2 Design of 1 Dimensional Linear Phase FIR Filter with Elementary Polynomials

2.1 Procedure

In this chapter we present an approach to realize FIR filters using elementary polynomials. A stepwise description to design the FIR filter is discussed below.

**Step 1.** Choose a polynomial, \( f(x) \), with following properties

1. It should have zeros near the origin; that is, \( 0 \leq |x| \leq 1 \), and
2. It should increase sharply for real values of \( x \), which are far away from \( x = 1 \); that is, \( x > 1 \).

Such a polynomial is shown in Figure 2.1. From figure we can comprehend that as \( x \) varies from 0 to some point \( x_0 \), function \( f(x) \) traces out a pattern of several side bands and one major band. The side bands are \( 1/b \) times down from the major band. This ratio can be selected at will by choosing the value of \( x_0 \).

**Step 2.** We use the transformation discussed in Chapter 1 for mapping the polynomial variable \( x \) to frequency variable \( \omega \) of filter characteristics. The transformation is repeated below for convenience

\[
x = x_0 \cos(\omega/2)
\]  
(2.1)

where, \( x_0 \) represents the maximum value of \( x \). At the value when \( x \) is \( x_0 \) the polynomial has the value \( b \), Figure 2.1.

**Step 3.** The zeros of the transfer function of FIR filter are calculated next. Let these zeros be represented by \( z_i = \exp(j\omega_i) \), where \( i = 1, 2, \ldots n \). The \( \omega_i \)s are calculated using the inverse of the transformation of Equation (2.1); that is,
\[ \omega_i = 2 \cos^{-1} \left( \frac{x_i}{x_0} \right) \]  \hspace{1cm} (2.2)

Note that \(x_i\)'s are the zeros of the original polynomial \(f(x)\). The FIR filter transfer function is given by

\[ H(z) = \prod_{i=1}^{n} (z - z_i) \]  \hspace{1cm} (2.3)

or,

\[ H(\omega) = \prod_{i=1}^{n} (e^{j\omega} - e^{j\omega_i}) \]  \hspace{1cm} (2.4)

![Graph](image)

**Figure 2.1: Example Polynomial.**

It will be clear in the next section, where we consider some specific cases of filter design, that the values of \(x_0\) and \(b\) are related to the bandwidth of the filter and the ripples in the stop-band, respectively.
2.2 Application

In this section, we consider different types of object functions which, in turn, will give us low pass filters having different magnitude and phase characteristics. Standard frequency transformation routines [2] can be applied on the resulting low pass filter to get the other type of filters; that is, high pass, band pass, etc. The examples below give a perspective of design from object function's side, but once one understood this he can design the filter other way round as well.

2.2.1 Design 1

Let us consider a polynomial where the values of $x_i$'s are all equal to 0; that is, all the zeros of the object function lie on the origin. In this case the polynomial is

$$f(x) = x^n$$ (2.5)

This polynomial satisfies all the conditions specified in Step 1. Figure 2.2 shows this polynomial for $n = 6$.

The value of the function, where we want our stop band to start, can be taken at $x = 1$. This value is arbitrary, but $x = 1$ gives good results. Therefore, at the start of the stop band

$$x^n = 1$$ (2.6)

Looking at the transformation $x = x_0 \cos(\omega/2)$ we observe that $x = 0$ transforms to $\omega = \pi$, $x = 1$ transforms to $\omega = 2 \cos^{-1}(1/x_0)$ which is the frequency where the stop-band starts.

2.2.1.1 Calculation of stop band frequency, $\omega_s$, and pass band frequency, $\omega_p$

Using above mentioned ideas, we desire that at $x = x_0$ the value of the function $f(x_0)$ should be $'b'$ times its value than it has at the stop band. Thus,

$$x_0^n = b$$ (2.7)
Figure 2.2: Polynomial $x^6$.

\[ x_0 = b^{1/n} \]  \hspace{1cm} (2.8)

We use the transform, Equation (2.1), to calculate stop band frequency, $\omega_s$. The stop band starts when the value of $x$ is 1, as discussed above, and $x_0 = b^{1/n}$. Therefore, Equation (2.1) converts to

\[ 1 = b^{1/n} \cos(\omega_s/2) \]  \hspace{1cm} (2.9)

and,

\[ \omega_s = 2 \cos^{-1}(1/b^{1/n}) \]  \hspace{1cm} (2.10)

The pass band frequency, $\omega_p$, of filter characteristic occur when function is $b/\sqrt{2}$ times, or $3\,dB$ down, of its peak value than that at the stop band. Thus,

\[ x_p^n = b/\sqrt{2} \]  \hspace{1cm} (2.11)

and,
\[ x_p = \frac{b^{1/n}}{2^{1/2n}} \]  
(2.12)

where, \( x_p \) is the point on the object function which corresponds to the pass band frequency, \( \omega_p \).

Using the transform, Equation (2.1), with Equation (2.12) we can calculate the value of pass band, \( \omega_p \), as follows

\[ b^{1/n} = \frac{b^{1/n}}{2^{1/2n}} = b^{1/n} \cos(\omega_p/2) \]  
(2.13)

therefore,

\[ \omega_p = 2 \cos^{-1}(1/2^{1/2n}) \]  
(2.14)

All the zeros of \( f(x) \) lie at \( x = 0 \), so there is no need of calculation involved in Step 3 in this case.

2.2.1.2 Calculation of \( b \)

As discussed above the maximum value of the object function is \( b \) times its value than that at the stop band. Thus, if we want our stop band to be, let us say, \( p \) dB down when compared with the maximum value of pass band, the value of \( b \) is absolute value of \( p \) dB; that is,

\[ b = 10^{p/20} \]  
(2.15)

Suppose in the polynomial discussed above, the value of \( n \) is 6; that is, our filter is of the order of 6. We want stop band to be 40 dB down; that is, \( b = 10^{40/20} = 100 \) (Equation (2.15)). The value of \( \omega_s \) is 2.17622 radians and \( \omega_p \) is 0.6733 radians, calculated using Equations (2.10) and (2.14), respectively. The transfer function for this FIR filter, where \( \omega_i = \pi \) and \( i = 1, 2, \ldots 6 \), is

\[ H(\omega) = [x_0 \cos(\omega/2)]^6 \]  
(2.16)

The polynomial, magnitude response in dB, and phase response for low pass FIR filter of Equation (2.16) is shown in Figures 2.2, 2.3 and 2.4, respectively. As expected the phase comes out to be zero. The magnitude characteristics is not very good but it gives us an insight into – what our transformation is going to do with the object function after application.
Figure 2.3: Magnitude response in dB for Polynomial $x^6$.

Figure 2.4: Phase response for Polynomial $x^6$. 
2.2.2 Design 2

Let us consider another polynomial with zeros not concentrated at the origin. The polynomial is defined in Equation (2.17) and it follows the characteristics discussed in Step 1, as well as it has zeros at locations other than \( x = 0 \) except one (Figure 2.5).

\[
f(x) = (x - 0)(x - 0.1)(x - 0.2)(x - 0.3)(x - 0.4)(x - 0.5) \quad (2.17)
\]

Order of the filter is 6. Suppose we want our stop band to be 40\(dB\) down; that is, \( b = 100 \).

We calculate the value of stop band for the above polynomial by equating, as in previous case, \( f(x) \) to 1; that is,

\[
(x - 0)(x - 0.1)(x - 0.2)(x - 0.3)(x - 0.4)(x - 0.5) = 1 \quad (2.18)
\]

When we solve Equation (2.18) we get six different values of \( x \). Any of the values of \( x \) can be used, apart from the fact that every value will give rise to a different value of the frequency where the stop band ends, we choose 1.2646, this point correspond to \( \omega_s \) and we call it by name \( x_s \).

To calculate the value of \( x_0 \) we equate \( f(x) \) to \( b (=100) \); that is,

\[
(x - 0)(x - 0.1)(x - 0.2)(x - 0.3)(x - 0.4)(x - 0.5) = 100 \quad (2.19)
\]

The solution of Equation (2.19) gives us six different values of \( x \), here again we consider any one positive value amongst them (see Figure 2.1), we retain \( x_0 = 2.4112 \).

The value of stop band, \( \omega_s \), can be calculated by using the transform discussed in Step 2; that is,

\[
1.2646 = 2.41121 \cos(\omega_s/2) \quad (2.20)
\]

\[
\omega_s = 2.0374 \text{ radians} \quad (2.21)
\]

Similarly, the value of pass band, \( \omega_p \), can be calculated by equating the
polynomial to \( b/\sqrt{2} \) or \( 100/\sqrt{2} \); that is,

\[
(x - 0)(x - 0.1)(x - 0.2)(x - 0.3)(x - 0.4)(x - 0.5) = 100/\sqrt{2} \tag{2.22}
\]

Above equation gives us six different values, we consider any one value. In this case every value will correspond to a different value of the frequency where the stop band starts; that is, \( 3\,\text{dB} \) down from the maximum value of the pass band of the filter. The value of the solution is \( x_p = 2.29069 \). We use the same transformation as discussed in Step 2 to get the pass band frequency; that is, \( \omega_p \)

\[
2.29069 = 2.41121 \cos(\omega_p/2) \tag{2.23}
\]

\[
\omega_p = 0.6350 \, \text{radians} \tag{2.24}
\]

We follow Step 2 for the transformation and Step 3 to calculate the position of zeros for this polynomial. The zeros (in radians) for this polynomial are calculated from Equation (2.2):

\[
\omega_1 = 2\cos^{-1} \frac{0}{x_0} = \pi, \quad \omega_2 = 2\cos^{-1} \frac{0.1}{x_0} = 3.0586,
\]

\[
\omega_3 = 2\cos^{-1} \frac{0.2}{x_0} = 2.9755, \quad \omega_4 = 2\cos^{-1} \frac{0.3}{x_0} = 2.8921,
\]

\[
\omega_5 = 2\cos^{-1} \frac{0.4}{x_0} = 2.8083, \quad \omega_6 = 2\cos^{-1} \frac{0.5}{x_0} = 2.7238. \tag{2.25}
\]

The transfer function for this FIR filter is (Step2)

\[
H(w) = x_0 \cos(\omega_1/2) \cos(\omega_2/2) \cos(\omega_3/2) \cos(\omega_4/2) \cos(\omega_5/2) \cos(\omega_6/2) \tag{2.26}
\]

Figures 2.5 and 2.6 show the polynomial and magnitude response in \( \text{dB} \) for this low pass FIR filter. The magnitude characteristics in \( \text{dB} \) show that the transition band starts becoming steep when compared with the previous design. It is noteworthy that the phase, in this case, will not be linear because zeros are not symmetrical.
Figure 2.5: Polynomial \((x - 0)(x - 0.1)(x - 0.2)(x - 0.3)(x - 0.4)(x - 0.5)\).

Figure 2.6: Magnitude response in dB of \(H(z)\) of Equation (2.26).
2.3 Conclusion

In this chapter an insight into the connection of object function with filter, using the transformation function, is presented. With this technique we can design linear phase FIR filter with user defined values of zeros of the filter function and decide how many $dB$ down the stop band should be with respect to the pass band; that is, the value of $b$. 