CHAPTER VIII

Measurement of Permeability from the Change of Frequency of a Right Triangular Cavity Due to the Introduction of a Cylindrical Sample.
8.1 Introduction.

Cavity resonators are used for tuning microwave sources, for generating driving fields in particle accelerators etc. Various normal shaped cavities such as rectangular, cylindrical, spherical, stripline have been studied in detail \([1,2,3]\). Here we study the prismatic cavity with cross-section as right-angled triangle. We consider various T.M. modes and also the frequency changes in T.M\(_{210}\) mode, when a cylindrical sample is introduced in the cavity.

8.2 Normal modes of T.M. Waves:

Consider the cavity as a space enclosed between the conducting planes \(x=0, y=0\) and \(x+y=a\) (Fig.1) where \(x, y, z\) form a rectangular co-ordinate system. With complex exponential time dependence, the wave equation for the transverse magnetic wave i.e. \(H_z = 0, E_z \neq 0\), the wave equation for \(E_z\) is

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \varepsilon_0 \mu_0 \right] E_z = 0 \quad \ldots \quad (8.2.1)
\]

The transverse field components will be obtained with the help of Maxwell's equations containing curl operators, given by
\[
K^2 H_x = \frac{\partial^2 H_z}{\partial z \partial x} + j \omega \epsilon_0 \frac{\partial E_z}{\partial y} \\
K^2 H_y = \frac{\partial^2 H_z}{\partial z \partial y} - j \omega \mu_0 \frac{\partial E_z}{\partial x} \\
K^2 E_x = \frac{\partial^2 E_z}{\partial z \partial x} - j \omega \mu_0 \frac{\partial H_z}{\partial y} \\
K^2 E_y = \frac{\partial^2 E_z}{\partial z \partial y} + j \omega \mu_0 \frac{\partial H_z}{\partial x} \\
\]

where \( K^2 = \omega^2 \epsilon_0 \mu_0 \)

Due to the continuity of the tangential component of \( E \) gives the boundary conditions as:

\[
E_z \bigg|_{x=0} = 0 \\
E_z \bigg|_{y=0} = 0 \\
E_z \bigg|_{x+y=a} = 0
\]

Vanishing of \( E \) at the perfect conductor surface gives the boundary conditions as:

\[
E_x \bigg|_{z=0} = E_y \bigg|_{z=0} = E_x \bigg|_{x=c} = E_y \bigg|_{z=c} = 0 \\
\]
The latter conditions as a result of equation (8.2.2) imply that

\[
\left. \frac{\partial E_z}{\partial z} \right|_{z=0} = \left. \frac{\partial E_z}{\partial z} \right|_{z=c} = 0 \quad \ldots \quad (8.2.5)
\]

Now equation (8.2.1) can be written as

\[
\left( -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E_z + \frac{\partial^2 E_z}{\partial z^2} + w^2 \varepsilon_0 \mu_0 E_z = 0
\]

Applying the integral transform defined by Ugile suitable for such problems, we have,

\[
- \lambda \bar{E}_z(m, n) + \frac{\partial^2 \bar{E}_z}{\partial z^2} + w^2 \mu_0 \bar{E}_z = 0
\]

Hence

\[
\frac{\partial^2 \bar{E}_z}{\partial z^2} + \left( w^2 \varepsilon_0 \mu_0 - \lambda \right) \bar{E}_z = 0 \quad \ldots \quad (8.2.6)
\]

where \( \lambda = \frac{n^2}{a^2} (m^2 + 2mn + n^2) \)

Boundary conditions (8.2.5) are transformed as

\[
\left. \frac{\partial \bar{E}_z}{\partial z} \right|_{z=0} = \left. \frac{\partial^2 \bar{E}_z}{\partial z^2} \right|_{z=1} = 0 \quad \ldots \quad (8.2.7)
\]
General solution of equation (8.2.6) can be written in the form

\[ \overline{E}_z = A \sin(\sqrt{\frac{\omega^2 \epsilon_0 \mu_0}{\epsilon_0}} - \lambda) Z + B \cos(\sqrt{\frac{\omega^2 \epsilon_0 \mu_0}{\epsilon_0}} - \lambda) Z. \]

The use of boundary conditions (8.2.7) gives

\[ A = 0 \quad \text{and} \quad -B \sqrt{\frac{\omega^2 \epsilon_0 \mu_0}{\epsilon_0}} - \lambda \cdot \sin(\sqrt{\frac{\omega^2 \mu_0}{\epsilon_0}} - \lambda) l = 0 \]

as \( B \neq 0 \) we have

\[ \sin(\sqrt{\frac{\omega^2 \mu_0}{\epsilon_0}} - \lambda) l = 0. \]

Hence

\[ (\sqrt{\frac{\omega^2 \mu_0}{\epsilon_0}} - \lambda) l = p\pi \]

where \( p \) is an integer so the solution is

\[ \overline{E}_z = B \cos(p\pi l) Z. \]

Any linear combination for different \( Z \) is also a solution, hence

\[ \overline{E}_z = \sum_{p} B_p \sin(p\pi z / l). \]
Now applying the inverse transform we have,

\[ E_z = \frac{4}{a^2} \sum_{m} \sum_{n} \sum_{p} B \cos \frac{p m z}{l} \psi_{m,n}(x,y) \]

For a particular mode, \( E_z \) is given by

\[ E_z = \frac{4}{a^2} B \cos \left( \frac{p m z}{l} \right) \psi_{m,n}(x,y) \]

putting the value of \( \psi_{m,n}(x,y) \)

\[ E_z = \frac{4B}{a^2} \left[ \sin(m+n)(\pi x/a) \sin(\pi y/a) - (-1)^m \sin(m+n)(\pi y/a) \sin(n \pi x/a) \right] \cos \left( \frac{p m z}{l} \right) \]  \hspace{1cm} \ldots \hspace{1cm} (8.2.8)

Where \( m \) and \( n \) are positive integers. Thus the normal modes of TM waves will be given by equation (8.2.8) with different values of \( m, n \) and \( p \).

The characteristic equation for this cavity is

\[ (m+n)^2 \frac{\pi^2}{a^2} + \frac{n^2 \pi^2}{a^2} + \frac{p^2 \pi^2}{l^2} = \omega^2 \epsilon_o \mu_o \]
or

\[
\left( m^2 + 2n^2 + 2mn \right) \frac{\pi^2}{a^2} + \frac{p^2 \pi^2}{l^2} = \omega^2 \varepsilon_0 \mu_0 = k^2 \quad \ldots \quad (8.2.9)
\]

The transverse components in any mode can be obtained with the help of Maxwell's equations containing curl operators by equation (8.2.2)

The complete set of field components for \( TM_{m, np} \) mode is

\[
E_x = \frac{4B}{a^2 k^2} \left[ (m+n)(\pi/a) \cos(m+n)(\pi x/a) \sin(n \pi y/a) \right. \\
- (-1)^m (n \pi/a) \cos(n \pi x/a) \sin(m+n)(\pi y/a) \left. \right] \left[- (\pi n/l) \sin \frac{p \pi z}{l} \right]
\]

\[
E_y = \frac{4B}{a^2 k^2} \left[ (n \pi/a) \sin(m+n)(\pi x/a) \cos(n \pi y/a) \right. \\
- (-1)^m (m+n)(\pi/a) \sin(n \pi x/a) \cos(m+n)(\pi y/a) \left. \right] \left[- \frac{p \pi}{l} \sin \frac{p \pi z}{l} \right]
\]

\[
E_z = \frac{4B}{a^2} \left[ \sin(m+n)(\pi x/a) \sin(n \pi y/a) \right. \\
- (-1)^m \sin(n \pi x/a) \sin(m+n)(\pi y/a) \left. \right] \cos \frac{p \pi z}{l}
\]
\[ H_x = j\omega_0 \frac{4B}{a^2 k^2} \left[ \frac{m+n}{a} \sin(m+n)\left(\frac{\pi x}{a}\right) \cos(m+n)\left(\frac{\pi y}{a}\right) \right] \cos \frac{\pi z}{l} \]

\[ H_y = -j\omega_0 \frac{4B}{a^2 k^2} \left[ \frac{m+n}{a} \cos(m+n)\left(\frac{\pi x}{a}\right) \sin(m+n)\left(\frac{\pi y}{a}\right) \right] \cos \frac{\pi z}{l} \]

\[ ... (8.2.10) \]

8.3. **Frequency change due to a cylindrical sample.**

Let us take a flat cylindrical sample kept at the surface \( z=0 \), axis of the cylinder passing through a point \( x = a/4 \), \( y = a/4 \) (Fig. 1). With the introduction of the sample, the frequency of the cavity resonator will change by the amount given by (Waldron pp. 296).

\[
\frac{\delta w}{w} = \frac{\iint_{V_0} (\vec{E}_1 \cdot \vec{D}_0 - \vec{E}_0 \cdot \vec{D}_1) - (\vec{H}_1 \cdot \vec{B}_0 - \vec{H}_0 \cdot \vec{B}_1) dV}{\iint_{V_0} (\vec{E}_0 \cdot \vec{D}_0 - \vec{H}_0 \cdot \vec{B}_0) dV} \quad ... (8.3.1)
\]

Where \( \vec{E}_0, \vec{D}_0, \vec{H}_0 \) and \( \vec{B}_0 \) are unperturbed values of fields and \( \vec{E}_1, \vec{D}_1, \vec{H}_1 \) and \( \vec{B}_1 \) are perturbations produced in these fields due to the introduction of a sample in the cavity, \( V_0 \) being the initial volume of the cavity.
8.4 | Frequency change for T.M\(_{210}\) mode:

We consider the frequency change for T.M\(_{210}\) mode in particular. The field components for this mode are obtained by putting \(m=2\), \(n=1\) and \(p=0\) in the general expression (8.2.10) hence we have

\[
E_x = 0
\]

\[
E_y = 0
\]

\[
E_z = \frac{4\pi}{a^2} \left[ \sin(3\pi x/a) \sin(\pi y/a) - \sin(\pi x/a) \sin(3\pi y/a) \right]
\]

\[
H_x = -j w \mu_0 \frac{4\pi}{a^2} \left[ \frac{\pi}{a} \sin(3\pi x/a) \cos(\pi y/a) - (3\pi/a) \sin(\pi x/a) \cos(3\pi y/a) \right]
\]

\[
H_y = -j w \mu_0 \frac{4\pi}{a^2} \left[ \frac{3\pi}{a} \cos \frac{3\pi x}{a} \sin \frac{\pi y}{a} - \frac{\pi}{a} \cos \frac{\pi x}{a} \sin \frac{3\pi y}{a} \right] \ldots (8.4.1)
\]

Now we consider the first term of the denominator of equation (8.3.1) which is

\[
\iint_{V_0} \sqrt{(E_x^2 + E_y^2)} \, dv = \int_0^a \int_0^{a-x} \int_0^1 (E_x^2 + E_y^2) \, dx \, dy \, dz
\]

\[
= \int_0^a \int_0^{a-x} \int_0^1 \left[ \epsilon_o E_x^2 + \epsilon_o E_y^2 + \epsilon_o E_z^2 \right] \, dx \, dy \, dz
\]

\[
= \int_0^a \int_0^{a-x} \int_0^1 \epsilon_o E_z^2 \, dz \text{ as } E_x \text{ and } E_y = 0
\]

for T.M\(_{210}\) mode.
Substituting the value of \( E_z \) from equation (8.4.1) in the above equation we get

\[
\iiint_{V_0} (\vec{E}_0 \cdot \vec{D}_0) \, dv = \int_0^a dx \int_0^{a-x} dy \int_0^1 dz \left[ \frac{16E_0^2}{a^4} \left( \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{a} + \sin^2 \frac{\pi x}{a} \sin^2 \frac{3\pi y}{a} - 2 \sin \frac{3\pi x}{a} \sin \frac{\pi y}{a} \sin \frac{\pi y}{a} \right) \right]
\]

Solving the integrals and substituting in the above equation we have

\[
\iiint_{V_0} (\vec{E}_0 \cdot \vec{D}_0) \, dv = \frac{16E_0^2}{a^4} \left( \frac{a^2}{8} + \frac{a^2}{8} \right)
\]

\[
= \frac{4B_0^2}{a^2} \frac{1}{\varepsilon_0^2} \quad \ldots \quad (8.4.2)
\]

Similarly substituting the values of \( H_x, H_y, H_z \) for \( T.M_{210} \) mode from equation (8.4.1) in the second term in the denominator of equation (8.3.1) we have
\[ \iiint_{V_0} (\mathbf{H}_o \cdot \mathbf{B}_o) \, dv = \int_0^a dx \int_0^{a-x} dy \int_0^1 dz \left( \mu_o H_x^2 + \mu_o H_y^2 \right) \]

\[ = \frac{16B^2 \mu_o j^2 w^2 e_1^2}{a^4 k^4} \left[ \int_0^a dx \int_0^{a-x} dy \int_0^1 dz \left( \frac{\pi^2}{a^2} \sin^2 \frac{3\pi x}{a} \cos \frac{2\pi y}{a} \right. \right. \]

\[ + \frac{9\pi^2}{a^2} \sin \frac{2\pi x}{a} \cos \frac{23\pi y}{a} \left. \right) \sin \frac{3\pi x}{a} \cos \frac{\pi y}{a} \sin \frac{\pi x}{a} \]

\[ + \left. \frac{6\pi^2}{a^2} \cos \frac{3\pi x}{a} \cos \frac{\pi x}{a} \sin \frac{3\pi y}{a} \sin \frac{\pi y}{a} \right] \]

After solving each integral in the above equation we get

\[ \iiint_{V_0} (\mathbf{H}_o \cdot \mathbf{B}_o) \, dv = \frac{16B^2 \mu_o j^2 w^2 e_1^2}{a^4 k^4} \left[ \frac{\pi^2}{8} + \frac{9\pi^2}{8} + 0 \right] \]

\[ = \frac{16B^2 \mu_o j^2 w^2 e_1^2}{a^4 k^4} \frac{5\pi^2}{2} \cdot \frac{2}{2} \]

\[ = - \frac{4OB^2 l}{a^4} \frac{w^2 \mu_o}{k^4} \pi^2 e_0 \cdot \]
Now we have

\[ K^2 = \mu_0 \varepsilon_0 w^2 = \frac{\pi^2}{a^2} (m^2 + 2mn + n^2) + \frac{p^2 \pi^2}{l^2} \]

\[ = 10 \frac{\pi^2}{a^2} \]

using the value in above equation we have

\[ \iint \int \nabla \cdot (\vec{H}_0 \cdot \vec{B}_0) \, dv = \frac{-40 \pi^2 B^2 l \varepsilon_0}{a^4 k^2} \]

\[ = \frac{-40 \pi^2 B^2 l \varepsilon_0}{a^4 10 \pi^2 / a^2} . \]

Hence

\[ \iint \int \nabla \cdot (\vec{H}_0 \cdot \vec{B}_0) \, dv = \frac{-4 \pi^2 \varepsilon_0 l}{a^2} \ldots \quad (8.4.3) \]

From equation (8.4.2) and (8.4.3) we have the complete denominator of equation (8.3.1) for T.M_{210} mode as

\[ \iint \int \nabla \cdot (\vec{B}_0 \cdot \vec{B}_0 - \vec{H}_0 \cdot \vec{B}_0) \, dv = \frac{4 \pi B^2 \varepsilon_0 l}{a^2} + \frac{4 \pi B^2 \varepsilon_0 l}{a^2} \]

Hence
\[ \iiint_{V_1} (\vec{E}_0 \cdot \vec{D}_0 - \vec{H}_0 \cdot \vec{B}_0) dV = \frac{8\beta^2}{9} \frac{1}{a^2} \] \hfill (3.4.4)

Now consider that the cylindrical specimen of volume \( V_1 \) is placed on \( z = 0 \) plane with the axis passing through \( x = a/4 \), \( y = a/4 \). Assuming that the fields over this specimen to be uniform and same as that at \( x = a/4 \), \( y = a/4 \) and \( z = 0 \), the perturbations in the magnetic field and in the magnetic induction will be

\[ \vec{H}_1 = -\vec{H}_0 \left( \frac{\mu-1}{\mu+1} \right) \]

and \( \vec{B}_1 = \mu_0 \vec{H}_0 \left( \frac{\mu-1}{\mu+1} \right) \) respectively.

Substituting these values in the numerator of equation (8.3.1) and taking into consideration that the first term \( (\vec{E}_1 \cdot \vec{D}_0 - \vec{H}_0 \cdot \vec{B}_1) \) vanishes at \( x = a/4 \), \( y = a/4 \) and \( z = 0 \), the numerator of equation 8.3.1 becomes

\[ \iiint_{V_1} (\vec{H}_1 \cdot \vec{B}_0 - \vec{H}_0 \cdot \vec{B}_1) dV = \int_{V_1} -\mu_0 \vec{H}_0^2 \left[ \frac{(\mu-1)}{(\mu+1)} \right] dV \]

\[ = \iiint_{V_1} 2\mu_0 \vec{H}_0^2 \left( \frac{\mu-1}{\mu+1} \right) dV \]

\[ = 2\mu_0 \left[ \left( \frac{\mu-1}{\mu+1} \right) \frac{a_0}{a_4} \right] \int_{V_1} \left( \frac{\mu-1}{\mu+1} \right) \frac{a_0}{a_4} dV \] \hfill (8.4.5)
Where $\Pi_0^2 = \Pi_x^2 + \Pi_y^2$ which from equation (8.4.1) is

$$H_0 = \frac{16\pi^2}{a^4 k^4} j w e_0^2 \left[ \frac{\pi^2}{2} \frac{\sin^2 2 \Pi x}{a} \cos \frac{\Pi y}{a} + \frac{9}{a^2} \frac{\sin 2 \Pi x}{a} \cos \frac{3 \Pi y}{a} \right. $$

$$- \frac{6}{a^2} \frac{\sin \Pi x}{a} \cos \frac{\Pi y}{a} \sin \frac{\Pi x}{a} \cos \frac{3 \Pi y}{a} + \frac{9}{a^2} \frac{\cos \Pi x}{a} \cos \frac{3 \Pi y}{a} \sin \frac{\Pi y}{a}$$

$$+ \frac{\pi^2}{a^2} \frac{\cos \Pi x}{a} \sin \frac{2 \Pi x}{a} \sin \frac{2 \Pi y}{a} - \frac{6}{a^2} \frac{\cos \Pi x}{a} \sin \frac{\Pi y}{a} \cos \frac{\Pi x}{a} \sin \frac{3 \Pi y}{a} \right]$$

Therefore

$$H_0^2 \bigg|_{a^2 \over 4^4 0^4} = \frac{16\pi^2}{a^4 k^4} \frac{2 2 2 2}{4 4 0} \left[ \frac{\pi^2}{a^2} \frac{1}{4} + \frac{9}{a^2} \frac{1}{4} + \frac{6}{a^2} \frac{1}{4} + \frac{9}{a^2} \frac{1}{4} \right.$$

$$+ \frac{\pi^2}{a^2} \frac{1}{4} + \frac{6}{a^2} \frac{1}{4} \right]$$

$$= \frac{16\pi^2}{a^4 k^4} j w e_0^2 \frac{32}{4} \frac{\pi^2}{a^2}$$

$$H_0^2 \bigg|_{a^2 \over 4^4 0^4} = - \frac{128 \pi^2}{a^6 k^4}$$
Hence equation (8.4.5) becomes

$$
\int_{v_1} - (\vec{H}_1 \cdot \vec{B}_0 + \vec{n} \cdot \vec{B}_1) dv = - 256 \frac{B^2 \pi^2 \mu w^2 \epsilon_0^2}{a^6 k^4} \left( \frac{\mu-1}{\mu+1} \right) v_1
$$

$$
= - 256 \frac{\pi^2 B^2 \epsilon_0}{a^6 k^2} \left( \frac{\mu-1}{\mu+1} \right) v_1
$$

Hence the relative change in frequency due to the introduction of the sample from equation (8.4.4), (8.4.6) and (8.3.1) is

$$
\frac{\delta w}{w} = \frac{-256 \pi^2 B^2 \epsilon_0 / a^2 \epsilon_k^2}{8 B^2 \epsilon_0 l/a^2} \left( \frac{\mu-1}{\mu+1} \right) v_1
$$

Hence

$$
\frac{\delta w}{w} = \frac{-16}{10a^2 \frac{1}{L}} \left( \frac{\mu-1}{\mu+1} \right) v_1
$$

or

$$
\frac{\delta w}{w} = - \frac{8}{5} \frac{v_l}{v_o} \left( \frac{\mu-1}{\mu+1} \right)
$$

(8.4.7)

Where $\frac{1}{2} a^2 l = v_o$ the volume of the triangular cavity.
From the knowledge the comparison between the quantities for right triangular prismatic cavity and those of the rectangular cavity can be made, they are given in Table 1.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Right triangular prismatic cavity with T.M(_{210}) mode, sample at ((\frac{a}{2}, \frac{a}{4}, 0))</th>
<th>Rectangular cavity with T.M(_{210}) mode sample being at ((\frac{a}{2}, \frac{a}{2}, 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) (\frac{\delta w}{w})</td>
<td>(-\frac{8}{5} \left(\frac{\mu-1}{\mu+1}\right) \frac{v_1}{v_o})</td>
<td>(-\frac{16}{5} \left(\frac{\mu-1}{\mu+1}\right) \frac{v_1}{v_o})</td>
</tr>
<tr>
<td>2) (K^2)</td>
<td>(\frac{10\pi^2}{a^2})</td>
<td>(\frac{5\pi^2}{a^2})</td>
</tr>
<tr>
<td>3) (\omega)</td>
<td>(\frac{10\pi^2}{a^2} \frac{1}{\mu_0 \varepsilon_0})</td>
<td>(\frac{5\pi^2}{a^2} \frac{1}{\mu_0 \varepsilon_0})</td>
</tr>
<tr>
<td>4) (\delta w)</td>
<td>(-8 \sqrt{\frac{2}{5}} \left(\frac{\mu-1}{\mu+1}\right) \frac{v_1}{v_o} \frac{\pi}{a}) (\cdot \frac{1}{\sqrt{\mu_0 \varepsilon_0}})</td>
<td>(-\frac{16}{\sqrt{5}} \left(\frac{\mu-1}{\mu+1}\right) \frac{v_1}{v_o} \frac{\pi}{a} \frac{1}{\sqrt{\mu_0 \varepsilon_0}})</td>
</tr>
</tbody>
</table>
8.5 Losses in Cavity Walls:

The loss in the cavity due to the finite conductivity of the metal walls, can be calculated by means of perturbation theory. The unperturbed cavity is taken as vacuum region bounded by a surface $S$ outside which is a perfect conductor. The perturbation consists of replacing the material outside the surface $S$ by a metal with $\epsilon = \frac{-j\sigma}{\omega e_0}$ and $\mu = 1$. The volume $V_0$ in equation (8.3.1) is the space within $S$, while the metal filled region outside $S$ constitute $V_1$.

We shall find the right hand side of the equation (8.5.1) is complex, hence to balance this we must have a complex frequency shift on the left hand side. According to Waldron, for any kind of cavity

$$\frac{\delta w}{w} + \frac{4}{2Q} = \frac{\mu_0 (1-j)}{2D} \int_S H_0^2 \, ds.$$ 

Where $D = L.H.S.$ of equation (8.4.4) and $d = \sqrt{2/\omega e_0 \sigma}$, the skin depth and $\int_S H_0^2 \, ds$ is the integration over the surface of cavity $Q$ is the factor of the cavity with lossy walls. Equating the imaginary parts of the equation, we get

$$Q = \frac{-D}{\mu_0 \int_S H_0^2 \, ds} \quad \cdots \quad (8.5.1)$$
8.6 Q. Factor for right triangular cavity in $T_{210}$ mode:

The surface integral on the denominator of equation (8.5.1) for the right triangular cavity is

$$
\iint_{z=0} \text{plane } H_0^2 ds + \iint_{z=1} \text{plane } H_0^2 ds + \iint_{x=0} \text{plane } H_0^2 ds
$$

$$
+ \iint_{y=0} \text{plane } H_0^2 ds + \iint_{x+y=a} \text{plane } H_0^2 ds \quad (8.6.1)
$$

Where $H_0$ is the tangential component of the magnetic field to the cavity surface.

Now for $z = 0$ plane using equation (8.4.1) we have the first integral in equation (8.6.1) as

$$
\iint_{z=0} H_0^2 ds = \int_0^a \int_0^{a-x} (H_0^2 x + H_0^2 y) dx dy
$$

$$
= -\frac{16\pi^2}{4\pi} \int_0^a \int_0^{a-x} \left[ \frac{2}{a^2} \sin^{\frac{2\pi x}{a}} \cos^{\frac{2\pi y}{a}} - \frac{2\pi^2}{a^2} \sin^{\frac{3\pi x}{a}} \cos^{\frac{\pi x}{a}} \cos^{\frac{3\pi y}{a}} \cos^{\frac{\pi y}{a}} \right]
$$

$$
+ \left[ \frac{2\pi^2}{a^2} \cos^{\frac{\pi x}{a}} \sin^{\frac{\pi y}{a}} + \frac{\pi^2}{a^2} \cos^{\frac{2\pi x}{a}} \sin^{\frac{2\pi y}{a}} - \frac{6\pi^2}{a^2} \cos^{\frac{3\pi x}{a}} \cos^{\frac{\pi x}{a}} \sin^{\frac{3\pi y}{a}} \sin^{\frac{\pi y}{a}} \right]
$$
Solving the integrals in the above equation we get

\[ \iiint_{z=0} H_o^2 ds = \frac{-16B_w^2 2^2 2}{4_k^4} \left[ \frac{\pi^2}{8} + \frac{9\pi^2}{8} + 0 + \frac{9\pi^2}{8} + \frac{\pi^2}{8} + 0 \right] \]

\[ = \frac{-40B_w^2 2^2 2}{4_k^4} \text{ ... (8.6.2)} \]

Similarly the second integral in equation (8.6.1) comes to

\[ \iiint_{z=1} H_o^2 ds = \frac{-4\pi^2 2^2 2}{4_k^4} \text{ ... (8.6.3)} \]

The third integral in equation (8.6.1) is

\[ \iiint_{x=0} H_o^2 ds = \frac{-16B_w^2 2^2 2}{4_k^4} \left[ \int_0^a dy \left( \frac{2\pi^2}{2^2} \sin \frac{\pi y}{a} + \frac{\pi^2}{2^2} \sin \frac{3\pi y}{a} \right) \right] \times \int_0^1 dz \]

after solving the integrals we have

\[ \iiint_{x=0} H_o^2 ds = \frac{-16B_w^2 2^2 2}{4_k^4} \left[ \frac{9\pi^2}{2a} + \frac{\pi^2}{2a} + 0 \right] \]

\[ = \frac{-82^2 2^2 2}{4_k^4} B \text{ ... (8.6.4)} \]
Now the forth integral in equation (8.6.1) is

\[ \iint_{y=0} H^2 \, ds = \frac{-16b^2 c^2}{a^4} \left[ \int \frac{dx}{a} \int \frac{dz}{a} \left( \frac{\pi^2}{a^2} \sin^2 \frac{3\pi x}{a} + \frac{9\pi^2}{a^2} \sin^2 \frac{\pi y}{a} - \frac{6\pi^2}{a^2} \sin \frac{3\pi x}{a} \sin \frac{\pi y}{a} \right) \right] \]

which comes to

\[ \iint_{y=0} H^2 \, ds = \frac{-16b^2 c^2}{a^4} \left[ \frac{\pi^2}{2a^2} + \frac{9\pi^2}{a^2} + 0 \right] \]

\[ = \frac{-80\pi^2 b^2 c^2}{a^4} \]

... (8.6.5)

The last integral in equation (8.6.1) is

\[ \iint_{x+y=a} H^2 \, ds = \frac{-16b^2 c^2}{a^4} \left[ \int \frac{dx}{a} \int \frac{dz}{a} \left( \frac{\pi^2}{a^2} \sin^2 \frac{2\pi x}{a} \cos \frac{\pi x}{a} \right) + \frac{9\pi^2}{a^2} \right] \]

\[ \sin \frac{2\pi x}{a} \sin \frac{3\pi (a-x)}{a} - \frac{6\pi^2}{a^2} \sin \frac{3\pi x}{a} \sin \frac{\pi x}{a} \]

\[ \cos \frac{\pi (a-x)}{a} \cos \frac{3\pi (a-x)}{a} + \frac{9\pi^2}{a^2} \]

\[ + \frac{\pi^2}{a^2} \cos^2 \frac{2\pi x}{a} \sin^2 \frac{2\pi (a-x)}{a} \]

\[ - \frac{6\pi^2}{a^2} \frac{3\pi x}{a} \frac{\pi (a-x)}{a} \frac{\pi x}{a} \frac{3\pi (a-x)}{a} \]

\[ - \frac{6\pi^2}{a^2} \cos \frac{\pi x}{a} \cos \frac{\pi (a-x)}{a} \cos \frac{\pi x}{a} \cos \frac{\pi (a-x)}{a} \]
Which after solving the integrals comes to

$$\iint_{x+y=a \text{ plane}} H_0^2 ds = \frac{-8\pi^2 B^2 w^2 e^2 l}{a^4 k^4} \quad \ldots \quad (8.6.6)$$

By substituting the values from equations (8.6.2), (8.6.3), (8.6.4), (8.6.5) and (8.6.6) in equation (8.6.1) we have

$$\iint s H_0^2 ds = \frac{-8\sigma B^2}{a^4} \frac{\pi^2 w^2 e^2}{k^4} (1 + \frac{3l}{a}) \quad \ldots \quad (8.6.7)$$

By using this value in equation (8.5.1) for the $Q$ factor of right triangular cavity in $T.M_{210}$ mode we have

$$Q_{\text{tri}} = \frac{-D}{d \mu_0 \iint s H_0^2 ds} = \frac{\frac{\varepsilon a^2 l}{\sigma^2}}{16 B^2} \frac{1}{a^4} \frac{1}{\mu^4} \frac{8\sigma B^2}{a^4} \frac{\pi^2 w^2 e^2}{k^4} (1 + \frac{3l}{a})$$

Where $D = \frac{8\sigma B^2 e^2 l}{a}$

and $d = \sqrt{2/\mu_0 w^2}$
After simplification we have
\[ Q_{\text{tri}} = \sqrt{\frac{w \mu_0 \sigma}{2}} \frac{V_0}{a^2} \left(1 + \frac{3}{a^2} \right) \]

Where \( V_0 = \frac{1}{2} a^2 \) the volume of cavity. If \( a = 1 \) then we have
\[ Q_{\text{triangle}} = 2\sqrt{\frac{\mu_0 \sigma}{2}} \frac{V_0}{a^2} \cdot \frac{(\text{tri})}{4} \quad \ldots \quad (8.6.8) \]

Now the \( Q \) factor of a rectangular cavity in \( T.M_{210} \) mode is given by (Waldron pp. 315 equation 67).

\[ Q_{\text{rect.}} = \sqrt{\frac{w \mu_0 \sigma}{2}} V_0 \text{rect.} \left[ \frac{5\pi^2 / a^2}{4\pi (a^2 + 2a^2 l) + \frac{\pi^2}{a^2} (a^2 + 2s^2 l)} \right] \]

Where \( a' \) is the side of the square rectangle. \( V_0 \) being the volume of the cavity. The above equation can be simplified as
\[ Q_{\text{rect}} = \sqrt{\frac{w \mu_0 \sigma}{2}} \frac{V_0 \text{rect.}}{a' (a' + 2l)} \]

when \( a' = 1 \). We have
\[ Q_{\text{rect}} = \sqrt{\frac{\mu_0 V_0}{2}} \frac{V_{o,\text{rect}}}{a'} \]  
\[ (3.6.9) \]

Now if the frequency of \( T.M_{210} \) mode of triangular and rectangular cavity is same, then

\[ w_{\text{tri}}^2 = w_{\text{rect}}^2 \]

hence

\[ \frac{10n^2/a^2}{\mu_0 \epsilon_0} = \frac{5n^2/a^2}{\mu_0 \epsilon_0} \]

which gives

\[ 2a' = a^2 \]

or

\[ a = \sqrt{2} \ a' \]

Now we have

\[ V_{o,\text{tri}} = \frac{a'^2}{2} \text{ and } V_{o,\text{rect}} = a'^2 \]

Putting the value of \( a \) in terms of \( a' \) in \( V_{o,\text{tri}} \)

\[ V_{o,\text{tri}} = \frac{2a'^2}{2} = a'^2 \]
Therefore

\[ V_{\text{tri}} = V_{\text{rect}} \]  
\[ \text{(8.6.10)} \]

Substituting the values of \( V_{\text{tri}} \) and \( a' \) in terms of \( V_{\text{rect}} \) and \( a' \) in equation (8.6.8) we get

\[ Q_{\text{tri}} = \frac{2\sqrt{w} u_o \sigma}{\sqrt{2}} \cdot \frac{V_{\text{rect}}}{a' \sqrt{2} (a' \sqrt{2} + 3 \ell)} \]

\[ = \frac{V_{\text{rect}}}{a' (a' + 2 \ell)} \cdot \frac{2a' (a' 2 \ell)}{a' \sqrt{2} (a' \sqrt{2} + 3 \ell)} \]

Therefore

\[ Q_{\text{tri}} = Q_{\text{rect}} \left( \frac{a' + 2 \ell}{a' + 3 \sqrt{2}} \right) \]

\[ Q_{\text{tri}} = \frac{a'}{L} + 2 \]

\[ \frac{Q_{\text{tri}}}{Q_{\text{rect}}} = \frac{a'}{L} + \frac{3}{\sqrt{2}} \]

\[ \text{(8.6.11)} \]
The values of \( \frac{Q_{\text{tri}}}{Q_{\text{rect}}} \) for different values of \( \frac{a'}{I} \) are calculated and they are listed in Table No.2.

<table>
<thead>
<tr>
<th>( \frac{a'}{I} )</th>
<th>( \frac{Q_{\text{tri}}}{Q_{\text{rect}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) 0</td>
<td>0.9428</td>
</tr>
<tr>
<td>2) 0.5</td>
<td>0.9541</td>
</tr>
<tr>
<td>3) 1.0</td>
<td>0.9612</td>
</tr>
<tr>
<td>4) 1.5</td>
<td>0.9668</td>
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<td>5) 2.0</td>
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</tr>
<tr>
<td>6) 2.5</td>
<td>0.9740</td>
</tr>
<tr>
<td>7) 3.0</td>
<td>0.9763</td>
</tr>
<tr>
<td>8) 3.5</td>
<td>0.9784</td>
</tr>
<tr>
<td>9) 4.0</td>
<td>0.9802</td>
</tr>
</tbody>
</table>

These variations are plotted in Fig. 2.
Variation of $\frac{Q_{\text{tri}}}{Q_{\text{rect}}}$

with $\frac{l'}{a}$

(fig 2.)
8.7 **Conclusions:**

1) If the 'a' sides of the cavities are same, then the change in frequency &w for right prismatic cavity with T.M\(_{210}\) mode and sample at (a/4, a/4, 0) is \(\sqrt{2}\) times that for rectangular cavity with T.M\(_{210}\) mode and sample at (\(\frac{a}{2}\), \(\frac{a}{2}\), 0).

2) From equation (8.64) we arrive to the conclusion that for same frequency in the T.M\(_{210}\) mode

\[
\frac{Q_{\text{triangular}}}{Q_{\text{rectangular}}} < 1
\]

i.e. \(Q_{\text{triangular}} < Q_{\text{rectangular}}\).

If we allow the variation of \(\frac{a}{l}\) for both the cavities then

\(Q_{\text{triangular}}\) varies from 0.9428 to 1. This indicates that \(Q\) values

are almost equal. The slight variation is as indicated in Fig. 2.
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