CHAPTER I

INTRODUCTION
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1.1) Nature of Boundary Value Problems:

Preliminary Remarks:

In Physics, there are many problems in which the fundamental object under investigation is a Vector or a Scalar field, and the ultimate aim is to know the behaviour of this field throughout the space for all the times, when initially the local behaviour of the field and all the time behaviour of the field at the boundary region of the space is given. In the problem of the wave guide, it is the electric field Vector or the magnetic field Vector that is to be determined inside the wave guide with its specified behaviour at the wave guide surface. In the problem of quantum mechanics, the Scalar Complex valued function $\Psi$ is to be found out specifying the required boundary conditions depending on the details of the problem. In the study of temperature distribution in a solid, it is the temperature that we wish to know with certain conditions on the temperature imposed at the boundary of the solid. In a similar way in vibration problems, the displacement vector field is to be investigated.

In the above stated problems and also in similar other problems of determination of Scalar or Vector fields, the field should satisfy certain relations throughout the space for all times. Generally such relations are those which interrelate at any given point and time the rate of //contd.....
change of the field in different directions and the rate of change with time. Such equations are called as partial differential equations in space and time or in short partial differential equations in four dimensional space namely three dimensions of space and fourth of time. The Scalar field should in addition to the above differential equations, satisfy certain conditions at the boundary of the space. When one wants to know the field every-where at all times, he must obtain the solution of partial differential equations such that the solution also satisfy the required conditions at the boundary.

Such problems of finding a Scalar or a Vector field which satisfies certain partial differential equations, inside certain space together with some boundary conditions at the boundary of the space are called as boundary value problems.

Mathematically this can be expressed as

\[
\text{Partial differential equation for the inner region.} \quad \left| \begin{array}{c}
\text{Boundary} \\
+ \text{Conditions} \\
\text{Boundary}
\end{array} \right| = \left| \begin{array}{c}
\text{value} \\
\text{problems.}
\end{array} \right|
\]

1.2) **Partial differential equations in boundary value problems**:

The partial differential equation vary, depending upon the field and the nature of boundary value problems. The various partial differential equations useful in different situations are listed as follows:
a) **Problems on Mechanical Vibrations**: (70)

1) The wave equation of the string is
\[
\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = -\frac{a(x,t)}{T}, \text{ where } v = \sqrt{\frac{T}{\rho}}
\]
and \(U(x,t)\) denotes the displacement of a point of the string at the position \(x\) and time \(t\), under the external load \(q(x,t)\) calculated per unit length \(T\) and \(\rho\) being the tension and the linear density of the string respectively.

2) **The equation of the longitudinal vibrations of a rod of constant cross-section**:
\[
\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{where } v = \sqrt{\frac{E}{\rho}}
\]
where \(u(x,t)\) denotes the displacement of the Section of a rod with abscissa \(x\) at time \(t\), \(E\) and \(\rho\) being the Young's modulus and density of the material of the rod respectively.

3) **The equation of the transverse vibrations of a rod, (beam)**: (36, 59).

The equation is
\[
\frac{\partial^4 u}{\partial x^4} + \frac{1}{a^4} \frac{\partial^2 u}{\partial t^2} = \frac{a(x,t)}{EJ} \quad \text{where } a^2 = \sqrt{\frac{ES}{\rho}}
\]
and \(u(x,t)\) denotes the displacement of the particles of the midline of the rod, \(q(x,t)\) denotes the external load per unit length, \(E\) is Young's modulus, \(J\) is the moment of inertia of the Cross-Section, \(\rho\) is the density of the material of the rod and \(S\) is the Cross-Sectional area.
4) The wave equation of the membrane: (70)
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = -\frac{q(x,y,t)}{T}, \quad v = \sqrt{T/\varrho}
\]
where \(u(x,y,t)\) is the displacement of the point \((x,y)\) of the membrane at time \(t\), under the external load per unit area, \(q(x,t)\) and the surface tension \(T\), \(\varrho\) being the surface density.

5) The equation of the transverse vibration of an elastic thin plate: (36, 59).
\[
\Delta \cdot \Delta \cdot u + \frac{1}{b^4} \frac{\partial^2 u}{\partial t^2} = \frac{q(x,y,t)}{D}, \quad b^2 = \sqrt{\frac{D}{\varrho \cdot h}}
\]
where \(u(x,y,t)\) is the displacement of the point \((x,y)\) of the mid plane of the plate at time \(t\), \(q(x,y,t)\) is the density of the external load, \(D\) is the flexural rigidity, \(h\) is the thickness, \(\varrho\) is the density of the material of the plate, and \(\Delta\) is the two dimensional Laplacian operator.

\[
\Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

The corresponding equations for static Deflection may be derived from the preceeding equations, if it is assumed that the external load and the unknown value of the displacements \(u\) are independent of time \(t\). For example the equation of equilibrium for a membrane will be
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{q(x,y)}{T}
\]

The well known equation of the torsion of prismatic rod is
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2
\]
where \(u(x,y)\) is a torsion function.
b) **Diffusion Problems**: (68, 69)

When heat diffuses through a solid with sources of heat inside, due to chemical or atomic reactions or electric current flow, the differential equations satisfied is

$$\frac{1}{\rho c} \left[ \frac{\partial}{\partial x} (k \frac{\partial \theta}{\partial x}) + \frac{\partial}{\partial y} (k \frac{\partial \theta}{\partial y}) + \frac{\partial}{\partial z} (k \frac{\partial \theta}{\partial z}) \right] + \frac{q}{\rho c} = \frac{\partial \theta}{\partial t}$$

where \(k\) is the thermal conductivity of the material which varies from point to point, \(\rho\), \(c\) and \(q\) being the density, specific heat and the rate of heat generation respectively.

When neutrons diffuse through a fissional material producing more neutrons per fission after the absorption of the original neutrons, the partial differential equation applicable for such diffusion process is

$$D \nabla^2 n + \frac{(K - 1) n}{\lambda_c} = \frac{\partial n}{\partial t}$$

where \(n = n(x, y, z, t)\) neutron density,

\(v = \) neutron speed,

\(D = \) Diffusion Constant,

\(\lambda_c = \) Total neutron Capture mean free path,

\(K = \) Average neutrons produced per capture by fission.

c) **Quantum mechanical problems**: (48)

In quantum mechanical problems every particle is supposed to be associated with a complex valued wave function whose knowledge is supposed to give all information regarding the properties of the particle, the square of the wave function giving the probability density of finding the particle at a given position at a given instant.
of time. The partial differential equation satisfied by
the wave function is the Schrodinger's equation.

\[-i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi\]

where \( V \) is the potential in which the particle of mass \( m \)
moves and \( \hbar \) is equal to \( \hbar/2\pi \), \( \hbar \) being the Planck's
constant.

1.3) **Types of Boundary Conditions**: (17, 49)

Frequently four types of boundary conditions arise
in Physics. Either the function or its normal derivative
or their combination is specified at the boundary.

a) **Dirichlet Conditions:**

In this condition the value of the Scalar or Vector
Field is specified at the boundary. In the case of a
vibrating string tied at its two ends, the boundary condi-
tion is of this type. It can be expressed as

\[ Y \bigg|_{x=0,L} = 0 \]

where \( Y \) is the transverse displacement of the string, and
\( x = 0 \) and \( x = L \) are the two ends of the string.

b) **Neumann Conditions:**

In this type of conditions the value of the gradient
of the Scalar or Vector field is specified at the boundary.
In the case of heat entering into a certain solid from its
boundary, it is specified as the gradient of the temperature,
equated to a known constant. In one dimensional form this
can be written as \( \frac{d\theta}{dx} \big|_{\text{boundary}} = \text{known constant.} \)
c) **Third type of boundary conditions:**

In this type of conditions, the value of linear combination of the normal derivatives (i.e. gradient of the field) and the field itself is specified at the boundary. In the case of hot solid losing heat according to Newton's law of cooling this type of condition is specified. It is expressed as

\[ \frac{dT}{dn} + hT \bigg|_{\text{at Surface}} = hT_0 \]

where \( n \) is the outgoing normal to the surface, \( h \) is the co-efficient of heat exchange and \( T_0 \) is the temperature of the surroundings.

d) **Periodic boundary conditions:**

When some situation is repeated at two ends of the space interval, the conditions are said to be periodic. In this case the values of the field at the two ends are equal so also are the gradients of the field. The suitable example of this is the behaviour of an electron in the periodic lattice.

1.4) **Integral Transform Technique:**

a) **Preliminary remarks:** The oldest technique for the solution of a partial differential equations of mathematical physics is the method of separation of variables introduced by 'a' Alembert, Daniel Bernali and Euler. This method has its own value today also in the use of Integral Transforms.
In the middle of the last century, the symbolic
logic or operational calculus was developed systemati-
cally and Heaviside applied it successfully to the solutions
of certain problems, connected with the theory of electro-
magnetic oscillations. The modern treatment of the opera-
tional calculus is now known as Laplace transform, which is
usually used by engineers as a tool of the solutions of
engineering problems. The original operator viewpoint
of Heaviside was substantially displaced by the work of
Carson, Doetseh, Van-del-Bole and others who took either
the Laplace transform or Mellin Transform as the basis of
their investigations.

b) Introduction to the technique:

Every student of Physics is familiar with Fourier
transform which is used in transforming Position Space to
wave propagation space or time space to frequency space. Also
he comes across the Laplace transform in the problems of
electric resonance. However, these are not only transforms
which are used and can be used to the problems in Physics.
Since we have mainly made use of integral transform technique,
we discuss it in brief.

Depending upon the nature of the differential equation
and the boundary conditions, a kernel function $K(x,s)$ and
the corresponding weight function $W(x)$ are used to define
the integral transform of the function $F(x)$ as

$$\mathcal{F}(s) = \mathcal{W}[f(x)] = \int_{a}^{b} K(x,s) W(x) f(x)dx$$

provided the integral exists over the range $(a,b)$. 
The integral transform so defines maps function of $x$ into function $S$ and it has importance only when its inverse transform is also defined. So that whenever required we can transform back the functions of $S$ into the functions $X$.

Further, it should satisfy some operational property so that if a function operated by the operator $L$ is operated by this integral transform, it should reduce to some parameter times the transformed function i.e.

$$T \left[ L \left[ f(x) \right] \right] = \lambda T \left[ f(x) \right]$$

or

$$\alpha \int_{a}^{b} L \left[ f(x) \right] K(x,s) \ W(x) \ dx = \lambda \ T \left[ f(s) \right]$$

From the above discussion it is obvious that for certain boundary value problems, the special integral transforms are to be developed.

c) Procedure to solve the boundary value problems by using integral transform technique:

The procedure to be followed in solving the boundary value problems, consisting of a differential equal with assigned boundary conditions is as follows:

1) Select an appropriate transform.

2) Multiply the differential equation and the boundary conditions by the selected Kernal and the corresponding weight function and integrate between the appropriate limits with respect to the variable or variables selected for exclusion.

3) The associated terms, which comes upto the integration by parts are to be covered by proper boundary conditions, while substituting values of limits of integrations.
4) Solve the resulting 'auxillary equation' by applying another integral transform or by using algebraic equation. Thus obtaining the transform of the desired function.

5) Finally apply the inversion theorem to get original function.

d) **Merits of the integral transform technique:**

   The merit of the integral transform method is solving boundary value problems lies in the transformation of the differential equation and the corresponding boundary conditions into an algebraic equation which is solvable by algebraic methods, and those solutions can be transformed to the boundary value problems by application of inverse transform.

   The merits of this method is:

   1) the possibility of applying it to the homogenous as well as non-homogenous boundary value problems.

   2) Simplification of calculations due to reduction to algebraic equations.

   3) The separation of principal and purely calculative parts of the solution.

   4) Possibility of forming an operational calculus by means of construction of tables of direct and inverse transform of functions of frequent use.

   5) The helpfulness in desiring unified solutions in many problems in physics and engineering.

   6) Usefulness in determination of Green's Functions.
The Integral Transform is the outcome of the works of various authors (7, 12, 14, 15, 16, 18, 21, 23, 27, 28, 29, 30, 32, 38, 41, 42, 51, 54, 55, 65, 71). Some of the recent works solving boundary value problems of different fields by the use of Integral Transforms are (1, 2, 3, 4, 5, 6, 7, 8, 11, 19, 20, 24, 25, 26, 31, 33, 34, 37, 39, 43, 44, 45, 46, 47, 48, 56, 62, 64, 66).

1.5) Boundary value problems with triangular boundaries:

Boundary value problems with triangular boundaries is also basically a combination of differential equation and boundary conditions. In case of boundaries other than triangular boundaries, the boundaries are expressed in terms of a single variable, for example, in case of a sphere the boundary is expressed by the equation $r = a$ where $a$ is the radius of the sphere. Or in the case of a cylinder, the circular boundary expressed by the condition $R = a$, where $a$ is the radius of the cylinder, or the top surface of the cylinder is expressed by the condition $z = 1$ where $1$ is the height of the cylinder and the bottom surface is expressed by $Z = 0$. But in the case of right triangular prism, all the boundaries are not expressible in terms of single variable, for example, in the case of isosceles right triangular boundary, the surfaces can be expressed by $Z = 0$, $Z = h$, $x = 0$, $y = 0$ and $x + y = a$. Here one surface is expressed in terms of the combination of $x$ & $y$. In the case of equilateral triangular boundary, the surface can be expressed by $y = 0$, $y = 3x + a\sqrt{3}$, $y = -3x + a\sqrt{3}$, $Z = 0$ and $Z = 1$. Here also the two surfaces are expressed as
a combination of two variables. Hence the solution for the triangular boundary becomes more difficult than in the case of other boundary problems.

An integral transform suitable for equilateral cross-section with Dirichlet type conditions has been introduced by Bhonsle and Patil (10) and the problem of elastic vibration is studied. Ugile (63) has developed a similar transform but for the third type of boundary conditions and used it to study the vibrations of triangular membra with boundaries attached to elastic medium. Further, an integral transform suitable for right angled triangular cross-section with Dirichlet type of boundary condition is developed by Ugile and used in the analysis of forced vibration of a plate on elastic foundation and carrying concentrated mass. He has also used the transform to workout the energy states of a particle in two dimensional infinite potential box inside a triangular cross-section. A problem of neutron diffusion for right angled triangular cross-sectional prism has been worked out by Patil & Patil (45).

Taking into consideration the scope for the boundary value problems with triangular boundaries, we have solved these problems in electrodynamics, neutron diffusion, etc. using an integral transform technique or integral transforms and finite difference technique together.
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