Chapter - 1

LCR-CONTINUOUS AND GLCR-CONTINUOUS FUNCTIONS
ABSTRACT

Singal and Singal [5] introduced and studied almost continuous mappings which inversely map regular open sets onto open sets. Ganster and Reilly [3] introduced and studied LC-continuous mappings using locally closed [2] sets. Our purpose is to investigate and study another weaker form of continuous functions, namely LCR-continuous functions, which turns out to be another even weaker than the form of almost continuous mappings [5]. Moreover GLCR-continuous functions are found properly weaker than LCR-continuous functions. Apart from studying some properties inter-relations of new functions with existing functions are also depicted in an implication diagram.
Chapter - 1

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INTRODUCTION

A subset $A$ of a topological space $(X, \tau)$ is defined as locally closed [2] if it is the intersection of an open set and a closed set. Synonymously, Stone [6] has used the term FG for a locally closed subset. By making use of the concept of a locally closed set Ganster and Reilly [3] introduced LC-irresoluteness, LC-continuity and sub-LC-continuity along with discussion of some properties of these functions. K. Balachandran et al [1] introduced and investigated the concept of generalized locally closed sets and hence the classes of GLC-irresolute functions and GLC-continuous functions.

2. BASIC DEFINITIONS

Throughout this work $(X, \tau)$ denotes a topological space with a topology $\tau$ on which no separation axioms are assumed unless explicitly stated and for a subset $A$ of $X$, $\text{cl}(A)$ or $\bar{A}$ and $\text{int}(A)$ or $A^0$ denote the closure of $A$ and interior of $A$ respectively. A subset $A$ of $X$ is said to be regular open (resp. regular closed) if $\text{int}$(cl$(A))=A$ (resp. cl$(\text{int}(A))=A$). $P(X)$ is taken as power set of $X$. The following definitions are prerequisite in this regard.

**Definition 2.1:** A subset $A$ of $(X, \tau)$ is called **g-closed** [4] if $\text{cl}(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is open in $(X, \tau)$. A subset $A$ of $(X, \tau)$ is called **g-open** if its complement $X-A$ is g-closed.
It has been proved [4] that closed (resp. open) sets are g-closed (resp.
g-open) sets but g-closed(resp. g-open) sets need not be closed (resp. open)
sets.

**Definition 2.2:** A subset $A$ of $(X, \tau)$ is called **locally closed** [2] if $A=\text{G} \cap \text{F}$
(or equivalently $A=\text{G} \cap \text{cl}(A)$ [3]) where $\text{G}$ is open and $\text{F}$ is closed in $(X, \tau)$.
It is remarkable that

(i) A subset $A$ of $(X, \tau)$ is locally closed if and only if its complement $X-A$ is
the union of open set and a closed set.

(ii) Every open (resp. closed) subset of $X$ is locally closed.

(iii) The complement of a locally closed set need not be locally closed.

**Definition 2.3:** A subset $A$ of $(X, \tau)$ is called **generalized locally closed (glc) set** [1] if $A=\text{G} \cap \text{F}$
where $\text{G}$ is g-open and $\text{F}$ is g-closed in $(X, \tau)$. Every g-closed
(g-open) set is generalized locally closed (glc) set.

The collection of all regular open (resp. regular closed, locally closed,
generalized locally closed) sets in $X$ will be denoted by $\text{RO}_X(X, \tau)$ (resp.
$\text{RC}(X, \tau), \text{LC}(X, \tau), \text{GLC}$).

The collections $\text{GLC}^*(X, \tau)$ and $\text{GLC}^{**}$ are introduced as follows:

**Definition 2.4 [1]**: For a subset $A$ of $(X, \tau)$

(i) $A \in \text{GLC}^*(X, \tau)$ if and only if there exists a g-open set $G$ and a closed set $F$
of $(X, \tau)$ such that $A=\text{G} \cap \text{F}$.

(ii) $A \in \text{GLC}^{**}(X, \tau)$ if and only if there exists an open set $G$ and a g-closed set
$F$ of $(X, \tau)$ such that $A=\text{G} \cap \text{F}$.
**Definition 2.5 [7]:** A set \( G \) is said to be \( \delta \)-open if for each \( x \in G \), there exists a regular open set \( H \) such that \( x \in H \subseteq G \), or equivalently, if \( G \) is expressible as an arbitrary union of regular open sets.

A set is \( \delta \)-closed if and only if its complement is \( \delta \)-open.

**Definition 2.6 [7]:** A point \( x \in X \) is said to be a \( \delta \)-adherent point of \( A \subseteq X \) if the interior of every closed neighbourhood of \( x \) intersect \( A \) or equivalently every regular open set containing \( x \) has non-empty intersection with \( A \). The set of all \( \delta \)-adherent points of \( A \) is called \( \delta \)-closure of \( A \) and is denoted by \( \delta \text{-cl}(A) \).

**Proposition 2.7 [1]:** Let \( A \) be subset of \((X, \tau)\)

(i) If \( A \) is locally closed then \( A \in \text{GLC}^*(X, \tau) \) and \( A \in \text{GLC}^{**}(X, \tau) \), however not conversely.

(ii) If \( A \in \text{GLC}^*(X, \tau) \) or \( A \in \text{GLC}^{**}(X, \tau) \) then \( A \) is generalized locally closed glc-set.

### 3. LCR-CONTINUOUS FUNCTIONS

Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function between topological spaces \((X, \tau)\) and \((Y, \sigma)\). Ganster and Reilly [3] defined distinct notions of locally closed continuity that means LC-continuity, sub-LC-continuity and LC-irresoluteness. Then, Balachandran et al [1] generalized these concepts to, GLC-irresoluteness, \( \text{GLC}^* \)-irresoluteness, \( \text{GLC}^{**} \)-irresoluteness, GLC-continuity, \( \text{GLC}^* \)-continuity and \( \text{GLC}^{**} \)-continuity. Here in this section we find a few more generalizations which are found useful in further study.

**Definition 3.1 [1]:** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called **GLC-irresolute**
(resp. **GLC*-irresolute, GLC**-irresolute) if $f^{-1}(V) \in \text{GLC}(X, \tau)$ (resp. $f^{-1}(V) \in \text{GLC}^*(X, \tau)$, $f^{-1}(V) \in \text{GLC}^{**}(X, \tau)$) for each $V \in \text{GLC}(Y, \sigma)$ (resp. $V \in \text{GLC}^*(Y, \sigma)$, $V \in \text{GLC}^{**}(Y, \sigma)$).

**Definition 3.2:** A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) **Locally closed continuous i.e. LC-continuous** [3] if $f^{-1}(V)$ is locally closed in $(X, \tau)$ for every $V \in (Y, \sigma)$.

(ii) **Generalized locally closed continuous i.e. GLC-continuous** (resp. **GLC*-continuous, GLC**-continuous) [1] if $f^{-1}(V) \in \text{GLC}(X, \tau)$ (resp. $f^{-1}(V) \in \text{GLC}^*(X, \tau)$, $f^{-1}(V) \in \text{GLC}^{**}(X, \tau)$) for each $V \in \sigma$.

**Definition 3.3:** A function $f : (X, \tau) \to (Y, \sigma)$ is called **LCR-continuous** (resp. **GLCR-continuous, GLC*R-continuous, GLC**R-continuous) if $f^{-1}(V) \in \text{LC}(X, \tau)$ (resp. $f^{-1}(V) \in \text{GLC}(X, \tau)$, $f^{-1}(V) \in \text{GLC}^*(X, \tau)$, $f^{-1}(V) \in \text{GLC}^{**}(X, \tau)$) for every $V \in \text{RO}(Y, \sigma)$.

**Definition 3.4 [7]:** A mapping $f : (X, \tau) \to (Y, \sigma)$ is called **almost continuous** if and only if $f^{-1}(V)$ is open set for every regular open set $V$.

Since every open set is locally closed therefore every almost continuous mapping is LCR-continuous but the converse is not true as Example 3.10 shows.

**Proposition 3.5:** Let $f : (X, \tau) \to (Y, \sigma)$ be a function

(i) If $f$ is LCR-continuous, then it is GLC*R continuous and GLC**R-continuous.

(ii) If $f$ is GLC*R-continuous or GLC**R continuous then it is GLCR-
(iii) If \( f \) is GLC-irresolute (resp. GLC*-irresolute, GLC**-irresolute) then \( f \) is GLCR-continuous (resp. GLC*R-continuous, GLC**R-continuous).

**Proof:**
(i) Suppose that \( f \) is LCR-continuous. Let \( V \) be regular open set in \((Y, \sigma)\), then \( f^{-1}(V) \) is locally closed in \((X, \tau)\) by definition. Thus \( f \) is GLC*R-continuous and GLC**R-continuous by proposition 2.7(i).

(ii) By using proposition 2.7(ii) above, the proof is obvious.

(iii) Since every regular open set is open, every open set is locally closed and every locally closed set belongs to GLC* as well as GLC**(Y, \( \sigma \)) by proposition 2.7 (i) and hence belong to GLC(Y, \( \sigma \)) by proposition 2.7 (ii).

**Theorem 3.6:** Let \( f : (X, \tau) \to (Y, \sigma) \) be a function. If \( f \) is LC-continuous; then it is LCR-continuous but the converse is not true.

**Proof:** Since every regular open set is open, the proof is obvious. The converse is not true as following.

**Example 3.7:** Let \( X=Y=\{a, b, c, d\}\), \( \tau=\{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b, c, d\}\} \) and \( \sigma =\{\phi, Y, \{c, d\}, \{b, c, d\}\} \). Let \( f : X \to Y \) be identity function then \( f \) is LCR-continuous but not LC-continuous, since for the open set \( \{c, d\} \) in \((Y, \sigma)\) \( f^{-1}\{c, d\}=\{c, d\} \) is not locally closed in \((X, \tau)\).

**Theorem 3.8:** If \( f : X \to Y \) is LCR-continuous, then it is GLC*R-continuous as well as GLC**R-continuous, and hence GLCR-continuous, but the converses are not true as following.

**Proof:** See proposition 3.5, above. The converses are not true as following counter examples show.
Example 3.9:- Let \( X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}\} \) and \( \sigma = P(Y) \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity mapping. Since \( \text{GLC}(X, \tau) = \text{GLC}^*(X, \tau) = \text{GLC}^{**}(X, \tau) = P(X) \), \( \text{LC}(X, \tau) = \{\phi, X, \{a\}, \{b, c\}\} \) and \( \text{LC}(Y, \sigma) = \text{GLC}^*(Y, \sigma) = \text{GLC}^{**}(Y, \sigma) = P(Y) \). Therefore \( f \) is GLC*R-continuous, GLC**R-continuous and hence GLCR-continuous but not LCR-continuous. Various subsets are regular open sets in \((Y, \sigma)\) but not locally closed in \((X, \tau)\).

Example 3.10:- Let \( X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{a, b\}\} \) and \( \sigma = \{\phi, Y, \{a\}, \{b, c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a mapping defined by \( f(a) = f(c) = a \) and \( f(b) = b \) then \( \text{GLC}(X, \tau) = \text{GLC}^*(X, \tau) = \text{GLC}^{**}(X, \tau) = P(X) \) and \( \text{LC}^*(X, \tau) = \text{GLC}^*(Y, \sigma) = \text{GLC}^{**}(Y, \sigma) = P(Y) \). Then \( f \) is GLCR-continuous as well as GLC**R-continuous but not almost continuous and neither GLC*R-continuous nor LCR-continuous since \( \{a\} \) is regular open is \((Y, \sigma)\) but \( f^{-1}\{a\} = \{a, c\} \) is not open and also is neither in \( \text{GLC}^*(X, \tau) \) nor in \( \text{LC}(X, \tau) \).
Thus we have the following implication diagram.

Almost continuous function
\[\downarrow\uparrow\]
LC-continuous function
\[\downarrow\uparrow\]
LCR-continuous function
\[\downarrow\]
GLC*R-continuous function
\[\downarrow\]
GLC**R-continuous function
\[\downarrow\]  
GLCR-continuous function

Where \(A \implies B\) (resp. \(A \nRightarrow B\)) represents that \(A\) implies \(B\) (resp. \(A\) does not always imply \(B\)).

4. SOME PROPERTIES OF LCR-CONTINUOUS FUNCTIONS

**Definition 4.1 [1]**: A topological space \((X, \tau)\) is called **g-submaximal** if every dense subset of \(X\) is g-open.

**Corollary 4.2 [1]**: A topological space \((X, \tau)\) is g-submaximal if and only if \(P(X)=\text{GLC}^*(X, \tau)\) holds.
The following theorem is an immediate consequence of above mentioned definition and corollary.

Proposition 4.3:- A topological space \((X, \tau)\) is g-submaximal if and only if every function from the space \((X, \tau)\) as its domain is GLC*R-continuous.

Proof (Necessity):- Suppose \(f : (X, \tau) \to (Y, \sigma)\) is a function and \((X, \tau)\) is g-submaximal, to prove that \(f\) is GLC*R-continuous. For this let \(V\) be any regular open set in \((Y, \sigma)\), then \(f^{-1}(V) \subseteq \text{P}(X) = \text{GLC}^*(X, \tau)\) by the corollary 4.2 above thus \(f\) is GLC*R-continuous.

Sufficiency:- Let \(Y=\{0, 1\}\) with topology \(\sigma = \{\emptyset, Y, \{0\}\}\) and \(f : (X, \tau) \to (Y, \sigma)\) be a function defined by \(f(x) = 0\) if \(x \in V\) and \(1\) if \(x \notin V\) for any subset \(V\) in \((X, \tau)\). It follows from the assumption that \(f\) is GLC*R-continuous and hence \(f^{-1}(\{0\}) = V \subseteq \text{GLC}^*R(X, \tau)\). Thus \((X, \tau)\) is g-submaximal by above corollary 4.2.

Proposition 4.4:- If \(f : (X, \tau) \to (Y, \sigma)\) is GLC**R-continuous and a subset \(B\) is closed in \((X, \tau)\), then the restriction of \(f\) to \(B\), say \(f/B : (B, \tau_B) \to (Y, \sigma)\) is GLC**R-continuous where \(\tau_B\) is relativization of \(\tau\) with respect to \(B \subseteq X\).

Proof:- Let \(V\) be any regular open set of \((Y, \sigma)\). Then \(f^{-1}(V) = G \cap F\) for some open set \(G\) and some g-closed set \(F\) in \((X, \tau)\). By using Theorem 2.9 of Levine [4]; we have \((f/B)^{-1}(V) = (G \cap (F \cap B)) \subseteq \text{GLC}^{**}(B, \tau_B)\). Thus \(f/B\) is GLC**R-continuous.

We recall the definition of the combination of two functions. Let \(X = A \cup B\), \(f : A \to Y\) and \(h : B \to Y\) be two functions. We say that \(f\) and \(h\) are compatible if \((f/A \cap B) = (h/A \cap B)\). The we can define a function \(f \nabla h : X \to Y\), called the combination of \(f\) and \(h\) as follows \(f \nabla h(x) = f(x)\) for every \(x \in A\).
and \( h(x) \) for every \( x \in B \).

**Proposition 4.5 [1]**: Let \( \{ Z_i: i \in \Delta \} \) be a finite \( g \)-closed cover of \( (X, \tau) \) that mean \( X = \bigcup \{ Z_i: i \in \Delta \} \) and let \( A \) be a subset of \( (X, \tau) \). If \( A \cap Z_i \in \text{GLC}^* (Z_i, \tau_i) \) for each \( i \in \Delta \), then \( A \subseteq \text{GLC}^*(X, \tau) \).

**Theorem 4.6**: Let \( A \) and \( B \) are \( g \)-closed sets in \( (X, \tau) \) and \( X = A \cup B \) and \( f : (X, \tau_A) \to (Y, \sigma) \) and \( h : (B, \tau_B) \to (Y, \sigma) \) be compatible functions. If \( f \) and \( h \) are \( \text{GLC}^* \)-continuous then \( f \lor h : (X, \tau) \to (Y, \sigma) \) is \( \text{GLC}^* \)-continuous.

**Proof**: Let \( V \) be any regular open set in \( (Y, \sigma) \). Then \( (f \lor h)^{-1}(V) \cap A = f^{-1}(V) \) and \( (f \lor h)^{-1}(V) \cap B = h^{-1}(V) \) hold. By assumption we have \( (f \lor h)^{-1}(V) \cap A \in \text{GLC}^*(A, \tau_A) \) and \( (f \lor h)^{-1}(V) \cap B \in \text{GLC}^*(B, \tau_B) \). Therefore it follows that \( (f \lor h)^{-1}(V) \in \text{GLC}^*(X, \tau) \) and hence \( f \lor h \) is \( \text{GLC}^* \)-continuous.
REFERENCES


