01},c_{02},\ldots,c_{09}) \) and \( c_i = (c_{i1},c_{i2},\ldots,c_{i9}) \) respectively.
After introducing fuzziness in the crisp parameters, the problem in a fuzzy environment becomes

\[
\begin{align*}
\text{Minimize } & \quad g_o(x, \tilde{c}_o) \\
\text{subject to the constraints } & \quad g_i(x, \tilde{c}_i) \leq \tilde{b}_i, \quad i=1,2,3,\ldots,m \\
& \quad x \geq 0
\end{align*}
\]

In fuzzy set theory the objective, coefficients and constraints are defined by their membership functions which may be linear or non-linear. According to Bellman and Zadeh \(\ldots\) and following Carlsson and Krohonen \(\ldots\) and Trappey, et al. \(\ldots\), problem is transformed to

\[
\begin{align*}
\text{Maximize } & \quad \alpha \\
\text{subject to } & \quad g_i(x, \mu^{-1}_c(\alpha)) = \mu^{-1}_i(\alpha), \quad i=1,2,3,\ldots,m \\
& \quad x \geq 0
\end{align*}
\]

where membership functions of fuzzy coefficients are \(\mu_c(x) = (\mu_{c1}(x), \mu_{c2}(x), \ldots, \mu_{cn}(x))\) and those of fuzzy objective and fuzzy constraints are \(\mu_i(x), \quad (i=1,2,3,\ldots,m)\). Here \(0 \leq \alpha \leq 1\) is an additional variable which is known as aspiration level.

1.6.2 Multi-Objective Mathematical Programming Problem:

Development of single objective mathematical programming problems and methods for their solutions has been presented in 1.6.1. But, the world has become more complex and almost every important real-world problem involves more than one objective. In such cases, decision makers find it imperative to evaluate solution alternatives according to multiple criteria. A multiple objective programming problem (MOPP) can be defined as
Maximise \( f_1(x) = z_1 \)  
Maximise \( f_2(x) = z_2 \)  
\[ \text{...} \]  
\[ \text{...} \]  
Maximise \( f_k(x) = z_k \)  
such that \( x \in S \)  

where \( S \) is the feasible region in decision space. \( S \) may also be defined by constraints  
\[ S = \{ x \in \mathbb{R}^n : g_i(x) \leq a_i, i = 1,2,3,\ldots, m, h_j(x) = b_j, j = 1,2,3,\ldots, l, x \geq 0, b \in \mathbb{R}^n \} \]  

if \( f_k \)'s and \( g_i \)'s are linear, the corresponding problem is called Multiobjective Linear Programming Problem (MOLPP). When all or any one of the above functions \( (f_k \)'s and \( g_i \)'s) are non-linear, it is referred to as Multi-objective Non-linear Programming Problem (MONLPP).

To solve these problems developed in different environments, several methods like linear and non-linear goal programming techniques. Fuzzy linear and non-linear goal programming techniques, etc. are available in the literature. Some of these methods which have been used in this thesis are presented in this section.

- **Fuzzy Programming technique to solve crisp multi-objective problem**

  To solve multi-objective programming problem, the first step is to assign upper, and lower bounds for each objective. Let \( U_r \) and \( L_r \) are the upper and lower bounds of \( r^{th} \) objective for each \( r = 1,2,3,\ldots,k \). Here \( L_r \) = aspired level of achievement, \( U_r \) = higher acceptable level of
achievement and \( dr = Ur - Lr \) the degradation allowance. The steps of the fuzzy programming technique are as follows:

Step-1: Solve the multi-objective programming problem as a single objective problem using only one objective at a time and ignoring the others. Let the optimal solution for the \( i \)th objective \( f_i(x) \) be \( x^i \).

Step-2: From the results of step-1, determine the corresponding values for every objective at each solution derived. According to each solution and value for every objective, pay off matrix can be formulated as follows:

<table>
<thead>
<tr>
<th></th>
<th>( f_1(x) )</th>
<th>( f_2(x) )</th>
<th>( \ldots )</th>
<th>( f_k(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^1 )</td>
<td>( f_1(x^1) )</td>
<td>( f_2(x^1) )</td>
<td>( \ldots )</td>
<td>( f_k(x^1) )</td>
</tr>
<tr>
<td>( x^2 )</td>
<td>( f_1(x^2) )</td>
<td>( f_2(x^2) )</td>
<td>( \ldots )</td>
<td>( f_k(x^2) )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x^k )</td>
<td>( f_1(x^k) )</td>
<td>( f_2(x^k) )</td>
<td>( \ldots )</td>
<td>( f_k(x^k) )</td>
</tr>
</tbody>
</table>

where \( x_1, x_2, \ldots, x_k \) are the idea solutions of the objectives \( f_1(x), f_2(x) \) \( \ldots \) \( f_k(x) \) respectively. As \( Ur = \max \{ f_1(x^1), f_2(x^2), \ldots, f_k(x^k) \} \), \( Lr = \{ f_1(x^1), f_2(x^2), \ldots, f_k(x^k) \}, r = 1, 2, 3, \ldots, k \) is restated as

\[
\begin{align*}
\text{Minimize} & \quad f(x) = f_1(x) f_2(x) \ldots f_n(x) \\
\text{Subject to} & \quad x \in S, \text{ where} \\
S & = \{ x : x \in \mathbb{R}^n : g_i(x) \leq a_i, i = 1, 2, 3, \ldots, m, h_j(x) = b_j, j = 1, 2, 3, \ldots, l, \ x \geq 0, b \in \mathbb{R}^n \}
\end{align*}
\]
Here $x$ is an $n$-dimensional vector of decision variables $f_1(x), f_2(x), \ldots, f_k(x)$ are distinct objective functions and $S$ is the set of feasible solutions.

Find $x = (x_1, x_2, x_3, \ldots, x_n)^T$ so as to satisfy

- $f_r(x) \leq L_r, \quad r = 1, 2, 3, \ldots, k$
- $g_i(x) \leq a_i, \quad i = 1, 2, 3, \ldots, m$
- $h_j(x) = b_j, \quad j = 1, 2, 3, \ldots, l$

for this multi-objective mathematical problem, a membership function $\mu_r(x)$ corresponding to the $r^{th}$ objective is defined as

$$
\mu_r(x) = \begin{cases}
1 & \text{if } f_r(x) \leq L_r \\
\frac{U_r - f_r(x)}{U_r - L_r} & \text{if } L_r \leq f_r(x) \leq U_r \\
0 & \text{if } f_r(x) \geq U_r
\end{cases}
$$

Step -3: From step -2, we may find for each objective, the aspiration values $L_r$ and $U_r$ corresponding to the set of solutions. Using Zimmermann [113] method, the equivalent non-linear programming problem for the multi-objective programming problem may be stated as

Maximize $\alpha$

such that

$$
\alpha \leq \frac{U_r - f_r(x)}{U_r - L_r} \quad \text{for } r = 1, 2, 3, \ldots, k
$$

- $g_i(x) \leq a_i, \quad i = 1, 2, 3, \ldots, m$
- $h_j(x) = b_j, \quad j = 1, 2, 3, \ldots, l$
- $x \geq 0$
1.7 Historical Review On Crisp Inventory Models

The analysis of an inventory system was first developed by Harris [56] of the Westing House Corporation. USA. He derived the classical lot size formula. This formula was also developed independently by R. H. Wilson [101] after a few years and hence it has been named as Harris Wilson Formula or Wilson’s formula. This model was formulated under certain simple assumption such as the demand rate is uniform and known, shortages are not allowed, replenishment rate is infinite and lead time is negligible etc. Since then the above-mentioned model has been modified and extended by several researchers changing the assumptions to make it more and more realistic.

The first full-length book to deal with inventory problems was written by F. E. Raymond [82]. It contains no theory or derivations and only attempts to explain how various extensions of the simple lot size models can be used in practice.

Interest in the study of inventory problems has increased since world war-II and numerous publications have been devoted solely to this subject. Arrow, Karlin and Scarf [4] published a book considering the mathematical properties of inventory systems. Five years later, an excellent collection on this subject was published by Hadley and Whitin [53] in their book. Devoting an interesting review on the development of the subject, Naddor [76] published a book named, “Inventory Systems“. Before Second World War most of the studies in this area were deterministic in nature.
After the war, management sciences along with various operation research techniques were gradually developed. Some inventory problems were studied considering the stochastic process. Whitin [100] developed a stochastic version of the classical lot size model. It is interesting to note that though the engineers were seeking to solve the practical inventory problems arising in industry by using analytical techniques, the economists were not motivated initially to take an active interest in inventory problems even knowing the fact that inventory problems play an important role in the studies of dynamic economic behavior.

The first attempt by economists to study the inventory problems was made in a paper of Arrow, Harris and Marschak [3]. They presented an article providing a rigorous mathematical analysis of a simple inventory model. Their work was further investigated by a abstract paper by mathematicians- Dvoretzky, Kiefer and Wolfowitz [43]. Since then researchers were mainly engaged in the classical EOQ model developed by Harris [56] under various conditions up to the late sixties. After that period, many research papers, newsletters and reviews dealing with various aspects of inventory system have been published in the journals throughout the world. In reality, there are many situations where the demand rate is not constant. It may depend on time, on hand inventory level or initial stock level, selling price, expenditure, frequency of advertisement, etc. Silver and Meal [91] published a lot size model taking time varying demand. After that the models with time dependent demand rates have been studied by several researchers such as, Donaldson [41], Silver [92], Mitra et. al. [74],
Dave and Patel [37], Murdeshwar [75], Goswami and Choudhari [49, 50], Datta and Pal [36], Goyal, et. al. [51], Bhunia and Maiti [14], and others. Again the demand rate is influenced by the selling price of an item as whenever the selling price of an item increases, the demand for that decreases and vice-versa. Generally, this type of demand is seen for furnished goods. Several authors like Cohen [29], Goyal and Gunasekaran [52], Das et. al. [33] have investigated this type of inventory model. In the wholesale or retail business the quantity discounts plays an important role between the customer and retailer as well as the retailer and whole seller. It is our common experience that whole seller as well as retailer gives price discount to encourage the customer to buy more. In this area, several researchers such as Hadley and Whitin [53], Benton [11], Das [31], published their research works.

In the real world, deterioration is a natural phenomenon. There are some Physical goods, which deteriorate with the progress of time during their normal storage. This natural phenomenon is very significant in the inventory systems. In this area, a lot of research papers have been published by several researchers viz. Ghare and Schrader [45], Misra [71], Covert and Philip [30], Aggarwal and Jaggi [2], Bahari-Kashani [5], Mandal and Phaujdar [69], Raafat [81], Datta and Pal [35], Goswami and Choudhari [49], Cheng and Ting [25], Goyal and Gunasekaran [52], Hariga and Benkherouf [54], Pakkala and Achary [78], Bhunia and Maiti [13], Giri and Chaudhari [47] and others.

Inventory models with two warehouse facilities (one is existing storage owned by the management at the place of
business, named as own warehouse ‘OW’ and another is hired on rental basis little away from business place, named as rented warehouse ‘RW’ have been discussed by Sarma [88], Dave [38], Pakkala and Achary [78], Bhunia and Maiti [12, 14], Benkerouf [10], Kar et. al. [58] and others.

Multi-item classical inventory models under resource constraints such as capital investment, available storage area, number of orders and available set-up time, etc. is available in well-known books such as Naddor [76], Hadley and Whitin [53], Churchman, et. al. [28], etc.

The above-mentioned inventory models have been developed either in the crisp environment or probabilistic environment. In real-life world, there is another type of environment called fuzzy environment in which inventory parameters / data are vague and imprecise. Practical inventory problems demand the formulation and solution of the models in this environment also.

1.8 Historical Review On Fuzzy Inventory Models

The first thought about fuzzy concept by Bertrad Russell is quoted very often: “All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial one. The law of excluded middle is true when precise symbols are employed but it is not true when symbols are vague as, in fact all symbols are.”
“All languages are vague” and “vagueness” clearly is a matter of degree.”

An important step towards dealing with vagueness was introduced by the philosopher Max Black who introduced the concept of vague set.

The first publication in fuzzy set theory by Zadeh [109] shows the intention of introducing new techniques to accommodate uncertainty in the non-stochastic sense. Bellman and Zadeh [8] first introduced fuzzy set theory in decision-making processes. After that Zimmermann [110], Dubois and Prade [42] etc. are developed the fuzzy set approach and it is widely used in many applied subjects. Few research papers have been developed in fuzzy inventory. In this system inventory costs (such as holding cost, shortage cost, etc.), purchasing cost, goals, resources, etc. are not defined precisely but are imprecise i.e., fuzzy in nature.

Sommer [94] applied fuzzy dynamic programming to an inventory and production-scheduling problem in which the management wishes to fulfill a contract for providing a product and then withdraws from the market. Hence the objectives of the problem can be represented as fuzzy statements:

a) The production should decrease as continuously as possible
b) The stock should be at best zero at the planning horizon.

The planning horizon is separated into discrete planning periods and the problem becomes one to determining the production levels and hence the stock on hand for each of the planning periods gives as the input of amounts of the product that must be provided during each period. The fuzzy objectives can in term be expressed
as a membership function of the production levels for each period. This membership function represents the set of desirable production levels to fulfill the objectives. He noted that an optimal set of production levels is that set which maximize this membership function.

Kaeprzyk and Stainiewski [57] considered the fuzziness inherent in the inventory level, the demand, the replenishment and the constraint imposed on replenishment and the goods for desirable inventory levels in an infinite horizon problem. The output of the algorithm developed for solving this fuzzy model is the firm’s optimal time-invariant replenishment rule as represented by a fuzzy conditional statement. The algorithm employs a procedure used in fuzzy mathematical programming in which a fuzzy optimal decision is viewed as the intersection of fuzzy constraints and a fuzzy goal. Park [80] examined the EOQ formula in the fuzzy set theoretic perspective associating the fuzziness with the cost data. Here inventory costs were represented by Trapezoidal Fuzzy Numbers (TrFN) and the fuzzy EOQ model was transformed to a crisp optimization problem by defuzzification. Instead of solving it, he gave an equivalent crisp expression of average inventory cost.

Lee, Kramer and Hwang [66] proposed a way to understand the effects of demand, fuzziness in Material Requirement Planning (MRP) system on classical inventory model. They also made a comparative study on this model using three algorithms which are (1) Part period Balancing (PPB) lot-sizing method (2) Heuristic for selecting lot-sizing and (3) Dynamic programming method on economic lot-size.
Bector et. al. [7] considered a lot size inventory model with variable demand in crisp and fuzzy environments. Lam and Wong [62] applied a fuzzy mathematical programming to solve the joint economic lot-size problem with multiple price breaks.

Roy and Maiti [83, 87] solved the classical EOQ models in fuzzy environment with fuzzy objective goal and constraints by fuzzy non-linear programming / fuzzy geometric programming and fuzzy goal programming techniques. Roy and Maiti [86] considered another multi-item fuzzy inventory model and solved it by fuzzy geometric programming technique. They [85] also formulated a multi-objective inventory model of deteriorating items in fuzzy environment and solved it by two different fuzzy non-linear programming techniques.

Recently, some inventory models are also discussed using different type of fuzzy numbers. Vujosevic, et. al. [98] considered a modification of EOQ formula in the presence of imprecisely estimation of holding and ordering costs. Here, costs are represented by Trapezoidal Fuzzy Number (TrFN). To obtain optimum order quantity yao et. al. [102,104,105,106,108] discussed fuzzy inventory with and without backorder models. In [102,106], costs represented by triangular and trapezoidal fuzzy number then fuzzy total cost is obtained by using extension principle. Centroid method is used for defuzzification. Only in two models of yao et. al. [105,107] Signed distance method is used to defuzzify. In [105], both Centroid and Signed distance methods are used to defuzzify the total cost in without backorder inventory model, where total demand and the holding cost per unit per day

33
are represented by triangular fuzzy numbers. Currently Chen S.H. and Wang C.C. [22] consider the backorder inventory model with fuzzy yearly demand, fuzzy order cost fuzzy inventory cost and fuzzy backorder cost. Function [principle method is used to defuzzify the total cost. Similarly in [23,24], Chen S.H. used the same Function principle method but these models are also for deteriorating items.

1.9 Motivation and Objectives of the thesis

In the existing literature, classical inventory models were developed generally under the assumption of constant inventory parameters including demand rate in both manufacturing and non-manufacturing environments. In reality, there are many situations where the demand rate is not constant but varies. It may depend on time, initial stock-level, on-hand inventory level, selling price of the item, number of advertisements etc. Now days, more attention is paid to inventory models with stock-dependent demand and frequency of advertisement-dependent demand as it is related with the motivation of the customers to buy more.

Deterioration of an item is one of the best concepts, which attracted the attention of the researchers in the inventory management. Till now, very few literatures are available on the inventory problems with deteriorating items. Similarly multi-item, multi-warehouse and multi-objective criteria are available in very rare cases.

In real life, the inventory parameters such as demand, holding cost, set-up cost, purchasing cost, storage area, and rate
of deterioration are normally imprecise, vague and flexible in nature. There values instead of being fixed may vary within some ranges. To depict this nature they may be represented by fuzzy parameters with appropriate membership functions. In this environment both linear and non-linear membership functions are proposed.

In last two decades, new researchers have worked with inventory problems in fuzzy environment but most of them have represented the impressions of the parameters like demand, inventory costs etc. by fuzzy numbers. Recently Yao et. al. [105,106] and Chen et. al. [22,23] represents such parameters by triangular and trapezoidal fuzzy numbers and Centroid, Signed distance and Function principle methods are used for defuzzification. But deterioration of an item is not considered in these problems. Some fuzzy inventory problems are formulated as optimization problems in fuzzy environment and applied the fuzzy mathematical programming techniques to solve them. In this area Maiti et. al. [83,84,86,87] have developed some inventory models with constraints in fuzzy environments and solved by applying fuzzy mathematical programming techniques, i.e. fuzzy non-linear programming technique, fuzzy goal programming technique etc. Rather than it there is a need and scope of development in such problems, which depicts real life situations.

In thesis, we have tried to fill up the gaps in the development and solution of the inventory models in fuzzy environments as mentioned above. The purpose is to solve some inventory control problems, which have not been referred so far in the literature. All
the models of this thesis have been devoted to the formulation and solution of the inventory models of deteriorating items. Various types of demand are used to formulate models. All the models are developed in fuzzy environment in which some new mathematical techniques have been developed and introduced.

1.10 Organization of the Thesis:

In this proposed thesis, some realistic inventory models are formulated and solved in crisp and fuzzy environments. This study of EOQ models for deteriorating items is divided in seven chapters as follows:

Chapter-1: Introduction

This chapter contains a brief review literature on inventory models. Further developments in inventory control are updated. Also the chapter reviews the inventory models under fuzziness. Finally, it is concluded with the recent development of inventory models under fuzzy environment along with summary.

Chapter-2: Fuzzy EOQ Models For Deteriorating Items

Defuzzification By Centroid and Signed Distance Method

In this chapter, an inventory model with shortages is developed under fuzzy environment. Deterioration rate, holding cost and shortage cost are represented by triangular fuzzy numbers in the total cost. Using the Centroid and Signed distance method to defuzzify the total cost and
optimum order quantity in the fuzzy sense is estimated. Model is illustrated with a numerical example and its results are compared. The sensitivity analysis of the optimum solution with respect to the changes in the different parameter values is also discussed.

Chapter-3: Fuzzy EOQ Model for Deteriorating Items where
Demand depends upon Selling Price and
Frequency of Advertisements
The third chapter deals with fuzzy inventory model with selling price of the item and the frequency of advertisements dependent demand. Deterioration rate, holding cost, shortage cost and advertisement cost are represented by trapezoidal fuzzy numbers in the total cost. Function principle method is used to defuzzify the total cost.

Chapter- 4: Fuzzy EOQ Model For Deteriorating Items With Two Warehouses
In this chapter, an EOQ model for deteriorating items with two warehouses is developed in fuzzy sense. Deterioration rates of two warehouses are considered to be different due to change in environment. Crisp model is expressed in fuzzy environment and solved by Signed distance and function principle method of defuzzification.
Chapter- 5 : Multi-Item Fuzzy EOQ Model For Deteriorating Items

In this chapter, a cost minimization deteriorating multi-items inventory model with price and frequency of advertisements dependent demand is formulated in fuzzy environment. The available warehouse space is limited for inventory storage. Total cost, warehouse space, inventory costs and deteriorating rate are represented by linear and non-linear \((n^{th}\) parabolic) membership functions. The model is solved by fuzzy non-linear programming (FNLP) method and illustrated with a numerical example.

Chapter- 6 : Fuzzy EOQ Model For Deteriorating Items with Some Non-Linear Membership Functions

This Chapter deals with multi-item fuzzy EOQ model with stock dependent demand for deteriorating items. Various Inventory costs are represented by different non-linear membership functions such as exponential, hyperbolic and inverse hyperbolic membership functions. Inventory goal, deteriorating rate and total investment constraint are represented by linear membership functions. The model is solved by fuzzy non-linear programming (FNLP) method. Results are presented along with those of corresponding crisp model and a sensitivity analysis.
Chapter-7: Multi-Objective Multi-Item Fuzzy EOQ Model For Deteriorating Items

In this chapter multi-objective multi-item inventory model for deteriorating items with price dependent demand under limited investment and storage area is developed in fuzzy environment. Profit goal, wastage cost, budget and storage area are represented by linear membership functions. The model is solved by fuzzy non-linear programming (FNLP) method with the help of a numerical example.

Finally, the scope of further study is discussed. At the end of the thesis a comprehensive bibliography on inventory models under fuzziness is listed.