5 Evolution of strange nonchaotic attractors in a parametrically modulated and externally driven, damped, rotating parabola system

5.1 Introduction

In this Chapter, we focus our attention on the strange nonchaotic attractors admitted by the quasiperiodically forced dynamical system discussed in Chapter IV, sec.4.3.2. Several routes (described in Chapter II), including the standard ones by which the appearance of strange nonchaotic attractors take place, are shown to be realizable in the same model over a two-parameter \((f-e)\) domain of the system. In particular, we demonstrate the existence of at least five different routes to chaos via strange nonchaotic attractors in this single dynamical system of quasiperiodically forced rotating parabola over the two-parameter
(f-\epsilon) space. To start with, the birth of the strange nonchaotic attractors associated with two important routes, namely, (i) torus breaking [159], and (ii) torus doubling [173], have been studied in our model. In low dimensions, Bier and Bountis [238] have shown that a dynamical system that undergoes one or more period doubling need not complete the entire infinite Feigenbaum cascade, and that there are possibilities of having only a finite number of period doubling, followed by, for example, undoubling or other bifurcations. The aim of the present Chapter is to look at the possibilities of such a novel remerging bifurcation phenomenon of the torus doubling sequence, and associated routes to chaos via SNAs, in the quasiperiodically forced system [196, 197]. Since the system that we consider possesses more than one control parameter and remains invariant under the reflection symmetry, the reemergence is likely to occur as is the case with low dimensional systems. To confirm such a possibility, our numerical studies show that in some regions of the (f-\epsilon) parameter space torus doubled orbit emerges and remerges from a single torus orbit at two different parameter values of \epsilon to form a torus bubble. Such a remerging bifurcation can lead to a taming of the growth of the torus doubled trees, and the development of the associated universal route to further chaos. However, the nature of remerging torus doubled trees or more specifically torus bubbling ensures the existence of different routes for the creation of SNA, when the full range of parameters are taken into account. To illustrate these possibilities in our system, we enumerate two new types of routes as: (1) two frequency quasiperiodicity \rightarrow torus doubling \rightarrow torus merging followed by the gradual fractalization of torus to chaos, (2) two frequency quasiperiodicity \rightarrow torus doubling \rightarrow wrinkling \rightarrow SNA \rightarrow chaos \rightarrow SNA \rightarrow wrinkling \rightarrow inverse torus doubling \rightarrow torus \rightarrow torus bub-
bles followed by the onset of torus breaking to chaos via SNA or followed by the onset of torus doubling route to chaos via strange nonchaotic attractor. Finally, we also show the occurrence of period doubling bifurcations of the destroyed torus (strange nonchaotic attractor) in our model.

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Here, we consider the motion of a freely sliding particle on a parabolic wire with parametrically changing angular velocity. That is, we consider the combined effect of both the external and parametric forcing in Eq. (4.46). We note that the system (4.46) remains invariant under the reflection symmetry $(x, y, f) \to (-x, -y, -f)$, (or equivalently (4.42) under the transformation $(x, f) \to (-x, -f)$). In analogy with low dimensional systems involving more than one control parameter when period bubbles occur, one may expect remergence of torus doubling sequences to occur in this model, which we indeed show to be true in the following.

To be concrete, we consider the dynamics of (4.46) and numerically integrate it using the fourth order Runge-Kutta algorithm with adaptive step size with the values of the parameters fixed at $\omega_p=0.25$, $\lambda=0.5$, $\alpha=0.2$, $\Omega^2=6.7$, $\omega_p=1.0$ and $\omega_e=0.991$. Various characteristic quantities such as the winding numbers, Lyapunov exponents, power spectral measures, and dimensions as discussed in the Chapter II have been used to distinguish quasiperiodic, strange nonchaotic and chaotic attractors. Further, to identify the different attractors, the dynamical transitions are traced out by two scanning procedures: (i) varying $f$
at a fixed $\epsilon$ and (ii) varying $\epsilon$ at a fixed $f$. The resulting phase diagram in the $(f-\epsilon)$ parameter space is shown in Fig. 5.1. The various features indicated in the phase diagram are summarised and the dynamical transitions are elucidated in the following.
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5.2.1 Torus breaking bifurcations and the birth of strange nonchaotic attractors

For low \( f \) and low \( \epsilon \) values, the system exhibits two frequency quasiperiodic oscillations denoted by IT in Fig. 5.1. When the value of \( \epsilon \) exceeds a certain critical value for a fixed low \( f \), a transition from two frequency quasiperiodic (IT) to chaotic attractor (C) via strange nonchaotic attractor (S) occurs on increasing \( \epsilon \). For example, we fix the strength of the external forcing parameter value as \( f=0.302 \) and vary the modulation parameter \( \epsilon \). For \( \epsilon=0.03 \), Fig. 5.2a of the attractor has smooth branches and this indicates that the system is in a two frequency quasiperiodic state. As \( \epsilon \) increases, the branches in the Fig.5.2b start to wrinkle. As \( \epsilon \) increases further, the attractor becomes extremely wrinkled and has several sharp bends. The sharp bends appear to become actual discontinuities at \( \epsilon=0.0419 \) and ultimately result in fractal phenomenon. Such a phenomenon is essentially the result of the collision of stable and unstable torus in a dense set of points as was shown by Feudal et al. \cite{159} in their route to chaos via SNA. At such values, the nature of the attractor is strange (Fig.5.2c) even though the largest Lyapunov exponent in Fig.5.3 remains negative. For this attractor, the correlation dimension is 1.33 while the Fourier amplitude scaling constant \((\sigma)\) is 1.54. Winding number \( W \) does not satisfy the relation \( W = \frac{m}{n} \omega_p + \frac{l}{n} \omega_e \) for this attractor. Hence, these studies confirm further that the attractor shown in Fig. 5.2c is a strange nonchaotic attractor. As \( \epsilon \) increases further, an attractor visibly similar to Fig.5.2c appears (see Fig5.2d for \( f=0.042 \)). However, it has a positive Lyapunov exponent and hence it corresponds to a chaotic attractor.
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Figure 5.2: Projection of the two frequency quasiperiodic attractors of Eq. (4.46) for \( f=0.302 \): Poincaré plot with \( \phi \mod 2\pi \) in the \((x,\phi)\) plane, (a) two frequency quasiperiodic attractor at \( \epsilon=0.030 \), (b) torus wrinkled attractor for \( \epsilon=0.0405 \), (c) strange nonchaotic attractor for \( \epsilon=0.0419 \), (d) chaotic attractor for \( \epsilon=0.042 \). The other parameters are \( \omega_0^2=0.25 \), \( \lambda=0.5 \), \( \alpha=0.2 \), \( \omega_p=1.0 \), \( \Omega_0^2=6.7 \) and \( \omega_c=0.991 \).

5.2.2 Remerging torus doubling bifurcations: torus bubble and its consequences

5.2.2.A Torus bubbling

On increasing the forcing parameter \( f \) further, \( 0.305< f <0.325 \), the fascinating novel phenomenon of torus bubble appears within a range of values of \( \epsilon \). Within this range of \( f \), on increasing the value of \( \epsilon \) along the same line, the onset of chaos is realised via strange nonchaotic attractor. To be more specific, the parameter \( f \) is fixed at 0.32 and \( \epsilon \) is varied. For \( \epsilon=0.03 \), the attractor is a two frequency quasiperiodic attractor (Figs. 5.4a & 5.5a). As \( \epsilon \) is increased to \( \epsilon=0.0317 \), the attractor undergoes a torus doubling bifurcation (Figs. 5.4b & 5.5b). We note from Figs. 5.4 and 5.5 that the two strands in the \((x,\phi)\) projection become four strands when torus doubling bifurcation occurs. When we compute
\( \phi \) modulo \( 4\pi \) instead of \( 2\pi \) during integration, we notice from Figs. 5.5 that the two bifurcated strands of length \( 2\pi \) are actually a single strand of length \( 4\pi \). As a result, it can be concluded that the torus doubling is nonetheless a length doubling bifurcation. Further, it may be noted that this bifurcation is geometrically similar to that of period doubling bifurcation in the three dimensional flows. One then expects as \( \epsilon \) is increased further that the doubled attractor has to continue the doubling sequence as is the case with period doubling phenomenon. Instead, in the present case, interestingly the strands of the length doubled attractor begin to merge into that of a single attractor at \( \epsilon = 0.0353 \) as shown in Fig 5.6a, leading to the formation of a torus bubble (see Fig.5.1), reminiscent of period bubbles in low dimensional systems. On further increase of the value of \( \epsilon \), the transition from two frequency quasiperiodicity to chaos via strange nonchaotic attractor takes place due to torus breaking bifurcations as discussed in section 5.2.1 (see Figs 5.6 b,c,& d and Fig.5.7).

It has been argued in the case of period bubbling in low dimensional systems that the cause of formation of the period bubbles is essentially due to the presence of reflection symmetry, combined with more than one control parameter present in the system. It appears that similar arguments holds good for the case of higher dimensions for the formation of torus bubbles.

5.2.2.B Formation of multibubbles

As the forcing parameter \( f \) is increased further in the region \( 0.325 < f < 0.332 \), the evolution of attractor undergoes the following transition to chaos, wherein more than one bubble is formed on increasing the value of \( \epsilon \): two frequency quasiperiodicity \( \rightarrow \) torus doubling \( \rightarrow \) wrinkling \( \rightarrow \) inverse torus doubling (dou-
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Figure 5.3: Largest Lyapunov exponent $\lambda_{\text{max}}$ vs. $\epsilon$ corresponding to fig. 5.2

Figure 5.4: Projection of the two frequency quasiperiodic attractors of Eq. (4.46) for $f=0.302$: Poincaré surface of section in the $(x,y)$ plane (a) torus at $\epsilon=0.03$, (b) torus doubled attractor at $\epsilon=0.0317$. The other parameters are $\omega_0^2=0.25$, $\lambda=0.5$, $\alpha=0.2$, $\omega_p=1.0$, $\Omega_0^2=6.7$ and $\omega_c=0.991$. 
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Figure 5.5: Projection of the two frequency quasiperiodic attractors of Eq. (4.46) for \( f = 0.302 \): (a) Poincaré surface of section with \( \phi \) mod \( 2\pi \) in the \( (x, \phi) \) plane for \( \epsilon = 0.03 \), (b) torus doubled attractor at \( \epsilon = 0.0317 \), (c) & (d) same as (a) & (b) except \( \phi \) mod \( 4\pi \) during integration. The other parameters are \( \omega_0^2 = 0.25 \), \( \lambda = 0.5 \), \( \alpha = 0.2 \), \( \omega_p = 1.0 \), \( \Omega_0^2 = 6.7 \) and \( \omega_e = 0.991 \).

bbed torus) \( \rightarrow \) merged torus \( \rightarrow \) torus bubble \( \rightarrow \) merged torus \( \rightarrow \) wrinkling \( \rightarrow \) SNA \( \rightarrow \) chaos. To illustrate this possibility, let us fix the forcing parameter value as \( f = 0.328 \) and vary the \( \epsilon \) value. For \( \epsilon = 0.03 \) the attractor is two frequency quasiperiodic. As \( \epsilon \) is increased to \( \epsilon = 0.0313 \), the attractor undergoes torus doubling bifurcation. The doubled attractor begins to wrinkle when the \( \epsilon \) value is increased. However, this wrinkled attractor appears to become again a torus doubled attractor, instead of approaching the SNA while the \( \epsilon \) value is increased further. This doubled attractor merges into a single torus through inverse bifurcation on increasing the value of \( \epsilon \). The merged torus again forms a torus bubble, and then finally transits to chaos via wrinkling and SNA as the value of the \( \epsilon \) is increased further.

On increasing the forcing parameter \( f \) further, \( 0.332 < f < 0.335 \), the transition from two frequency quasiperiodicity to chaos via SNA takes place through
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the following route, wherein more than two bubbles are formed as $\epsilon$ increases: two frequency quasiperiodicity $\to$ torus doubling $\to$ wrinkling $\to$ inverse torus doubling $\to$ merged torus $\to$ torus bubble $\to$ torus $\to$ torus bubble $\to$ torus $\to$ wrinkling $\to$ SNA $\to$ chaos.

5.2.3 Strange nonchaotic and chaotic attractors within and outside the main torus bubble

On further increase of $f, f > 0.335$, inside the main torus bubble we observe interesting possibilities of strange nonchaotic and chaotic attractors via wrinkling as $\epsilon$ increases. Here two interesting possibilities arise inside the main bubble, and the dynamics outside the main bubble more or less follows the previous case 5.2.2.B. The details are as follows.
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5.2.3.A SNA within the main torus bubble

In a narrow region of $f$, $0.335 < f < 0.339$, the SNA undergoes an inverse bifurcation scheme leading to two frequency quasiperiodic attractor as $\epsilon$ increases through the following route: two frequency quasiperiodicity $\rightarrow$ torus doubling $\rightarrow$ wrinkling $\Rightarrow$ SNA $\rightarrow$ wrinkling $\rightarrow$ inverse torus doubling (doubled torus) $\rightarrow$ merged torus. For example, the forcing parameter $f$ is fixed at $f=0.337$ and $\epsilon$ is varied. For $\epsilon=0.03$ the attractor is a two frequency quasiperiodic (Fig. 5.8a). As $\epsilon$ is increased to $\epsilon=0.031$, the attractor undergoes a torus doubling bifurcation (as seen in Fig.5.8b). In lower dimensional systems, the period doubling occurs in an infinite sequence until the accumulation point is reached, beyond which chaotic behaviour appears. However, with tori, in the present case, the trun-
cation of the torus doubling begins when the two strands become extremely wrinkled when the \( \epsilon \) value is increased, as shown in Fig. 5.8c These strands lose their continuity as well as smoothness and become strange at \( \epsilon = 0.0339 \). At such values, the attractor possesses geometrically strange properties but does not obey the sensitivity to initial conditions (the maximal Lyapunov exponent is negative as seen in Fig. 5.9) and so, it is named as strange nonchaotic attractor (Fig. 5.8d). The emergence of such SNA is due to the collision of stable doubled torus and its unstable parent, as was shown by Heagy and Hammel [173] in their route. Interestingly, the SNA (Fig. 5.8e), instead of approaching a chaotic attractor as the \( \epsilon \) value increases, becomes wrinkled and then, torus doubled attractor (Fig. 5.8f). The doubled attractor again merges into a single torus (Fig. 5.8g) on further increasing the value of \( \epsilon \).

### 5.2.3.B Chaotic attractor within the main torus bubble

In rather large regions of \( f, f > 0.339 \), the SNA as formed above transits into a chaotic attractor on increasing the value of \( \epsilon \) further through the following route: two frequency quasiperiodicity \( \rightarrow \) torus doubling \( \rightarrow \) wrinkling \( \rightarrow \) SNA \( \rightarrow \) chaos \( \rightarrow \) SNA \( \rightarrow \) wrinkling \( \rightarrow \) inverse torus doubling (doubled torus) \( \rightarrow \) merged torus. To illustrate this possibility, let us choose the parameter \( f = 0.342 \), and vary the value of \( \epsilon \). For \( \epsilon = 0.03 \) the attractor is a two frequency quasiperiodic attractor (Fig. 5.10a). As \( \epsilon \) is increased to \( \epsilon = 0.0309 \), the attractor undergoes a torus doubling bifurcation (Fig. 5.10b). As \( \epsilon \) is increased further the strands of the doubled attractor begins to wrinkle, as shown in Fig. 5.10c. The formation of sharp bends in the strand of the attractor begins to appear as \( \epsilon \) is increased further. These bends tend to become actual discontinuities at \( \epsilon = 0.0337 \), as
shown in Fig. 5.10d. The emergence of such discontinuities on the torus is due to the collision of stable doubled torus and its unstable parent which is similar to the one found by Heagy and Hammel [173]. At such values, the attractor loses smoothness and becomes "strange". The attractor shown in Fig. 10d is nothing but strange nonchaotic as the maximum Lyapunov exponent works out to be $\lambda = -0.01213$ (Fig.5.11). Further the correlation dimension is 1.49, the scaling constant($\sigma$) is 1.38 and winding number $W$ does not satisfy the relation $W = \frac{m}{n} \omega_p + \frac{l}{n} \omega_e$ for this attractor. Hence, these characteristic studies confirm further that the attractor shown in Fig. 5.10e is strange but nonchaotic. On further increase of the value of $\epsilon$ to 0.034, we find the emergence of a chaotic attractor (Fig. 5.10e), which though visibly similar to the nonchaotic strange attractor Fig. 5.10d, has a positive Lyapunov exponent (see Fig.5.11). The chaotic attractor again becomes SNA when $\epsilon$ is further increased (Fig. 5.10f). As the value of $\epsilon$ is still increased, the SNA becomes torus doubled attractor (Fig.5.10g) via wrinkling. This doubled attractor then merges into a single torus (Fig. 5.10h) when the value $\epsilon$ is continuously increased.

5.2.3.C Dynamics outside the main torus bubble

There are two interesting transitions existing outside the main torus bubble, namely (i) torus breaking to chaos via SNA and (ii) torus doubling to chaos via SNA. The details are as follows.

In a narrow region of $f$, $0.335 < f < 0.345$, the transition from two frequency quasiperiodicity to chaos via SNA takes place outside the main bubble through the following route as $\epsilon$ increases: torus $\to$ torus bubble $\to$ torus $\to$ torus bubble $\to$ wrinkling $\to$ SNA $\to$ chaos. However in the region of $f$, $0.345<$
Figure 5.8: Projection of the two frequency quasiperiodic attractors of Eq. (4.46) for $f=0.337$: Poincaré plot with $\phi \mod 2\pi$ in the $(x, \phi)$ plane, (a) two frequency quasiperiodic torus at $\epsilon=0.03$, (b) doubled torus attractor for $\epsilon=0.0312$, (c) wrinkled doubled attractor for $\epsilon=0.0335$, (d) strange nonchaotic attractor for $\epsilon=0.0342$, (e) strange nonchaotic attractor for $\epsilon=0.0345$, (f) wrinkled attractor for $\epsilon=0.03461$ (g) doubled torus attractor for $\epsilon=0.0347$, (h) merged attractor for $\epsilon=0.036$. The other parameters are $\omega_0^2=0.25$, $\lambda=0.5$, $\alpha=0.2$, $\omega_p=1.0$, $\Omega_0^2=6.7$ and $\omega_c=0.991$. 
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Figure 5.9: Largest Lyapunov exponent $\lambda_{\text{max}}$ vs. $\epsilon$ corresponding to Fig. 5.8
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Figure 5.10: Projection of the two frequency quasiperiodic attractors of Eq.(4.46) for \( f=0.347 \): Poincaré plot with \( \phi \mod 2\pi \) in the \((x,\phi)\) plane, (a) two frequency quasiperiodic torus at \( \epsilon=0.03 \), (b) doubled torus attractor for \( \epsilon=0.0309 \), (c) wrinkled doubled attractor for \( \epsilon=0.033 \), (d) strange nonchaotic attractor for \( \epsilon=0.0337 \), (e) chaotic attractor for \( \epsilon=0.034 \), (f) strange nonchaotic attractor for \( \epsilon=0.0345 \), (g) doubled torus attractor for \( \epsilon=0.0347 \), (h) merged attractor for \( \epsilon=0.036 \). The other parameters are \( \omega_x=0.25 \), \( \lambda=0.5 \), \( \alpha=0.2 \), \( \omega_p=1.0 \), \( \Omega^2=6.7 \) and \( \omega_\epsilon=0.991 \).

\( f < 0.352 \), the transitions, particularly, from wrinkled two attractor to wrinkled one attractor, take place.

Higher values of the forcing strength \( f, 0.352 > f \), introduce the appearance of other kind of transitions; particularly, beyond the main bubbles, the transitions are quite different from the above discussed ones. When the forcing parameter value increase is coupled with increasing \( \epsilon \) value, the transition evolves in the following way: torus \( \rightarrow \) torus bubble \( \rightarrow \) torus \( \rightarrow \) merged torus \( \rightarrow \) torus doubling \( \rightarrow \) wrinkling \( \rightarrow \) inverse torus doubling \( \rightarrow \) merged torus \( \rightarrow \) torus
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Fig. 5.10 (continued)

Figure 5.11: Largest Lyapunov exponent $\lambda_{\text{max}}$ vs. $\epsilon$ corresponding to Fig. 5.10
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Figure 5.12: Projection of the two frequency quasiperiodic attractors of Eq.(4.46) for \( f = 0.34 \): Poincaré plot with \( \phi \) mod \( 2\pi \) in the \((x,\phi)\) plane, (a) two frequency quasiperiodic torus at \( \epsilon = 0.03 \), (b) doubled torus attractor for \( \epsilon = 0.0305 \), (c) strange nonchaotic attractor for \( \epsilon = 0.0334 \), (d) doubled strange nonchaotic attractor for \( \epsilon = 0.0337 \). The other parameters are \( \omega_o^2 = 0.25, \lambda = 0.5, \alpha = 0.2, \omega_p = 1.0, \Omega_o^2 = 6.7 \) and \( \omega_c = 0.991 \).

doubling \( \rightarrow \) wrinkling \( \rightarrow \) strange nonchaotic attractor \( \rightarrow \) chaos.

5.2.4 Period doubling bifurcations of destroyed torus within the main torus bubble

In the previous sections, we have observed that the period doubling bifurcation has been truncated by the destruction of the torus in certain regions of \((f-\epsilon)\) parameter regions. However, in some cross sections of \((f-\epsilon)\) parameter space, the period doubling bifurcation phenomena still continues in the destroyed torus, even though the doubling sequence of torus has been terminated by destroyed torus. Such a route has also been recently observed in coupled Duffing oscillators [167] and in certain maps [161]. Such doubling of destroyed tori has been observed in a long range of \( f \), \( 0.338 < f < 0.358 \). To be more specific, we choose
the parameters as $f=0.34$ and vary the value of $\epsilon$. For $\epsilon=0.03$, the attractor is a two frequency quasiperiodic torus (Fig. 5.12a). As $\epsilon$ is increased to $\epsilon=0.0305$, the system undergoes torus doubling bifurcations (Fig. 5.12b). Even on increase of the value of $\epsilon$, the attractor begins to wrinkle. On further increase of the value of $\epsilon$, the destroyed torus exhibits period doubling bifurcation (Fig. 5.12c &d). If the parameter $\epsilon$ increases continuously, the system exhibits chaotic behaviour.

5.3 Conclusion

In this chapter, we considered the dynamics of the specific example of the quasiperiodically forced velocity dependent nonpolynomial oscillator system (4.46) which illustrates many of the typical routes to chaos via strange nonchaotic attractors. It was found that the first two of these routes can be realized in the following ways.

1. two frequency quasiperiodicity $\rightarrow$ strange nonchaotic attractor $\rightarrow$ chaos.

   In this case, the emergence of SNA is essentially the result of interaction of stable and unstable torus in a dense set of points.

2. two frequency quasiperiodicity $\rightarrow$ torus doubling $\rightarrow$ strange nonchaotic attractors $\rightarrow$ chaos. Here, the birth of SNA is due to the collision of stable doubled orbit with its unstable parent torus.

Torus doubling bifurcations in dynamical systems often form finite sequence which 'merge' in some cross sections of the parameters space, inhibiting the onset of torus doubling route to chaos. Such remerging bifurcations having finite number of 'bubbles' occur only within certain range of parameter values.
An important consequence of such remerging is that the orbits become again stable and relatively large regions re-appear around them, where the motion is regular and predictable. On the other hand, torus doubling routes to chaos are generally observed when the full range of parameters is explored. To illustrate such remerging torus doubling bifurcations, our present study allows us to add two more routes [196, 197] which seem to be typical:

1. two frequency quasiperiodicity $\rightarrow$ torus doubling $\rightarrow$ torus merging followed by the gradual fractalization of torus to chaos

2. two frequency quasiperiodicity $\rightarrow$ torus doubling $\rightarrow$ wrinkling $\rightarrow$ SNA $\rightarrow$ chaos $\rightarrow$ SNA $\rightarrow$ wrinkling $\rightarrow$ inverse torus doubling $\rightarrow$ torus $\rightarrow$ torus bubbles followed by the onset of torus breaking to chaos via SNA or followed by the onset of torus doubling route to chaos via SNA.

From these routes, it can be concluded that prior to standard routes for transition to strange nonchaotic attractor, the possibilities of several bifurcations on the torus can be realised. The existence of such remerging bifurcation is due to symmetry properties of the system.

Finally, the period doubling bifurcations of destroyed tori have been observed in our model when a single parameter is varied:

1. two frequency quasiperiodicity $\rightarrow$ destroyed torus $\rightarrow$ period doubling of destroyed torus $\rightarrow$ chaos.