Chapter - 4

WEAKLY $b$-CONTINUOUS AND FAINTLY $b$-CONTINUOUS FUNCTIONS
ABSTRACT

In 1961, Levine [10] introduced weakly continuous functions and in 1987, Noiri [18] introduced and studied weakly $\alpha$-continuous functions. Later on Ekici [7], in 2008, introduced and studied BR-continuous and hence weakly BR-continuous functions in a similar fashion, by means of $b$-regular and $b$-open [4] sets. This prompted us to introduce and study weakly $b$-continuous, neatly weak $b$-continuous and faintly $b$-continuous functions in present chapter by making use of $b$-open sets. We studied several characterizations of weakly $b$-continuous functions and faintly $b$-continuous functions. Some basic properties including restrictions and compositions of such functions have also been studied. The next section is devoted to other existing related functions. Some relationships among them, along with a few counter examples supporting only one way relationship, are investigated. Implication diagram of such functions is also depicted. The $b$-convergence and $\Theta$-convergence of nets and filters have been introduced to characterize faintly $b$-continuous functions. In the last, the graph functions, separation and covering properties are also studied in reference of weakly $b$-continuous and faintly $b$-continuous functions.
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Weakly b-continuous and faintly b-continuous functions

INTRODUCTION

The concept of continuous functions play an important role in Mathematics. Several different forms of generalizations of such functions have been introduced and studied so far. Levine [10] introduced the concept of a weakly continuous function. In 2008 Ekici [7] has introduced and studied the class of functions namely BR-continuous functions and weakly BR-continuous functions by making use of b-regular sets. He obtained some characterizations of weakly BR-continuous functions and established relationships among such functions and several other existing functions. In a similar manner here our purpose is to introduce and study generalizations in form of new classes of functions namely weakly b-continuous and faintly b-continuous functions. The author [8] has already introduced and studied b-continuous functions.

Let \((X, \tau)\) and \((Y, \iota)\) (or \(X\) and \(Y\)) denote topological spaces. For a subset \(A\) of a space \(X\), the closure \(\text{cl}(A)\) and the interior of \(A\) are denoted by \(\text{cl}(A)\) and \(\text{int}(A)\) respectively. A subset \(A\) is said to be regular open (resp. regular closed) if \(A=\text{int} (\text{cl}(A))\) (resp. \(A=\text{cl} (\text{int}(A))\)). A subset \(A\) is said to be preopen [13] (resp. semi open [11], b-open [4], \(\alpha\)-open [15], semi preopen [3] or \(\beta\)-open [1]) if \(A\subseteq \text{int} (\text{cl}(A))\) (resp. \(A\subseteq \text{cl} (\text{int}(A))\), \(A\subseteq \text{int} (\text{cl}(A))\cup \text{cl} (\text{int}(A))\), \(A\subseteq \text{int} (\text{cl}(\text{int}(A)))\), \(A\subseteq \text{cl} (\text{int}(\text{cl}(A))))\). A subset \(G\) of \(X\) is called b-neighbourhood of \(x\in X\) if there exists a b-open set \(B\) containing \(x\) such that
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$B \subseteq G$.

A point $x \in X$ is said to be a $\theta$-cluster point of $A$ [24] if $A \cap \text{cl}(U) \neq \emptyset$ for every open set $U$ containing $x$. The set of all $\theta$-cluster points of $A$ is called $\theta$-closure of $A$ and is denoted by $\theta\text{-cl}(A)$. A subset $A$ is called -closed if $\theta\text{-cl}(A) = A$ [24]. The complement of a $\theta$-closed set is called $\theta$-open set. The complement of a $b$-open (resp. preopen, semi open, $\alpha$-open, semi preopen) set is called $b$-closed (resp. preclosed, semi closed, $\alpha$-closed, semi preclosed). The intersection of all $b$-closed (resp. preclosed, semi closed, $\alpha$-closed, semi preclosed) sets of $X$ containing $A$ is called $b$-closure (resp. preclosure, semi closure, $\alpha$-closure, semi preclosure) of $A$ and denoted by $b\text{-cl}(A)$ (resp. $p\text{-cl}(A)$, $s\text{-cl}(A)$, $\alpha\text{-cl}(A)$, $sp\text{-cl}(A)$). The union of all $b$-open (resp. preopen, semi open, -open, semi preopen) sets of $X$ contained in $A$ is called $b$-interior (resp. preinterior, semi interior, $\alpha$-interior, semi preinterior) of $A$ and denoted by $b\text{-int}(A)$ (resp. $p\text{-int}(A)$, $s\text{-int}(A)$, $\alpha\text{-int}(A)$, $sp\text{-int}(A)$). A subset $A$ is said to be $b$-regular [4] if it is $b$-open as well as $b$-closed. The family of all $b$-open (resp. $b$-regular) sets of $X$ is denoted by $\text{BO}(X)$ (resp. $\text{BR}(X)$). A point $x \in X$ is called $b$-$\theta$-cluster point [21] of a subset $A$ of $X$ if $b\text{-cl}(B) \cap A \neq \emptyset$ for every $b$-open set $B$ containing $x$. The set of all $b$-cluster points of $A$ is called $b$-$\theta$-closure of $A$ and is denoted by $b$-$\theta\text{-cl}(A)$. A subset $A$ of $X$ is said to be $b$-$\theta$-closed if $A = b\text{-}\theta\text{-cl}(A)$. The complement of a $b$-closed set is said to be $b$-$\theta$-open. A point $x \in X$ is called $b$-$\theta$-interior point of $A \subseteq X$ if there exists a $b$-regular set $U$ containing $x$ such that $U \subseteq A$ and is denoted by $x \in b\text{-}\theta\text{-int}(A)$. The graph of a function $f : X \to Y$, denoted by $G(f)$ is the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$, and graph function $g : X \to X \times Y$ of $f : X \to Y$ is defined by $g(x) = (x, f(x))$ for each $x \in X$. 

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2. DEFINITIONS AND CHARACTERIZATIONS

**Definition 2.1:** A function \( f : X \to Y \) is said to be **b-continuous** [8] (resp. strongly \( \theta\)-**b-continuous** [21]) if for each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists a b-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \) (resp. \( f(b-cl(U)) \subseteq V \)).

**Definition 2.2:** A function \( f : X \to Y \) is said to be **weakly continuous** [10] (resp. **weakly \( \alpha\)**-**continuous** [18]) if for each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists an open (resp. \( \alpha \)-open) set \( U \) containing \( x \) such that \( f(U) \subseteq cl(V) \).

**Definition 2.3:** A function \( f : X \to Y \) is said to be **weakly b-continuous** if for each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists a b-open set \( U \) containing \( x \) such that \( f(U) \subseteq cl(V) \).

**Theorem 2.4:** For a function \( f : X \to Y \), the following are equivalent:

(a) \( f \) is weakly b-continuous at \( x \in X \).

(b) for each neighbourhood \( V \) of \( f(x) \), there exists an b-open set \( U \) containing \( x \) (or b-neighbourhood \( U \) of \( x \)) such that \( f(U) \subseteq cl(V) \).

(c) \( b-cl(f^{-1}(int(cl(V)))) \subseteq f^{-1}(cl(V)) \) for every subset \( V \) of \( Y \).

(d) \( b-cl(f^{-1}(int(F))) \subseteq f^{-1}(F) \) for every regular closed subset \( F \) of \( Y \).

(e) \( b-cl(f^{-1}(V)) \subseteq f^{-1}(cl(V)) \) for every open set \( V \) of \( Y \).

(f) \( f^{-1}(V) \subseteq b-int(f^{-1}(cl(V))) \) for every open set \( V \) of \( Y \).

(g) \( b-cl(f^{-1}(V)) \subseteq f^{-1}(cl(V)) \) for each preopen set \( V \) of \( Y \).

(h) \( f^{-1}(V) \subseteq b-int(f^{-1}(cl(V))) \) for each preopen set \( V \) of \( Y \).

**Proof:** (a) (b) obvious by definition.

(a)\( \Rightarrow \) (c) Let \( V \subseteq Y \) and \( x \in X \)–\( f^{-1}(cl(V)) \). Then \( f(x) \not\subseteq Y - cl(V) \) and there exists
an open set $U$ containing $f(x)$, such that $U \cap V = \phi$. We have $\text{cl}(U) \cap \text{int}(\text{cl}(V)) = \phi$. Since $f$ is weakly b-continuous, so, there exists a b-open set $W$ containing $x$ such that $f(W) \subset \text{cl}(U)$. Then $W \cap f^{-1}(\text{int}(\text{cl}(V))) = \phi$ and $x \in X - \text{b-cl}(f^{-1}(\text{int}(\text{cl}(V))))$. Hence $\text{b-cl}(f^{-1}(\text{int}(\text{cl}(V)))) \subset f^{-1}(\text{cl}(V))$.

(c)$\Rightarrow$(d) Let $F$ be any regular closed set in $Y$. Then $\text{b-cl}(f^{-1}(\text{int}(F))) = \text{b-cl}(\text{f}^{-1}(\text{int}(\text{cl}(\text{int}(F))))) \subset \text{f}^{-1}(\text{int}(F)) = \text{f}^{-1}(F)$.

(d)$\Rightarrow$(e) Let $V$ be an open subset of $Y$. Since $\text{cl}(V)$ is regular closed in $Y$, then $\text{b-cl}(f^{-1}(V)) \subset \text{b-cl}(\text{int}(\text{cl}(V))) \subset \text{f}^{-1}(\text{cl}(V))$.

(e)$\Rightarrow$(f) Let $V$ be any open set in $Y$. Since $Y - \text{cl}(V)$ is open in $Y$, then $x \in X - \text{b-int}(f^{-1}(\text{cl}(V)))$. Take $W = \text{b-int}(f^{-1}(\text{cl}(V))) \subset \text{f}^{-1}(\text{cl}(V))$. Thus $f(W) \subset \text{cl}(V)$ and hence $f$ is weakly b-continuous at $x \in X$.

(a)$\Rightarrow$(g) Let $V$ be any preopen set in $Y$ and $x \in X - f^{-1}(\text{cl}(V))$. There exists an open set $G$ containing $f(x)$, such that $G \cap V = \phi$. We have, $\text{cl}(G \cap V) = \phi$. Since $V$ is preopen, then $V \cap \text{cl}(G) \subset \text{int}(\text{cl}(V)) \cap \text{cl}(G) \subset \text{cl}(\text{int}(\text{cl}(V)) \cap G) \subset \text{cl}(\text{int}(\text{cl}(V)) \cap G) \subset \text{cl}(V \cap G) = \phi$. Since $f$ is weakly b-continuous and $G$ is an open set containing $f(x)$, there exists a b-open set $W$ in $X$ containing $x$ such that $f(W) \subset \text{cl}(G)$. Then $f(W) \cap V = \phi$ and $W \cap f^{-1}(V) = \phi$. This implies that $x \in X - \text{b-cl}(f^{-1}(V))$ and thus $\text{b-cl}(f^{-1}(V)) \subset f^{-1}(\text{cl}(V))$.

(g)$\Rightarrow$(h) Let $V$ be any preopen set. Since $Y - \text{cl}(V)$ is open in $Y$, then $x \in X - \text{b-int}(f^{-1}(\text{cl}(V))) = \text{b-cl}(f^{-1}(Y - \text{cl}(V))) \subset f^{-1}(\text{cl}(Y - \text{cl}(V))) \subset X - f^{-1}(V)$. This shows that $f^{-1}(V) \subset \text{b-int}(f^{-1}(\text{cl}(V)))$.

(h)$\Rightarrow$(a) Let $x \in X$ and $V$ be any open set in $Y$ containing $f(x)$. We have $x \in f^{-1}(V) \subset \text{b-int}(f^{-1}(\text{cl}(V)))$. Take $W = \text{b-int}(f^{-1}(\text{cl}(V)))$. Then $f(W) \subset \text{cl}(V)$.
and hence $f$ is weakly $b$-continuous at $x$ in $X$.

**Theorem 2.5:-** For a function $f : X \rightarrow Y$ the following are equivalent:

(1) $f$ is weakly $b$-continuous at $x \in X$.

(2) $x \in b\text{-int}(f^{-1}(\text{cl}(U)))$ for each neighbourhood $U$ of $f(x)$.

**Proof:-** (1) $\Rightarrow$ (2) Let $U$ be any neighbourhood of $f(x)$. Then there exists a $b$-open $G$ containing $x$ such that $f(G) \subseteq \text{cl}(U)$. Since $G \subseteq f^{-1}(\text{cl}(U))$ and $G$ is $b$-open then $x \in G \subseteq b\text{-int} G \subseteq b\text{-int}(f^{-1}(\text{cl}(U)))$.

(2) $\Rightarrow$ (1) Let $x \in b\text{-int}(f^{-1}(\text{cl}(U)))$ for each neighbourhood $U$ of $f(x)$. Then $V = b\text{-int}(f^{-1}(\text{cl}(U)))$. This implies that $f(V) \subseteq \text{cl}(U)$ and $V$ is $b$-open. Hence $f$ is weakly $b$-continuous at $x \in X$.

**Theorem 2.6:-** If $f : X \rightarrow Y$ is a weakly $b$-continuous function and $Y$ is Hausdorff, then $f$ has $b$-closed point inverses.

**Proof:-** Let $y \in Y$ and $x \in X$ such that $f(x) \neq y$. Since $Y$ is Hausdorff, there exist disjoint open sets $G$ and $H$ such that $f(x) \notin G$ and $y \in H$. Also, $G \cap H = \emptyset$, implies $\text{cl}(G) \cap H = \emptyset$. We have $y \notin \text{cl}(G)$. Since $f$ is weakly $b$-continuous, so, there exists a $b$-open set $U$ containing $x$ such that $f(U) \subseteq \text{cl}(G)$. Assume that $U$ is not contained in $\{x \in X : f(x) = y\}$. If possible for some $u \in U$, $f(u) = y$, then $y = f(u) \in \text{cl}(G)$. This contradicts $\text{cl}(G) \cap H = \emptyset$. Hence $U \subseteq \{x \in X : f(x) \neq y\}$ and $U$ is $b$-open in $X$. Thus, set $\{x \in X : f(x) \neq y\}$ is $b$-open in $X$, equivalently, $f^{-1}\{y\} = \{x \in X : f(x) = y\}$ is $b$-closed in $X$.

**Theorem 2.7:-** For a function $f : X \rightarrow Y$, the following are equivalent:

(a) $f$ is weakly $b$-continuous.

(b) $f(b\text{-cl}(G)) \subseteq \theta\text{-cl}(f(G))$ for each subset $G$ of $X$.

(c) $b\text{-cl}(f^{-1}(H)) \subseteq f^{-1}(\theta\text{-cl}(H))$ for each subset $H$ of $Y$.

(d) $b\text{-cl}(f^{-1}(\text{int}(\theta\text{-cl}(H)))) \subseteq f^{-1}(b\text{-cl}(H))$ for every subset $H$ of $Y$. 
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Proof:- (a)⇒(b) Let $G \subseteq X$, $x \in b\text{-}\text{cl}(G)$ and $U$ be any open set in $Y$ containing $f(x)$. There exists a $b$-open $W$ containing $x$ such that $f(W) \subseteq \text{cl}(U)$. Since, $x \in b\text{-}\text{cl}(G)$, then $W \cap G = \emptyset$. This implies that $\emptyset \neq f(W) \cap f(G) \subseteq \text{cl}(U) \cap f(G)$ and $f(x) \in \theta\text{-}\text{cl}(f(G))$. Thus, $f(b\text{-}\text{cl}(G)) \subseteq \theta\text{-}\text{cl}(f(G))$.

(b)⇒(c) Let $H \subseteq Y$. Then $f(b\text{-}\text{cl}(f^{-1}(H))) \subseteq \theta\text{-}\text{cl}(H)$ and hence $b\text{-}\text{cl}(f^{-1}(H)) \subseteq f^{-1}(\theta\text{-}\text{cl}(H))$.

(c)⇒(d) Let $H \subseteq Y$. Since $\text{cl}(H)$ is closed in $Y$, then $b\text{-}\text{cl}(f^{-1}(\text{int}(\theta\text{-}\text{cl}(H)))) \subseteq f^{-1}(\text{int}(\text{cl}(\theta\text{-}\text{cl}(H)))) = f^{-1}(\text{cl}(\text{int}(\theta\text{-}\text{cl}(H)))) \subseteq f^{-1}(\theta\text{-}\text{cl}(H))$.

(d)⇒(a) Let $H$ be any open set of $Y$. We have $H \subseteq \text{int}(\text{cl}(H)) = \text{int}(\theta\text{-}\text{cl}(H))$. Thus, $b\text{-}\text{cl}(f^{-1}(H)) \subseteq b\text{-}\text{cl}(f^{-1}(\text{int}(\theta\text{-}\text{cl}(H)))) \subseteq f^{-1}(\theta\text{-}\text{cl}(H)) \subseteq f^{-1}(\text{cl}(H))$. This implies from Theorem 2.4(e) that $f$ is weakly $b$-continuous.

Theorem 2.8:- If $f^{-1}(\theta\text{-}\text{cl}(V))$ is $b$-closed in $X$ for every subset $V$ of $Y$, then $f$ is weakly $b$-continuous.

Proof:- Let $V \subseteq Y$. Since $f^{-1}(\theta\text{-}\text{cl}(V))$ is $b$-closed in $X$, then $b\text{-}\text{cl}(f^{-1}(V)) \subseteq b\text{-}\text{cl}(f^{-1}(\theta\text{-}\text{cl}(V))) = f^{-1}(\theta\text{-}\text{cl}(V))$. This implies from above Theorem 2.7 that $f$ is weakly $b$-continuous.

Theorem 2.9:- If $f : X \to Y$ is a function which is weakly $b$-continuous, then $f^{-1}(V)$ is $b$-closed in $X$ for every $\theta$-closed subset $V$ of $Y$.

Proof:- Follows directly from Theorem 2.7. Since $f$ is weakly $b$-continuous, so, $b\text{-}\text{cl}(f^{-1}(V)) \subseteq f^{-1}(\theta\text{-}\text{cl}(V)) = f^{-1}(V)$ for a $\theta$-closed set $V$ in $Y$. This implies that $b\text{-}\text{cl}(f^{-1}(V)) = f^{-1}(V)$. Thus, $f^{-1}(V)$ is $b$-closed if $V$ is $\theta$-closed.

Corollary 2.10:- Let $f : X \to Y$ be a weakly $b$-continuous function, then
$f^{-1}(V)$ is $b$-open in $X$ for every $\theta$-open subset $V$ of $Y$.

**Theorem 2.11:** Let $f : X \to Y$ be a function. If $Y$ is regular then following are equivalent:
(a) $f$ is weakly $b$-continuous.
(b) $f$ is $b$-continuous.
(c) $f$ is strongly $\theta$-$b$-continuous if and only if $f$ is continuous [21].

**Proof:** Let $x \in X$ and $V$ be an open set of $Y$ containing $f(x)$. Since $Y$ is regular, then there exists an open set $H$ of $Y$ containing $f(x)$ such that $H \subseteq \text{cl}(H) \subseteq V$. Since $f$ is weakly $b$-continuous, there exists a $b$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq \text{cl}(H) \subseteq V$. Thus $f$ is $b$-continuous. Converse is obvious.

**Lemma 2.12 [4]:** The intersection of an $\alpha$-open set and a $b$-open set is a $b$-open set.

**Lemma 2.13 [14]:** If $A$ is $\alpha$-open in $X$, then $B_0(A) = B_0(X) \cap A$.

**Lemma 2.14 [2]:** If $A \subseteq B \subseteq X$, $B \in B_0(X)$ and $A \in B_0(B)$, then $A \in B_0(X)$.

**Theorem 2.15:** Let $\{A_i : i \in I\}$ be an $\alpha$-open cover of a space $X$ and $f : X \to Y$ be a function, then following are equivalent:
(a) $f$ is weakly $b$-continuous.
(b) the restriction $f/A_i : A_i \to Y$ is weakly $b$-continuous for each $i \in I$.

**Proof:** (a) $\Rightarrow$ (b) Let $i \in I$ and $A_i$ be an $\alpha$-open set in $X$. Let $x \in A_i$ and $V$ be an open set in $Y$ containing $f/A_i(x) = f(x)$. Since $f$ is weakly $b$-continuous, so, there exists a $b$-open set $G$ containing $x$ such that $f(G) \subseteq \text{cl}(V)$. Moreover $G \cap A_i$ is $b$-open in $A_i$ containing $x$ and $f/A_i(G \cap A_i) = f(G \cap A_i)$.
\( \subset \text{cl}(G) \subset \text{cl}(V) \). Hence \( f/A_t \) is weakly b-continuous.

(b)\( \Rightarrow \) (a) Let \( x \in X \) and \( V \) be an open set in \( Y \) containing \( f(x) \). There exists \( i \in I \), such that \( x \in A_i \). Since \( f/A_i : A_i \rightarrow Y \) is weakly b-continuous, there exists a b-open set \( G \) in \( A_i \) containing \( x \) such that \( f/A_i(G) \subset \text{cl}(V) \). Since each \( A_i \) is \( \alpha \)-open in \( X \) then \( G \) is b-open in \( X \) containing \( x \) and \( f(G) \subset \text{cl}(V) \). Hence \( f \) is weakly b-continuous.

### 3. SOME RELATED FUNCTIONS

In this section we study some more functions related to the weakly b-continuous functions:

**Definition 3.1**: A function \( f : X \rightarrow Y \) is said to be

(1) **(b, s)-open** if it maps b-open sets onto semi open sets.

(2) **neatly weak b-continuous** if for each \( x \) in \( X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists a b-open set \( U \) containing \( x \) such that \( \text{int}(f(U)) \subset \text{cl}(V) \).

**Theorem 3.2**: If a function \( f : X \rightarrow Y \) is neatly weak b-continuous and (b, s)-open then \( f \) is weakly b-continuous.

**Proof**: Let \( x \) be in \( X \) and \( V \) be any open set in \( Y \) containing \( f(x) \). Since \( f \) is neatly weak b-continuous, there exists a b-open set \( U \) in \( X \) containing \( x \) such that \( \text{int}(f(U)) \subset \text{cl}(V) \). Since \( f \) is (b, s)-open, therefore \( f(U) \) is semi open in \( Y \). Hence \( f(U) = \text{cl}(\text{int}(f(U))) \subset \text{cl}(V) \). Thus \( f \) is weakly b-continuous.

We can similarly prove

**Corollary 3.3**: If \( f \) is neatly weak b-continuous carrying b-open sets onto
open sets, then also $f$ is weakly $b$-continuous.

**Definition 3.4:-** A function $f : X \to Y$ is said to be **faintly $b$-continuous** (resp. **faintly continuous** [12]) if for each $x$ in $X$ and each $\theta$-open set $V$ in $Y$ containing $f(x)$ there exists a b-open (resp. open) set $U$ containing $x$ such that $f(U) \subseteq V$.

**Theorem 3.5:-** Let $f : X \to Y$ be a function, then following are equivalent :
(a) $f$ is faintly $b$-continuous.
(b) $f^{-1}(V)$ is b-open in $X$ for each $\theta$-open set $V$ in $Y$.
(c) $f^{-1}(V)$ is b-closed in $X$ for each $\theta$-closed $V$ in $Y$.
(d) $f(b\text{-}cl(A)) \subseteq \theta\text{-}cl(f(A))$ for each subset $A$ of $X$.
(e) $b\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(\theta\text{-}cl(B))$ for each subset $B$ of $Y$.

**Proof:** Simple and hence omitted.

**Definition 3.6 [8]:-** A function $f : (X, \tau) \to (Y, \varnothing)$ is said to be **$b$-continuous** if and only if inverse image under $f$ of every open (closed) set is b-open (b-closed).

**Theorem 3.7:-** For a function $f : (X, \tau) \to (Y, \varnothing)$, the following are equivalent :
(a) $f$ is faintly $b$-continuous.
(b) $f : (X, \tau) \to (Y, \sigma_\varnothing)$ is b-continuous where $\sigma_\varnothing$ is the collection of $\theta$-open sets in space $(Y, \varnothing)$.
(c) $f : (X, \tau) \to (Y, \sigma)$ is b-continuous if $(Y, \sigma)$ is a regular space.

**Proof:** It is well known [24] that the collection $\sigma_\varnothing$ of all $\theta$-open sets in space $(Y, \sigma)$ is a topology on $Y$. So it is obvious from definitions. Moreover, $\sigma = \sigma_\varnothing$ if and only if $(Y, \varnothing)$ is a regular space.
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**Corollary 3.8:-** For a function $f : X \to (Y, \varnothing)$, the following are equivalent provided $(Y, \varnothing)$ is an almost regular space.

(a) $f$ is weakly $b$-continuous.

(b) $f$ is faintly $b$-continuous.

If ‘almost regular’ is replaced by ‘regular’ in above corollary then we may add even (c) $f$ is $b$-continuous and (d) $f$ is strongly $\theta$-$b$-continuous to the list of equivalences.

**Proof:-** (a) and (b) are obvious for (c) and (d) we have Theorem 3.10 [21] which states that if $Y$ is a regular space. Then $f : X \to Y$ is strongly $\theta$-$b$-continuous if and only if $f$ is $b$-continuous.

**Corollary 3.9:-** The restriction of faintly $b$-continuous function to an $\alpha$-set is faintly $b$-continuous.

**Proof:-** Let $f : X \to Y$ be faintly $b$-continuous. $A$ be $\alpha$-set in $X$. To show $f/A : A \to Y$ is faintly $b$-continuous. Let $V$ be $\theta$-open set in $Y$, then $f^{-1}(V)$ is $b$-open in $X$. Hence $(f/A)^{-1}(V) = f^{-1}(V) \cap A$ is $b$-open set in $A$, [Lemma 2.13]. Thus the restriction $f/A : A \to Y$ is faintly $b$-continuous.

The following results about composite of functions are easy to establish.

**Theorem 3.10:-** For $f : X \to Y$ and $g : Y \to Z$, the function $gof : X \to Z$ is

(a) $b$-continuous, whenever $f$ is $b$-continuous and $g$ is continuous.

(b) $b$-continuous whenever $f$ is faintly $b$-continuous and $g$ is strongly $\theta$-continuous.

(c) strongly $\emptyset$-$b$-continuous whenever $f$ is strongly $\theta$-$b$-continuous and $g$ is continuous.

**Theorem 3.11:-** If $f : X \to Y$ is weakly $b$-continuous and $g : Y \to Z$ is
continuous, then the composition \( g \circ f : X \rightarrow Z \) is weakly \( b \)-continuous.

**Proof:** Let \( x \in X \) and \( A \) be an open set of \( Z \) containing \( g(f(x)) \). We have \( g^{-1}(A) \) is an open set of \( Y \) containing \( f(x) \). Then there exists a \( b \)-open set \( B \) containing \( x \) such that \( f(B) \subset \text{cl}(g^{-1}(A)) \). Since \( g \) is continuous, so, \( (g \circ f)(B) \subset g(\text{cl}(g^{-1}(A))) \subset \text{cl}(A) \). Thus \( g \circ f \) is weakly \( b \)-continuous.

**Definition 3.12:** A space \( X \) is said to be \textbf{\( b \)-connected} [6] if it cannot be written as the union of two non-empty disjoint \( b \)-open sets.

**Theorem 3.13:** If \( f : X \rightarrow Y \) is weakly \( b \)-continuous surjection and \( X \) is \( b \)-connected, then \( Y \) is connected.

**Proof:** Suppose that \( Y \) is not connected. There exist non-empty open sets \( U \) and \( V \) of \( Y \) such that \( Y = U \cup V \) and \( U \cap V = \emptyset \). Then \( U \) and \( V \) are clopen in \( Y \). By Theorem 2.4(f), \( f^{-1}(V) \subset \text{b-int}(f^{-1}(\text{cl}(V))) \) for every open set \( V \) in \( Y \). Thus, \( f^{-1}(V) \subset \text{b-int}(f^{-1}(V)) \). This implies \( f^{-1}(V) \) is \( b \)-open in \( X \) and similarly \( f^{-1}(U) \) is \( b \)-open in \( X \). We have \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \), \( X = f^{-1}(U) \cup f^{-1}(V) \). Moreover \( f^{-1}(U) \) and \( f^{-1}(V) \) are non-empty. Hence \( X \) is not \( b \)-connected, a contradiction. Thus, \( Y \) must be connected.

**Lemma 3.14[14]:** Let \( A \) be a subset of \( X \) and \( B \) be a subset of \( Y \), then \( A \times B \in \text{BO}(X \times Y) \) provided \( A \in \text{BO}(X) \) and \( B \in \text{BO}(Y) \).

**Theorem 3.15:** Let \( \{X_i : i \in I\} \) and \( \{Y_i : i \in I\} \) be any two families of topological spaces. If \( f_i : X_i \rightarrow Y_i \) is a weakly \( b \)-continuous function for each \( i \in I \), then the function \( f : \prod X \rightarrow \prod Y \) defined by \( f((x_i)) = (f_i(x_i)) \) for each \( (x_i) \in \prod X \) is weakly \( b \)-continuous.

**Proof:** Let \( x = (x_i) \in \prod X \) and \( V \) be an open set containing \( f(x) \). There
exists an open set \( \prod U_i \) such that \( f(x) \in \prod U_i \times \prod Y_j \subset V \) where \( U_i \) is open in \( Y_i \). Since each \( f_i \) is weakly \( b \)-continuous there exists \( b \)-open set \( G_i \) in \( X_i \) containing \( x_i \) such that \( f_i(G_i) \subset \text{cl}(U_i) \) for each \( i = 1, 2, \ldots, n \). Take \( G = \prod_{j=1}^{i=n} G_j \times \prod_{j=1}^{i=n} Y_j \). Then \( G \) is \( b \)-open in \( \prod X_i \) containing \( x \) and \( f(G \subset \prod_{j=1}^{i=n} f_i(G_i) \times \prod_{j=1}^{i=n} Y_j \subset \prod_{j=1}^{i=n} \text{cl}(U_i) \times \prod_{j=1}^{i=n} Y_j \subset \text{cl} V \). Hence \( f \) is weakly \( b \)-continuous.

We recall that a space \( X \) is said to be submaximal [23] if each dense subset of \( X \) is open in \( X \). It is further shown [23] that a space is submaximal if and only if every preopen subset of \( X \) is open. A space \( X \) is said to be extremely disconnected [5] if the closure of each open set of \( X \) is open. We note [21] that an externally disconnected space is exactly the space where every semi open set is \( \alpha \)-open.

**Definition 3.16:-** A function \( f : X \to Y \) is said to be strongly \( \theta \)-continuous [17] (resp. strongly \( \theta \)-semi continuous [9], strongly \( \theta \)-precontinuous [19], strongly \( \theta \)-\( \beta \)-continuous [20]) if for each \( x \) in \( X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists an open (resp. semi open, preopen, semi preopen) set \( U \) of \( X \) containing \( x \) such that \( f(\text{cl}(U)) \subset V \) (resp. \( f(\text{s-cl}(U)) \subset V \), \( f(\text{p-cl}(U)) \subset V \), \( f(\text{sp-cl}(U)) \subset V \)).

**Theorem 3.17:-** The following are equivalent for a function \( f : X \to Y \) where \( X \) is submaximal externally disconnected space and \( Y \) is regular space.

(a) \( f \) is weakly \( b \)-continuous.
(b) \( f \) is faintly \( b \)-continuous.
(c) \( f \) is \( b \)-continuous.
(d) $f$ is strongly $\theta$-$b$-continuous.
(e) $f$ is strongly $\theta$-continuous.
(f) $f$ is strongly $\theta$-semi continuous.
(g) $f$ is strongly $\theta$-precontinuous.
(h) $f$ is strongly $\theta$-semi precontinuous or strongly $\theta$-$\beta$-continuous.

**Proof:** The proof is obvious using the fact that if $X$ is submaximal extremally disconnected space, then open set, preopen set, semi open set, $b$-open set and semi preopen (or $\beta$-open) sets are equivalent concepts [21].

### 4. $\theta$-CONVERGENCE AND $b$-CONVERGENCE OF NETS AND FILTERS

The notion of $\theta$-convergence and $b$-convergence of nets and filters, found useful here to characterize faintly $b$-continuous functions, is a generalization of the convergence of nets and filters which is obviously a fundamental concept in analysis and topology.

**Definition 4.1:** A net $\{x_a : a \in A\}$ in a topological space $(X, \tau)$ is said to be $\theta$-convergent [24] (resp. $b$-convergent) at $x$ in $X$ if the net is eventually in each $\theta$-open (resp. $b$-open) set containing $x$. In other words, if for each $\theta$-open (resp. $b$-open) set $G$ containing $x$ there exists an element $a_G$ in $A$ such that $a_G \in A$ and $a \geq a_G \implies x_a \in G$.

Following theorem is easy to prove:

**Theorem 4.2:** Let $A \subseteq X$. If $x$ in $X$, then $x \in \theta$-$\text{cl}(A)$ (resp. $x \in b$-$\text{cl}(A)$) if and only if there exists a net in $A$, $\theta$-converging (resp. $b$-converging) to $x$. 
**Definition 4.3:** A filter base $\beta = \{B_\alpha\}$ on a space $X$ is said to be $\theta$-convergent [24] (resp. $b$-convergent) to a point $x$ in $X$ if for each $\theta$-open (resp. $b$-open) set $U$ containing $x$, there exists a $B_\alpha$ in $\beta$ such that $B_\alpha \subset U$.

**Theorem 4.4:** If $A \subset X$ and $x \in X$, then $x \in \theta$-cl$(A)$ (resp. $x \in b$-cl$(A)$) if and only if there exists a filter base on $A$ $\theta$-converging (resp. $b$-converging) to $x$.

**Theorem 4.5:** For a function $f : X \to Y$, the following are equivalent:

(a) $f$ is faintly $b$-continuous.

(b) for each $x \in X$ and each net $\{x_\alpha : \alpha \in A\}$ $b$-converging to $x$, the net $\{f(x_\alpha : \alpha \in A\}$ $\theta$-converges to $f(x)$ in $Y$.

(c) for each $x \in X$ and each filter base $\beta = \{B_\alpha\}$ $b$-converging to $x$, $f(\beta)$, $\theta$-converges to $f(x)$ in $Y$.

**Proof:** (a) $\Rightarrow$ (b) Let $\{x_\alpha : \alpha \in A\}$ be a net $b$-converging to $x \in X$ and $V$ be any $\theta$-open set containing $f(x)$, then $x \in f^{-1}(V) \in bO(X)$, the family of $b$-open sets in $X$. Since the net $\{x_\alpha : \alpha \in A\}$ is eventually in $f^{-1}(V)$, so, the net $\{f(x_\alpha) : \alpha \in A\}$ is eventually in $V$. Thus the net $\{f(x_\alpha) : \alpha \in A\}$ $\theta$-converges to $f(x)$ in $Y$.

(b) $\Rightarrow$ (a) Let $A \subset X$ and $x \in X$ such that $x \in b$-cl$(A)$, then there exists a net $\{x_\alpha : \alpha \in A\}$ in $A$ $b$-converging to $x$ (Theorem 4.2). By hypothesis the net $\{f(x_\alpha) : \alpha \in A\}$ in $f(A)$ $\theta$-converges to $f(x)$. So $f(x) \in \theta$-cl$(f(A))$ (Theorem 4.2). Thus, $f(b$-cl$(A)) \subset \theta$-cl$(f(A))$ for each $A \subset X$. So $f$ is faintly $b$-continuous (Theorem 3.5(d)).

(a) $\Rightarrow$ (c) Let $V$ be $\theta$-open in $Y$. Then $f^{-1}(V)$ is $b$-open, so, there exists $B_\alpha \in \beta$ such that $B_\alpha \subset f^{-1}(V)$ or $f(B_\alpha) \subset V$.

(c) $\Rightarrow$ (a) Let $A \subset X$ and $x \in b$-cl$(A)$. Then there exists a filter base $\beta = \{B_\alpha\}$ on $A$ $b$-converging to $x$ (Theorem 4.4). By hypothesis, the filter base $f(\beta)$
on \( f(A) \), \( \theta \)-converges to \( f(x) \), so, \( f(x) \in \theta \text{-cl}(f(A)) \) (Theorem 4.4). Thus \( f(b \text{-cl}(A)) \subseteq \theta \text{-cl}(f(A)) \) for each \( A \subseteq X \). Hence \( f \) is faintly \( b \)-continuous (by Theorem 3.5(d)).

**5. COMPARISONS:**

Obviously a \( b \)-continuous function is weakly \( b \)-continuous but the converse is not true as following.

**Example 5.1:** Let \( X = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\} \) and \( Y = X \), \( \sigma = \{\phi, Y, \{c\} \}. \) Then identity function \( f : (X, \tau) \to (Y, \sigma) \) is weakly \( b \)-continuous at each \( x \in X \) but not \( b \)-continuous at \( x = c \).

Moreover, an open set is \( b \)-open as well as \( \alpha \)-open. Therefore, a weakly continuous function is weakly \( b \)-continuous as well as weakly \( \alpha \)-continuous but the converses are not true as following.

**Example 5.2:** Let \( X = \{a, b, c\} = Y \), \( \tau = \{\phi, X, \{a\} \) and \( \sigma = \{\phi, Y, \{b\}, \{c\}, \{b, c\} \). Then identity function \( i : (X, \tau) \to (Y, \tau) \) is weakly \( b \)-continuous as well as weakly \( \alpha \)-continuous since only \( b \)-open sets in \( (X, \tau) \) are, which are precisely \( \alpha \)-open but \( f \) is not weakly continuous at \( x = b \) nor at \( x = c \).

It is evident that a weakly \( b \)-continuous function is faintly \( b \)-continuous, which can be proved as following.

**Theorem 5.3:** If \( f : X \to Y \) is weakly \( b \)-continuous function then \( f \) is faintly \( b \)-continuous.

**Proof:** Let \( x \in X \) and \( V \) be a \( \theta \)-open set in \( Y \) containing \( f(x) \). Then there exists an open set \( W \) such that \( f(x) \in W \cap \text{cl}(W) \supseteq V \). Now, since \( f \) is weakly \( b \)-continuous so, there exists a \( b \)-open set \( U \) containing \( x \) such that
f(U) ⊆ cl(W) ⊆ V. Consequently, f is faintly b-continuous.

The converse of above Theorem is not true in general, as following.

**Example 5.4:** Let X = Y = {a, b, c}, τ an indiscrete topology on X, and σ = {ϕ, Y, {a}, {b}, {a, b}}. Then f : (X, τ) → (Y, σ) defined as identity function is faintly b-continuous but not weakly b-continuous at x = a as {a} is an open set containing f(x) at x = a but there is not b-open set A containing x such that f(A) ⊆ cl({a}) = {a, c}. Although the function is faintly b-continuous, since the only non-empty θ-open set in Y is Y itself.

Moreover, a faintly continuous function is faintly b-continuous because an open set is b-open but the converse is not true as following:

**Example 5.5:** Let X = Y = {a, b, c, d} and τ = {ϕ, X, {a}, {b}, {a, b, c}}. Then identity function from (X, τ) to (Y, ), where σ = {ϕ, Y, {a}, {b, c, d}} is faintly b-continuous as well as b-continuous but not faintly continuous as {b, c, d} is θ-open in (Y, σ) but not open in (X, τ), although b-open in (X, τ).

Also a b-continuous function is faintly b-continuous, since every θ-open set is open but the converse is not true as following:

**Example 5.6:** Let X = Y = {a, b, c}, τ = {ϕ, X, {b}, {a, c}} and σ = {ϕ, Y, {a}, {b}, {a, b}, {a, c}}. Then the identity function f : (X, τ) → (Y, σ) is faintly b-continuous but not b-continuous as {b} and {a, c} are θ-open in (Y, σ) also are open and hence b-open in (X, τ) but {a} is open in (Y, σ) but not b-open in (X, τ). This also proves that even faintly continuous function may fail to be b-continuous.
Thus, we have following implication diagram:

\[
\begin{array}{c}
\text{Weakly continuous} \iff \text{Weakly } \alpha\text{-continuous} \\
\downarrow & \downarrow \\
\text{b-continuous} & \iff \text{Weakly b-continuous} \\
\uparrow & \uparrow \\
\text{Faintly b-continuous} & \iff \text{Faintly continuous}
\end{array}
\]

Here \(\Rightarrow\) indicates ‘implies but not implied by’ and \(\iff\) indicates does not imply. Counter examples have been given on appropriate place.

### 6. SOME SEPARATION AND COVERING PROPERTIES

Let us recall,

A space is said to be \(\mathfrak{b}\)-\(T_0\) (resp. \(\mathfrak{g}\)-\(T_0\)) if for every pair of distinct points there exists a b-open (resp. \(\mathfrak{g}\)-open) set containing one of them but not the other. A space is said to be \(\mathfrak{b}\)-\(T_1\) (resp. \(\mathfrak{g}\)-\(T_1\)) if for every pair of distinct points, there exist two b-open (resp. \(\mathfrak{g}\)-open) sets each containing one of them but not the other. A space is said to be \(\mathfrak{b}\)-\(T_2\) (resp. \(\mathfrak{g}\)-\(T_2\)) if for every pair of distinct points, there exist two disjoint b-open (resp. \(\mathfrak{g}\)-open) sets each containing one of them. Equivalently, b-open (resp. \(\mathfrak{g}\)-open) sets separate every pair of distinct points.

**Theorem 6.1:** If \(f : X \to Y\) is faintly b-continuous injection and \(Y\) is \(\mathfrak{g}\)-\(T_0\), then \(X\) is \(\mathfrak{b}\)-\(T_0\).
Proof:- If \( x \) and \( y \) are distinct points of \( X \), then \( f(x) \) and \( f(y) \) are distinct in

\(-T_0\) space \( Y \), so, there exists a \( \Theta \)-open set \( V \) containing one of them say
\( f(x) \) but not \( f(y) \). Due to \( f \), there exists a \( b \)-open set \( U \) containing \( x \in X \) such
that \( f(U) \subseteq V \) and \( f(y) \in Y \setminus V \), indicates \( y \) is not in \( U \). Thus \( X \) is \( b\)-\( T_0 \).

Corollary 6.2:- If \( f : X \rightarrow Y \) is faintly \( b \)-continuous injection and \( Y \) is \( \Theta\)-\( T_0 \)
(resp. \( -T_1, \Theta\)-\( T_2 \)), then \( X \) is \( b\)-\( T_0 \) (resp. \( b\)-\( T_1, b\)-\( T_2 \)) space.

Theorem 6.3:- If for each pair of distinct points \( x_1 \) and \( x_2 \) in a space \( X \),
there exists a function \( f \) from \( X \) into a Urysohn space \( (Y, \sigma) \), with
\( f(x_1) \neq f(x_2) \) and \( f \) is weakly \( b \)-continuous at \( x_1 \) and \( x_2 \), then \( X \) is \( b\)-\( T_2 \).

Proof:- Let \( x_1 \) and \( x_2 \) be any pair of distinct points in \( X \). Then there exists a
function \( f : X \rightarrow Y \) such that \( Y \) is Urysohn, \( f(x_1) \neq f(x_2) \) and \( f \) is weakly
\( b \)-continuous at \( x_1 \) and \( x_2 \). Let \( y_i = f(x_i) \) for \( i = 1, 2 \). We have \( y_1 \neq y_2 \). Since \( Y \) is
Urysohn, therefore, there exist open sets \( V_1 \) and \( V_2 \) containing \( y_1 \) and \( y_2 \)
respectively such that \( \text{cl}(V_1) \cap \text{cl}(V_2) = \emptyset \). Since \( f \) is weakly \( b \)-continuous at
\( x_1 \) and \( x_2 \), so, there exist \( b \)-open sets \( U_i \) for \( i = 1, 2 \) containing \( x_i \) such that
\( f(U_i) \subseteq \text{cl}(V_i) \). This gives \( U_1 \cap U_2 = \emptyset \) and hence \( X \) is \( b\)-\( T_2 \).

Recall that a function is faintly continuous [12] if and only if
inverse image of every \( \Theta \)-open sets is open. A set \( N \subseteq X \) is called
\( b \)-neighbourhood of \( x \in X \) if there is a \( b \)-open set \( B \) such that \( x \in B \subseteq N \).

Proposition 6.4[4]:-
(a) The union of any family of \( b \)-open sets is a \( b \)-open set.
(b) The intersection of an open and a \( b \)-open set is a \( b \)-open set.

Theorem 6.5:- A set \( A \subseteq X \) is \( b \)-open if and only if \( A \) is \( b \)-neighbourhood of
each of its points.
Proof:- For each $x \in A$, there is a $b$-open set $B$ such that $x \in B \subseteq A$. Thus $A$ is union of $b$-open sets and hence $b$-open by above 6.4(a). Converse is obvious.

**Theorem 6.6:-** If $f : X \rightarrow Y$ is faintly continuous and $g : X \rightarrow Y$ is faintly $b$-continuous and $Y$ is $\theta$-$T_2$ space, then set $A\{x \in X : f(x)=g(x)\}$ is $b$-closed in $X$.

**Proof:-** Let $x \in X$ such that $x \in X - A = E$, so $f(x) \neq g(x)$ in $\theta$-$T_2$ space $Y$. Therefore, there exist two disjoint $\theta$-open sets $G$ and $H$ containing $f(x)$ and $g(x)$ respectively. By nature of $f$ and $g$, $f^{-1}(G)$ is open and $g^{-1}(H)$ is $b$-open, so, $F=f^{-1}(G) \cap g^{-1}(H)$ is also $b$-open (by 6.4(b) above) set containing $x$ and is contained in $E = X - A$. Since $x$ is arbitrary so $E$ is $b$-open (by 6.4(a) and 6.5). Thus $A$ is $b$-closed.

**Corollary 6.7:-** If $X$ is a $\theta$-$T_2$ space and $f$ is faintly $b$-continuous function of $X$ onto itself. Then $A\{x \in X : f(x)=x\}$ is a $b$-closed set in $X$.

**Proof:-** Taking $g$ as identity function hence faintly continuous in above Theorem.

**Corollary 6.8:-** With reference to above Theorem 6.6, if $f$ and $g$ agree on a $b$-dense subset $A$ of $X$ ($A \subseteq X$ is $b$-dense in $X$ if $b$-$cl(A) = X$), then $f=g$.

**Theorem 6.9:-** Let $f$ and $g$ be weakly $b$-continuous functions from a space $X$ to a Urysohn space $Y$. If $BO(X)$, the collection of $b$-open sets in $X$ is closed under finite intersection, then the set $\{x \in X : f(x)=g(x)\}$ is $b$-closed in $X$.

**Proof:-** Obvious.

**Theorem 6.10:-** If $f : X \rightarrow Y$ is weakly $b$-continuous and $Y$ is Urysohn then
the set \( A = \{(x, y) \in X \times X : f(x) = f(y)\} \) is \( b \)-closed in \( X \times X \).

**Proof:** Let \((x, y) \notin A\), then \( f(x) \neq f(y) \). Since, \( Y \) is Urysohn, then there exist open sets \( V \) and \( W \) of \( Y \) containing \( f(x) \) and \( f(y) \) respectively such that \( \text{cl}(V) \cap \text{cl}(W) = \emptyset \). Since \( f \) is weakly \( b \)-continuous, there exists a \( b \)-open set \( U \) and \( G \) in \( X \) containing \( x \) and \( y \), respectively, such that \( f(U) \subseteq \text{cl}(V) \) and \( f(G) \subseteq \text{cl}(W) \). Take \( N = U \times G \). Thus \( N \) is \( b \)-open set containing \( (x, y) \) and \( N \cap A = \emptyset \). Hence \( A \) is \( b \)-closed.

**Definition 6.11:** A subset \( A \) of a space \( X \) is said to be **\( b \)-compact** (resp. **\( \theta \)-compact**) if every cover of \( A \) by means of \( b \)-open (resp. \( \theta \)-open) sets has a finite subcover. A space \( X \) is **\( b \)-compact** (resp. **\( \theta \)-compact**) if it is \( b \)-compact (resp. \( \theta \)-compact) as a subset of itself.

Obviously, every \( b \)-closed (resp. \( \emptyset \)-closed) subset of a \( b \)-compact (resp. \( \emptyset \)-compact) space is \( b \)-compact (resp. \( \emptyset \)-compact).

**Theorem 6.12:** Let \( f : X \to Y \) be faintly \( b \)-continuous function and \( A \subseteq X \) is \( b \)-compact, then \( f(A) \) is \( \theta \)-compact.

**Proof:** Let \( f : X \to Y \) be faintly \( b \)-continuous function and \( \{G_\lambda : \lambda \in \Delta\} \) be any cover of \( f(A) \subseteq Y \) by \( \theta \)-open sets. Then \( \{f^{-1}(G_\lambda) : \lambda \in \Delta\} \) turns out to be a cover of \( A \subseteq X \) by \( b \)-open sets and \( A \) is \( b \)-compact. So, there exists a finite subfamily \( \{f^{-1}(G_{\lambda_i}) : i=1,2,3,\ldots,n\} \) which cover \( A \). It follows that \( \{G_{\lambda_i} : i=1,2,3,\ldots,n\} \) is a finite subcover of the cover \( \{G_\lambda : \lambda \in \Delta\} \). Thus \( f(A) \) is \( \theta \)-compact.

**Corollary 6.13:** If \( f : X \to Y \) is \( b \)-continuous and \( K \subseteq X \) is \( b \)-compact, then \( f(K) \) is a compact set in \( Y \).

**Corollary 6.14:** Every faintly \( b \)-continuous (resp. \( b \)-continuous) image of
a b-compact space is \( \theta \)-compact (resp. compact).

**Theorem 6.15:** Every faintly b-continuous function from a b-compact space into a \( \theta \)-T\(_2\) space maps inversely \( \theta \)-compact sets onto b-compact sets.

**Proof:** Let \( f : X \rightarrow Y \) be faintly b-continuous function from a b-compact space \( X \) into a \( \theta \)-T\(_2\) space \( Y \). If \( A \) is \( \theta \)-compact set in \( Y \), then \( A \) must be \( \theta \)-closed in \( Y \). So, \( f^{-1}(A) \) is b-closed subset of a b-compact space \( X \) and hence is b-compact.

**Theorem 6.16:** If a function \( f : X \rightarrow Y \) is from a b-compact and b-T\(_2\) space \( X \) to a \( \theta \)-compact, \( \emptyset \)-T\(_2\) space \( Y \). Then \( f \) is faintly b-continuous if and only if \( f \) maps \( \theta \)-compact sets inversely onto b-compact sets.

**Proof:** Let \( A \) be \( \theta \)-closed set in \( Y \). Since \( Y \) is \( \emptyset \)-compact so, \( A \) is \( \theta \)-compact and hence \( f^{-1}(A) \) is b-compact subset of \( X \), which is b-T\(_2\), so, \( f^{-1}(A) \) is b-closed set in \( X \). Thus \( f \) is faintly b-continuous. Converse part follows from Theorem 6.15.

**Theorem 6.17:** If a function \( f : X \rightarrow Y \) has a b-closed graph then \( f(K) \) is closed in \( Y \) for each b-compact subset \( K \) of \( X \).

**Proof:** Let \( K \) be a b-compact subset of \( X \) and \( y \in Y - f(K) \). Then for each \( x \in K \) we have \( (x, y) \notin G(f) \) and hence b-closed graph of \( f \) ensures the existence of a b-open set \( U_x \) in \( X \) containing \( x \) and an open set \( V_y \) in \( Y \) containing \( y \) such that \( f(U_x) \cap V_y = \emptyset \). The family \( \{U_x : x \in K\} \) is a cover of \( K \) by b-open sets of \( X \) and hence there exists a finite subset \( K_0 \) of \( K \) such that \( K \subseteq \bigcup\{U_x : x \in K_0\} \). Put \( V = \bigcap\{V_x : x \in K_0\} \). Then \( V \) is an open set containing \( y \) and \( f(K) \cap V \subseteq \bigcup\{f(U_x) : x \in K_0\} \cap V \subseteq \bigcup\{f(U_x) \cap V_x\} = \emptyset \). Therefore, we have \( y \notin f(K) \) and hence \( f(K) \) is closed in \( Y \).
Theorem 6.18:- Let $X$ be a submaximal extremally disconnected space. If $f : X \to Y$ has a $b$-closed graph then $f^{-1}(K)$ is $\emptyset$-closed in $X$ for each compact set $K$ of $Y$.

Proof:- Let $K$ be a compact set of $Y$ and $x \not\in f^{-1}(K)$. Then for each $y \in K$, have $(x, y) \in G(f)$ and hence there exists a $b$-open set $U_y$ in $X$ containing $x$ and an open set $V_y$ of $Y$ containing $y$ such that $f(U_y) \cap V_y = \emptyset$. The family $\{V_y : y \in K\}$ is an open cover of $K$, which is compact in $Y$, so, there exists a finite subset $K_0$ of $K$ such that $K \subseteq \bigcup \{V_y : y \in K_0\}$. Since $X$ is submaximal extremally disconnected space, so, each $U_y$ is open in $X$ and closure of an open set is open, so, hence, $cl(U_y)$ is also an open set containing $x$ set $U = \bigcap \{U_y : y \in K_0\}$, then $U$ is an open set containing $x$ and $f(cl(U)) \cap K \subseteq \bigcup \{f(cl(U_y)) \cap V_y \subseteq \bigcup f(U_y) - V_y \cap V_y = \emptyset$. Therefore, we have $cl(U) \cap f^{-1}(K) = \emptyset$ and hence $x \notin -cl(f^{-1}(K))$. This shows that $f^{-1}(K)$ is $\emptyset$-closed in $X$.

Definition 6.19:- A space $X$ is said to be almost compact [16] or quasi $H$-closed [22] if every cover of $X$ by open sets has a finite subcover whose closures cover $X$.

Theorem 6.20:- Let $f : X \to Y$ be a weakly $b$-continuous surjection. If $X$ is $b$-compact, then $Y$ is almost compact or quasi $H$-closed.

Proof:- Let $\{V_i : i \in I\}$ be a cover of $Y$ by open sets then for each point $x \in X$, there exists $i_x \in I$ such that $f(x) \in V_{i_x}$. Since $f$ is weakly $b$-continuous, there exists a $b$-open set $U_x$ in $X$ containing $x$ such that $f(U_x) \subseteq cl(V_{i_x})$. The family $\{U_x : x \in X\}$ is a cover of $X$ by $b$-open sets, so, there exists a finite subset $X_0$ of $X$ such that $X = \bigcup \{U_x : x \in X_0\}$. Thus $Y = f(X) = \bigcup \{cl(V_{i_x}) : x \in X_0\}$ and hence $Y$ is almost compact.

Theorem 6.21:- Let $f : X \to Y$ be a function having a $b$-graph $G(f)$. Then
f(N) is $\emptyset$-closed in Y for each b-compact subset N relative to X.

**Proof:-** Let N be b-compact relative to X and $y \in Y - f(N)$. Then $(x, y) \notin G(f)$ for each $x \in N$. This implies that there exists a b-open set $A_x$ of X containing $x$ and an open set $U_x$ of Y containing $y$ such that $f(A_x) \cap \text{cl}(U_x) = \emptyset$. The family $\{A_x : x \in N\}$ is a cover of N by b-open sets of X. Since N is b-compact relative to X, there exists a finite subset $N^* \subseteq N$ such that $N \subseteq \bigcup_{x \in N^*} U_x$. Then $U \subseteq \bigcup_{x \in N^*} U_x$ in Y containing $y$ and $f(N) \cap \text{cl}(U) \subseteq \bigcup_{x \in N^*} \text{cl}(U_x) = \emptyset$.

Thus $y \notin \text{cl}(f(N))$ and hence $f(N)$ is $\emptyset$-closed in Y.

### 7. GRAPHS OF WEAKLY b-CONTINUOUS AND FAINTLY b-CONTINUOUS FUNCTIONS

In this section we study various types of graphs with weakly b-continuous and faintly b-continuous functions.

**Definition 7.1:-** A function $f : X \to Y$ is said to have a **b-graph** if for each $(x, y) \notin G(f)$, there exists a b-open set $U$ in X and an open set $V$ in Y such that $(x, y) \in U \times V$ and $(U \times \text{cl}(V)) \cap G(f) = \emptyset$.

**Theorem 7.2:-** If Y is Urysohn space and $f : X \to Y$ is weakly b-continuous, then $G(f)$ is a b-graph.

**Proof:-** Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exist open sets $A$ and $B$ of $Y$ containing $f(x)$ and $y$ respectively, such that $\text{cl}(A) \cap \text{cl}(B) = \emptyset$. Since $f$ is weakly b-continuous, there exists a b-open set $G$ in X containing $x$ such that $f(G) \subseteq \text{cl}(A)$. This shows that $f(G) \cap \text{cl}(B) = \emptyset$ and $(G \times \text{cl}(B)) \subseteq \emptyset$. 

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G(f) = φ. Thus G(f) is a b-graph.

**Theorem 7.3:** If \( f: X \to Y \) is a weakly b-continuous injective function having \( G(f) \) a b-graph. Then \( X \) is b-T₂.

**Proof:** Let \( x_1 \) and \( x_2 \) be two distinct points of \( X \). We have \( f(x_1) \neq f(x_2) \) and \( (x_1, f(x_2)) \notin G(f) \). Since \( G(f) \) is a b-graph, so, there exists a b-open set \( U \) in \( X \) and an open set \( V \) in \( Y \) such that \( (x_1, f(x_2)) \in U \times V \) and \( (U \times \text{cl}(V)) \cap G(f) = \emptyset \). We have \( f(U) \cap \text{cl}(V) = \emptyset \). Since \( f \) is weakly b-continuous, so, there exists a b-open set \( W \) in \( X \) containing \( x_2 \) such that \( f(W) \subseteq \text{cl}(V) \). We have \( f(U) \cap f(W) = \emptyset \) and \( U \cap W = \emptyset \). Thus \( X \) is b-T₂.

**Definition 7.4:** The graph \( G(f) \) of \( f: X \to Y \) is said to be **extremely b-closed** if for each \( (x, y) \notin G(f) \), there exists a b-open sets \( U \) containing \( x \) and a \( \Theta \)-open set \( V \) containing \( y \) such that \( (U \times V) \cap G(f) = \emptyset \).

This gives us following.

**Theorem 7.5:** The graph of \( f: X \to Y \) is **extremely b-closed** if and only if for each \( x \) in \( X \) and each \( y \) in \( Y \) different from \( f(x) \), there exists a b-open set \( U \) containing \( x \) and a \( \Theta \)-open set \( V \) containing \( y \) such that \( f(U) \cap V = \emptyset \).

**Proof:** Obvious from above definition.

**Definition 7.6:** The graph of \( f: X \to Y \) is said to be **b-closed** if for each \( (x, y) \notin G(f) \), there exists a b-open set \( U \) containing \( x \) and an open set \( V \) containing \( y \) such that \( (U \times V) \cap G(f) = \emptyset \) (or equivalently \( f(U) \cap V = \emptyset \)). This gives.

**Theorem 7.7:** The graph of \( f: X \to (Y, \sigma) \) is extremely b-closed if and only if the graph of \( f: X \to (Y, \sigma_\Theta) \) is b-closed.
**Theorem 7.8**: If \( f : X \to (Y, \sigma) \) is faintly \( b \)-continuous and \((Y, \sigma_\theta)\) is Hausdorff, then \( f \) has an extremely \( b \)-closed graph.

**Proof**: It is evident that \( f : X \to (Y, \sigma_\theta) \) is \( b \)-continuous because \( f : X \to (Y, \sigma_\theta) \) is faintly \( b \)-continuous. Since \( \sigma_\theta \) is Hausdorff, so, the graph of \( f : X \to (Y, \sigma_\theta) \) is \( b \)-closed. Thus \( f : X \to (Y, \sigma) \) has an extremely \( b \)-closed graph by above Theorem 7.7.

**Theorem 7.9 [12]**: Let \( f : X \to Y \) be continuous. If \( V \subset Y \) is \( \theta \)-open, then \( f^{-1}(V) \) is \( \theta \)-open in \( X \).

**Theorem 7.10**: Let \( Y \) be completely Hausdorff and \( f : X \to Y \) be faintly \( b \)-continuous, then the graph \( G(f) \) of \( f \) is extremely \( b \)-closed.

**Proof**: Let \( x \in X \) and \( y \) be in \( Y \) different from \( f(x) \). Since \( Y \) is completely Hausdorff, so, there exists a continuous function \( g : Y \to R \) such that \( g(f(x)) \) is not equal to \( g(y) \). Thus there exists disjoint open sets \( W \) and \( G \) in \( R \) containing \( g(f(x)) \) and \( g(y) \) respectively such that \( g^{-1}(W) \) and \( g^{-1}(G) \) are disjoint. But \( g^{-1}(W) \) is \( \theta \)-open (by Theorem 7.9 and the fact that every open subset of \( R \) is \( \theta \)-open). Therefore, there exists a \( b \)-open set \( U \) containing \( x \) such that \( f(U) \subset g^{-1}(W) \). So, that \( f(U) \cap g^{-1}(G) = \emptyset \). Thus, the graph of \( f \) is extremely \( b \)-closed.

**Theorem 7.11 [12]**: Let \( X \) and \( Y \) be topological spaces. If \( U \subset X \) and \( V \subset Y \) are \( \theta \)-open, then \( U \times V \) is \( \theta \)-open in \( X \times Y \).

**Theorem 7.12**: If the graph map of \( f : X \to Y \) is faintly \( b \)-continuous, then \( f \) is faintly \( b \)-continuous.

**Proof**: Let \( x \) be in \( X \) and \( V \) be a \( \theta \)-open set in \( Y \) containing \( f(x) \). Then \( X \times V \) is \( \theta \)-open in \( X \times Y \) by Theorem 7.11 and contains \( g(x) = (x, f(x)) \). Since the
graph map is faintly $b$-continuous, there exists a $b$-open set $U$ containing $x$ such that $g(U) \subset X \times V$. This implies that $f(U) \subset V$. Thus $f$ is faintly $b$-continuous.

**Theorem 7.13:** If $f : X \to Y$ is weakly $b$-continuous, then the graph map $g : X \to X \times Y$ is faintly $b$-continuous.

**Proof:** Let $x \in X$ and $W$ be a $\theta$-open set in $X \times Y$, containing $g(x)$. Then there exists a closed neighbourhood, hence a closed basic open set $cl(U \times V)$ containing $g(x)$ and lying inside $W$. Thus $g(x) = (x, f(x)) \in cl(U \times V) = cl(U \times cl(V))$, so that $f(x) \in cl(V)$. Since $f$ is weakly $b$-continuous, there exists a $b$-open set $A$ in $U$ containing $x$ such that $f(A) \subset cl(V)$. Consequently $g(A) \subset cl(U) \times cl(V) \subset W$. This shows $g$ is faintly $b$-continuous.

**Theorem 7.14:** Let $f : X \to Y$ be a weakly $b$-continuous function and $K$ be a $\theta$-closed subset of $X \times Y$. If $BO(X)$, the collection of $b$-open sets is closed under finite intersection, then $p(K \cap G(f))$ is $b$-closed in $X$. Where $p$ is the projection of $X \times Y$ onto $X$, and $G(f)$, the graph of $f$.

**Proof:** Let $x \in b-cl(p(K \cap G(f)))$ and $G$ be any open subset of $X$ containing $x$ and $H$ be any open subset of $Y$ containing $f(x)$. Since, $f$ is weakly $b$-continuous therefore $x \in f^{-1}(H) \subset b-int(f^{-1}(cl(H)))$. This implies that $x \in G \cap b-int(f^{-1}(cl(H)))$. Since $x \in b-cl(p(K \cap G(f)))$ then $(G \cap b-int(f^{-1}(cl(H)))) \cap p(K \cap G(f))$ contains point $x_0 \in X$. We have $(x_0, f(x_0)) \in K$ and $f(x_0) \in cl(H)$. Then $\phi \neq (G \times cl(H)) \cap K \subset cl(G \times H) \cap K$ and $(x, f(x)) \in -cl(K)$. Since $K$ is $\theta$-closed, $(x, f(x)) \in K \cap G(f)$ and $x \in p(K \cap G(f))$. This shows that $p(K \cap G(f))$ is $b$-closed in $X$.

**Corollary 7.15:** Let $f : X \to Y$ be a function with the $\theta$-closed graph and
g : X → Y be a weakly b-continuous function. Suppose that BO(X) is closed under finite intersection. Then the set \( \{ x \in X : f(x) = g(x) \} \) is b-closed in X.

**Proof:** Let \( G(f) \) be \( \theta \)-closed. We have \( p(G(f) \cap G(g)) = \{ x \in X : f(x) = g(x) \} \). By Theorem 7.14 the set \( \{ x \in X : f(x) = g(x) \} \) is b-closed in X.

**Theorem 7.16:** Let \( f : X \rightarrow Y \) be a function, where BO(X) is closed under finite intersection. If for each \( (x, y) \notin G(f) \), there exists a b-open set \( U \) in X and an open set \( V \) in Y containing \( x \) and \( y \) respectively, such that \( f(U) \cap \text{int}(\text{cl}(V)) = \emptyset \). Then inverse image of each \( N \)-closed set of \( Y \) is b-closed in \( X \).

**Proof:** Suppose that there exists an \( N \)-closed set \( W \) in \( Y \) such that \( f^{-1}(W) \) is not b-closed in \( X \). We have a point \( x \in b-\text{cl}(f^{-1}(W)) \) - \( f^{-1}(W) \). Since \( x \notin f^{-1}(W) \) then \( (x, y) \notin G(f) \) for each \( y \in W \). So, there exist b-open sets \( U_y(x) \) in \( X \) and an open set \( V_y \) in \( Y \) containing \( x \) and \( y \) respectively such that \( f(U_y(x)) \cap \text{int}(\text{cl}(V_y)) = \emptyset \). The family \( \{ V_y : y \in W \} \) is a cover of \( W \) by open sets of \( Y \). Since \( W \) is \( N \)-closed, there exist a finite number of points \( y_1, y_2, \ldots, y_n \) in \( W \) such that \( W = \bigcup_{i=1}^{n} \left( \text{cl}(V_{y_i}) \right) \). Taking \( U = \bigcap_{i=1}^{n} U_{y_i} \), we have \( f(U) \cap W = \emptyset \). Since \( x \in b-\text{cl}(f^{-1}(W)) \), then \( f(U) \cap W \neq \emptyset \). This is a contradiction. Therefore inverse image of each \( N \)-closed set of \( Y \) is b-closed in \( X \).

**Theorem 7.17:** If the graph function \( g \) of a function \( f : X \rightarrow Y \) is weakly b-continuous, then \( f \) is weakly b-continuous.

**Proof:** Let \( g \) be weakly b-continuous at \( x \) in \( X \) and \( U \) be an open set of \( Y \) containing \( f(x) \). Then \( X \times U \) is an open set containing \( g(x) \). So, there exists a b-open set \( V \) containing \( x \) such that \( g(V) \subset \text{cl}(X \times U) = X \times \text{cl}(U) \). This implies that \( f(V) \subset \text{cl}(U) \) and hence \( f \) is weakly b-continuous.
Theorem 7.18:- If \( f : X \to Y \) is weakly b-continuous and \( Y \) is Hausdrorff then for each \((x, y)\) in \( X \times Y - G(f)\), there exists a b-open set \( U \) in \( X \) and an open set \( V \) in \( Y \) containing \( x \) and \( y \) respectively, such that \( f(U) \cap \text{int}(\text{cl}(V)) = \emptyset \).

Proof:- Let \((x, y)\) be not in \( G(f) \). Then \( y \neq f(x) \). Since \( Y \) is Hausdorff so, there exist disjoint open sets \( V \) and \( H \) containing \( y \) and \( f(x) \) respectively. We have \( \text{int}(\text{cl}(V)) \cap \text{cl}(H) = \emptyset \). Since \( f \) is weakly b-continuous, there exists a b-open set \( U \) containing \( x \) such that \( f(U) \subset \text{cl}(H) \). Hence \( f(U) \cap \text{int}(\text{cl}(V)) = \emptyset \).

Definition 7.19:- The graph \( G(f) \) of \( f : X \to Y \) is said to be strongly b-closed if for each \((x, y) \in G(f)\), there exists a b-open set \( U \) containing \( x \) and an open set \( V \) containing \( y \) such that \( (U \times \text{cl}(V)) \cap G(f) = \emptyset \) or, equivalently, \( f(U) \cap \text{cl}(V) = \emptyset \).

Theorem 7.20:- Let \( f : X \to Y \) be a function having an extremely b-closed graph, then \( f \) has strongly b-closed graph.

Proof:- Let \( x \in X \) and \( y \in Y \) such that \( f(x) \neq y \). Then according to hypothesis, there exists a b-open set \( U \) containing \( x \) and a \( \theta \)-open set \( V \) containing \( y \) such that \( f(U) \cap V = \emptyset \). Since \( V \) is \( \theta \)-open, there exists an open set \( A \) such that \( y \in A \subset \text{cl}(A) \subset V \), so that \( f(U) \cap \text{cl}(A) = \emptyset \). Thus the graph of \( f \) is strongly b-closed.

Thus in present context we have:

Extremely b-closed graphs

\[
\begin{array}{c}
\text{Strongly b-closed graphs} \\
\downarrow \\
b\text{-closed graphs}
\end{array}
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REFERENCES


