CHAPTER - 05

COMMUTATIVITY PRESERVING DERIVATIONS IN RINGS

§ 5.1 INTRODUCTION

This chapter has been devoted to the study of commutativity-preserving derivations in prime and semi-prime rings. Most of the results of this chapter are based on the work of Daif and Bell [41], Bell and Daif ([10], [9]), Deng and Ashraf [42] etc.

Section 5.2 begins with a result due to Daif and Bell [41] which states that if R is a semi-prime ring and I is a non-zero ideal of R and if R admits a derivation d for which either xy + d(xy) = yx + d(yx) or xy - d(xy) = yx - d(yx) for all x, y ∈ I, then I is a central ideal. In section 5.3 some results based on commutativity of prime and semi-prime rings admitting strong commutativity-preserving derivations and endomorphisms are given. Section 5.4 opens with a result due to Deng and Ashraf [42] in which the mappings F and G of a ring R satisfy [F(x), G(y)] = [x, y], for all x, y in a subset of R. Thus, they introduced a more general concept than the strong commutativity-preserving mappings and obtained certain results on commutativity of a ring R. In the last section of this chapter generalizations of the results of section 5.2 which are due to Bell and Daif [9] have been presented. Examples are also given at the proper places to justify the hypothesis of the various theorems.

§ 5.2

In the year 1992, Daif and Bell [41] investigated commutativity
of a ring $R$ with a derivation $d$ satisfying either of the properties $xy + d(xy) = yx + d(yx)$ or $xy - d(xy) = yx - d(yx)$, for all $x, y$ in an ideal of the ring $R$. In fact they proved the following:

**Theorem 5.2.1** ([41, Theorem 3]). Let $R$ be a semi-prime ring and $I$ a non-zero ideal of $R$. If $R$ admits a derivation $d$ such that either $xy + d(xy) = yx + d(yx)$ or $xy - d(xy) = yx - d(yx)$, for all $x, y$ in an ideal of the ring $R$. Then $I$ is a central ideal.

Following are the consequences of the above theorem.

**Corollary 5.2.1.** Let $R$ be a prime ring and $I$ a non-zero ideal of $R$. If $R$ admits a derivation $d$ such that either $xy + d(xy) = yx + d(yx)$ or $xy - d(xy) = yx - d(yx)$, for all $x, y$ in an ideal of the ring $R$. Then $R$ is commutative.

**Corollary 5.2.2.** Let $R$ be a semi-prime ring admitting a derivation $d$ for which either $xy + d(xy) - yx - d(yx)$ or $xy - d(xy) = yx - d(yx)$, for all $x, y$ in $R$. Then $R$ is commutative.

For developing the proof of the above theorem we need the following Lemmas. Lemma 5.2.1 contain some well-known results on prime and semi-prime rings.

**Lemma 5.2.1 (i).** If $R$ is a semi-prime ring, then the center of a non-zero one sided ideal of $R$ is contained in $Z(R)$. In particular, any commutative one sided ideal of $R$ is contained in $Z(R)$.

(ii) If $R$ is a semi-prime ring, then the centralizer of any non-zero one sided ideal is equal to the center of $R$. Thus, if $R$ has a non-zero central right ideal, then $R$ must be commutative.

(iii) Let $R$ be a prime ring. If $a, b$ are elements of $R$ such that $axb = bxa$, for all $x \in R$, and if $a \neq 0$, then $b = \lambda a$, for some $\lambda$ in the extended centroid of $R$. 

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(iv) Let \( R \) be a prime ring, \( I \) a non-zero right ideal of \( R \), and let \( T \) be an endomorphism of \( R \). If \( T(u) = u \) for all \( u \in I \), then \( T \) is the identity map on \( R \).

(v) If \( R \) is a semi-prime ring and \( I \) is any non-zero ideal of \( R \), then \( C_R(I) \) is equal to the centralizer of \( I \) in \( R \).

Lemma 5.2.2. Let \( R \) be a semi-prime ring and \( I \) be a non-zero ideal of \( R \). If \( z \) in \( R \) centralizes the set \([I, I]\), then \( z \) centralizes \( I \).

Proof. Let \( z \) centralize \([I, I]\). Then for all \( x, y \) in \( I \), we have
\[
z[x , xy] = [x , xy]z,
\]
which can be rewritten as
\[
zx[x , y] = x[x , y]z.
\]
Hence, \([z , x][x , y] = 0\), for all \( x, y \) in \( I \). Replacing \( y \) by \( yz \), we get
\([z , x] I[z , x] = 0\) for all \( x, y \) in \( I \). Since \( I \) is an ideal, it follows that \([z , x]IR[z , x] = 0\) for all \( x, y \) in \( I \).

Proof of Theorem 5.2.3. First we suppose that
\[
xy + d(xy) = yx + d(yx),
\]
for all \( x, y \) in \( I \). (5.2.1)

This can be rewritten as
\[
[x , y] = -d([x , y]),
\]
for all \( x, y \) in \( I \). (5.2.2)

Now, for all \( x, y, z \) in \( I \), we have \([x , y]z + d([x , y]z) = z[x , y] + d(z[x , y])\), which yields \([x , y]z + d([x , y])z + [x , y]d(z) = z[x , y] + d(z)[x , y] + zd([x , y])\). and applying (5.2.2) we conclude that
\[
[x , y]d(z) = d(z)[x , y],
\]
for all \( x, y, z \) in \( I \). (5.2.3)

By Lemma 5.2.2, we see that \( d(I) \) centralizes \( I \); and it follows from (5.2.1) that \([x , y]\) is in the center of \( I \), for all \( x, y \) in \( I \). Another application of Lemma 5.2.2 shows that the ideal \( I \) is commutative.

Hence by Lemma 2.5.1 (ii), \( I \) is in the center of \( R \). In the event that \( xy - d(xy) = yx - d(yx) \) for all \( x, y \) in \( I \), it is equally easy to establish (5.2.3). Hence, \( I \) is a central ideal. \( \square \)
In the year 1994, Bell and Daif [10] studied commutativity of rings admitting a special kind of commutativity preserving map which they called strong commutativity-preserving and they defined it as follows:

Let \( S \) be a non-empty subset of \( R \). A map \( f : R \to R \) is called strong commutativity-preserving on \( S \) if \( [f(x), f(y)] = [x, y] \), for all \( x, y \in S \).

In the mentioned paper authors also proved the following theorem on strong commutativity-preserving map.

**Theorem 5.2.4 ([10, Theorem 1]).** Let \( R \) be a semi-prime ring and let \( I \) be a non-zero right ideal in \( R \). If \( R \) admits a derivation \( d \) which is strong commutativity-preserving on \( I \), then \( I \subseteq Z(R) \).

**Proof.** For all \( x, y \in I \), we have \([x, xy] = [d(x), d(xy)]\), from which it follows easily that

\[
[d(x), x]d(y) + d(x)[d(x), y] = 0, \text{ for all } x, y \in I. \tag{5.2.4}
\]

Replacing \( y \) by \( yr \) gives

\[
[d(x), x](yd(r) + d(y)r) + d(x)(y[d(x), r] + [d(x), y]r) = 0,
\]

which on comparison with (5.2.4) yields

\[
[d(x), x]yd(r) + d(x)y[d(x), r] = 0, \text{ for all } x, y \in I \text{ and } r \in R. \tag{5.2.5}
\]

Let \( r = d(x) \), we see that

\[
[d(x), x]Id^2(x) = \{0\} = [d(x), x]IRd^2(x), \text{ for each } x \in I. \tag{5.2.6}
\]

Since \( R \) is semi-prime, it must contain a family \( A = \{P_\alpha : \alpha \in \Lambda\} \) of prime ideals such that \( \bigcap P_\alpha = \{0\} \). If \( P \) is a typical member of \( A \) and \( x \in I \), (5.2.6) shows that

\[
d^2(x) \in P \text{ or } [d(x), x]I \subseteq P. \tag{5.2.7}
\]

Suppose that \( d^2(x) \in P \). Then for all \( y \in I \), \([x, yd(x)] = [d(x), d(yd(x))]\).
and hence
\[ [x, y]d(x) + y[x, d(x)] = y[d(x), d^2(x)] + [d(x), y]d^2(x) +
[d(x), d(y)] d(x). \]
Therefore
\[ y[x, d(x)] = y[d(x), d^2(x)] + [d(x), y]d^2(x). \]
Thus \( I[x, d(x)] \subseteq P \) and \( I R [x, d(x)] \subseteq P \), so that either \( I \subseteq P \) or \([x, d(x)] \in P\). Either of these conditions implies \([x, d(x)] I \subseteq P\).

Now recalling (5.2.7), we get \([x, d(x)] I \subseteq P\), for all \( x \in I \) and \( P \in A \).

Since \( \cap P_a = \{0\} \), we have \([x, d(x)] I = \{0\}\), for all \( x \in I \). Now it follows from (5.2.5) that \( d(x)I R [d(x), r] = \{0\} \), for each \( x \in I \) and \( r \in R \). Hence for each \( P \in A \) and each \( x \in I \), \( d(x)I \subseteq P \) or \([d(x), R] \subseteq P\).

For fixed \( P \), the sets of all elements \( x \in I \) for which these two conditions hold are additive subgroups of \( I \) whose union is \( I \), therefore
\[ d(I)I \subseteq P \text{ or } [d(I), R] \subseteq P. \quad (5.2.8) \]

Suppose that \( d(I)I \subseteq P \). For arbitrary \( x, y, z \in I \), the condition
\[ [x, yz] = [d(x), d(yz)] \]
reduces to \([d(x), y]d(z) = -d(y)[d(x), z]\); and since the right side of the latter equation is in \( P \), we have \( yd(x)d(z) \in P \).

Thus \( I[d(x), d(z)] = I[x, z] \subseteq P \), for all \( x, z \in I \); and primness of \( P \) implies that either \( I \subseteq P \) or \([x, z] \in P \), for all \( x, z \in I \). In either event \([I, I] \subseteq P \). Returning to (5.2.8), we note that the second alternative gives \([d(I), d(I)] \subseteq P \) and hence \([I, I] \subseteq P \). Now using the fact that \( \cap P_a = \{0\} \), we conclude that \( I \) is a commutative right ideal; and since \( R \) is semi-prime. Lemma 5.2.1 (i) implies that \( I \subseteq Z(R) \).

Corollary 5.2.1. If \( R \) is a semi-prime ring admitting a derivation \( d \) which is strong commutativity-preserving on \( R \), then \( R \) is commutative.

Example 5.2.1. Let \( R \) be a 3-dimensional algebra over a field of
characteristic two, with basis \{u_0, u_1, u_2\} and multiplication defined by

\[ u_i u_j = \begin{cases} u_0 & \text{if } (i, j) = (1, 2) \\ 0 & \text{otherwise} \end{cases} \]

Let \( d \) be the linear transformation on \( R \) defined by \( d(u_0) = 0, \ d(u_1) = u_1 \) and \( d(u_2) = u_2 \). It is easily verified that \( d \) is a derivation which is strong commutativity-preserving on \( R \).

The derivation \( d \) is not an inner derivation. Indeed, it is easy to show that any ring \( R \) admitting an inner derivation which is strong commutativity-preserving on \( R \) must be commutative.

**Example 5.2.2.** Let \( R = R_1 \oplus R_2 \), where \( R_1 \) is a non-commutative prime ring with derivation \( d_1 \) and \( R_2 \) is a commutative domain. Define \( d : R \to R \) such that \( d((r_1, r_2)) = ((d_1(r_1), 0). \) Then \( R \) is a semi-prime, and \( d \) is a derivation which is strong commutativity-preserving on the ideal \( I \) consisting of elements of the form \((0, r_2)\). Thus, under the hypothesis of theorem 5.2.4, we cannot prove that \( R \) must be commutative.

**Remark 5.2.1.** Example 5.2.1 shows that in the hypothesis of theorem 5.2.4, \( R \) must be semi-prime.

§ 5.3

Over the last two decades, a lot has been explored about commutativity-preserving mappings. Inspired by these works, Bell and Daif [10] investigated commutativity of prime and semi-prime rings admitting derivations and endomorphisms, which are strong commutativity-preserving on its certain subsets. More recently in the year 1996, Deng and Ashraf [42] studied a more general concept than the strong commutativity-preserving mappings and they considered the situation when mappings \( F \) and \( G \) of a ring \( R \) satisfy...
\[ F(x), G(y) \] = \[ x, y \], for all \( x, y \) in some subset of \( R \). In fact, they obtained commutativity of \( R \), when the mapping \( G \) is assumed to be either a derivation or an endomorphism of \( R \).

**Theorem 5.3.1 ([42, Theorem 1])**. Let \( R \) be a semi-prime ring, and \( I \) a non-zero ideal of \( R \). If \( R \) admits a mapping \( F \) and a derivation \( d \) such that \( [F(x), d(y)] = [x, y] \), for all \( x, y \in I \), then \( R \) contains a non-zero central ideal.

For developing the proof of the above theorem, we begin with the following lemma.

**Lemma 5.3.1 ([13, Theorem 3])**. Let \( R \) be a semi-prime ring and \( I \) be a non-zero left ideal. If \( R \) admits a derivation \( d \) such that \( d \) is non-zero on \( I \) and \( [x, d(x)] \in Z(R) \), for all \( x \in I \), then \( R \) contains a non-zero central ideal.

**Proof of Theorem 5.3.1.** If \( d(I) = 0 \), then \( I \) is commutative and is a central ideal of \( R \). Hence onward we assume that \( d(I) \neq 0 \), for all \( x, y, z \in I \), we have \( [x, yz] = [F(x), d(yz)] \). This yields that
\begin{equation}
    d(y)[F(x), z] + [F(x), y]d(z) = 0. \tag{5.3.1}
\end{equation}
Replacing \( y \) by \( ry \) in (5.3.1) for \( r \in R \), and using (5.3.1), we find that
\begin{equation}
    d(r)y[F(x), z] + [F(x), r]yd(z) = 0. \tag{5.3.2}
\end{equation}
Let \( \Omega = \{ P_\alpha | \text{ } P_\alpha \text{ is a prime ideal of } R \text{ with } \cap P_\alpha = \{0\} \} \). For a fixed \( P_\alpha \in \Omega \), by (5.3.2), we obtain \( d^2(x)RI[F(x), z] = \{0\} \subseteq P_\alpha \). Thus, either \( d^2(x) \in P_\alpha \) or \( I[F(x), z] \subseteq P_\alpha \).

For a given \( x \in I \), if \( d^2(x) \in P_\alpha \), from \( [x, d(x)y] = [F(x), d(d(x)y)] \), we get \( d(x)[x, y] + [x, d(x)]y = [F(x), d^2(x)y] + \ldots \)
\[ [F(x), d(x)d(y)]. \] This implies that \([x, d(x)]y = [F(x), d^2(x)y] \in P_\alpha \]

i.e. \([x, d(x)]I \subseteq P_\alpha.\) On the other hand \(l[F(x) . I] \subseteq P_\alpha\) and \(lR[F(x) , I] \subseteq P_\alpha.\) Hence, either \(I \subseteq P_\alpha\) or \([F(x), y] \subseteq P_\alpha,\) for all \(y \in I.\) Thus, we find that \([F(x) , y] \subseteq P_\alpha,\) for all \(y \in I.\) Now, replacing \(y\) by \(ry\) for all \(r \in R\) yields that \([F(x) , r]y \in P_\alpha\) i.e. \([F(x) , R]Rl \subseteq P_\alpha\) and consequently, either \([F(x) , R] \subseteq P_\alpha\) or \(l \subseteq P_\alpha.\) But, if \(l \subseteq P_\alpha,\) then obviously \([x, d(x)]I \subseteq P_\alpha.\) Also, if \([F(x), R] \subseteq P_\alpha,\) then the relation \([F(x,d(ry))] = [x,ry]\) implies that \([x,ry] \in P_\alpha\) i.e. \([x,ry] = r[x, y] + r[x, y] \in P_\alpha.\) This together with \(r[x, y] = r[F(x), d(y)] \in P_\alpha,\) gives that \([x, ry]\) is in \(P_\alpha.\) Hence again \([x, d(x)]I \subseteq P_\alpha.\)

Therefore, in both the cases, we have \([x, d(x)]I \subseteq P_\alpha.\) So \([x, d(x)]I \subseteq P_\alpha = \{0\},\) and \([x, d(x)]I[x, d(x)] = 0.\) By semi-primness of \(I,\) we obtain \([x, d(x)] = 0,\) and hence by Lemma 5.3.1., \(R\) has a non-zero central ideal.

\[ \Box \]

**Corollary 5.3.1.** Let \(R\) be a semi-prime ring admitting a derivation \(d,\) and let \(I\) be a non-zero ideal of \(R.\) If for each \(x \in I\) there exists an integer \(n = n(x) > 1\) such that \([d^n(x), d(y)] = [x, y],\) for all \(y \in I,\) then \(R\) contains a non-zero central ideal.

**Corollary 5.3.2.** Let \(R\) be a semi-prime ring. If \(R\) admits mapping \(F\) and a derivation \(d\) such that \([F(x), d(y)] = [x, y]\) for all \(x, y \in R,\) then \(R\) is commutative.

**Proof.** Using similar arguments as used to get equation (5.3.2), we have \(d^2(x)R[F(x),y] = 0.\) Now, replace \(y\) by \(d(y),\) to get \(d^2(x)R[F(x), d(y)] = d^2(x)R[x, y] = 0.\) Thus, for any prime ideal \(P_\alpha\) of \(R,\) we
have either \( d^2(x) \in P_\alpha \) or \([x, y] \in P_\alpha\). But, since \([x, y] = [F(x), d(y)] = [F^2(x), d^2(y)]\). The case \( d^2(x) \in P_\alpha \) implies that \([x, y] \in P_\alpha\) again. Hence \([x, y] = 0\).

Bell and Daif \([10, \text{Theorem 2}]\) studied strong commutativity-preserving endomorphism and prove the following:

**Theorem 5.3.2.** Let \( R \) be a prime ring and \( I \) an essential right ideal of \( R \). If \( R \) admits a non-identity endomorphism \( T \) which is strong commutativity-preserving on \( I \), then \( R \) is commutative.

**Proof.** For all \( x, y \in I \), we have \([x, xy] = [T(x), T(xy)]\), from which it follows that \((T(x) - x)[x, y] = 0\). Replacing \( y \) by \( yr \), \( r \in R \) we get \((T(x) - x)[x, r] = \{0\} = (T(x) - x)I[R, r]\) for all \( x \in I \), \( r \in R \); thus, for \( x \in I \), either \( x \in Z(R) \) or \((T(x) - x)I = \{0\}\). The set of \( x \in I \) for which these alternatives hold are additive subgroups of \( I \). Hence either \( I \subseteq Z(R) \) or \((T(x) - x)I = \{0\}\), for all \( x \in I \). If \( I \subseteq Z(R) \), \( R \) is commutative by Lemma 5.2.1(ii). Thus, we can assume that

\[
(T(x) - x)I = \{0\}, \text{ for all } x \in I. \tag{5.3.3}
\]

Now use the fact that \([x, yx] = [T(x), T(yx)]\) that is \([x,y](T(x)-x) = 0\), for all \( x, y \in I \); and replace \( y \) by \( yw \), \( w \in I \), we obtain

\[
[x, y]I(T(x) - x) = \{0\} = [x, y]I[(T(x) - x), \text{ for all } x, y \in I. \tag{5.3.4}
\]

By Lemma 5.2.1(iv), \( T \) cannot be the identity on \( I \); and it follows easily from (5.3.4) that

\[
[x, y]I = \{0\}, \text{ for all } x, y \in I. \tag{5.3.5}
\]

Let \( V = I \cap T^{-1}(I) \), and note that \( V \) contains all commutators \([x, y]\) for all \( x, y \in I \). If \( I \) is commutative, \( R \) is commutative by Lemma 5.2.1(ii); hence we may assume that \( I \) is not commutative and \( V \neq \{0\}\).

Consider any \( b \in V/\{0\} \). By (5.3.5), we have \([bx, by]b = 0\) for all
\(x, y \in \mathbb{R}\) i.e. \(bxbxb = bxbxb\), for all \(x, y \in \mathbb{R}\). Thus for fixed \(x \in \mathbb{R}\), Lemma 5.2.1(iii) gives us an element \(\lambda = \lambda(x)\) in the extended centroid of \(\mathbb{R}\) such that \(bxb = \lambda b\). It follows that \([bxb, b] = 0 = b[xb, b]\), for all \(x \in \mathbb{R}\). Now if \(b\) is not a left zero divisor, then \(b\) centralizes the non-zero left ideal \(Rb\); hence by Lemma 5.2.1(ii), \(b\) is central and therefore regular. But by (5.3.5), \(b\) is a right zero divisor; consequently \(b\) must be left zero divisor and \(\Lambda_r(b) \neq \{0\}\). Since \(I\) is an essential right ideal, there exists \(a \in I \setminus \{0\}\) for which \(ba = 0\). The fact that \(T\) is strong commutativity-preserving on \(I\) gives \(ab = T(a)T(b)\), and by (5.3.3), we get \(ab = aT(b)\) or \(a(b-T(b)) = 0\). Since \(a\) may be replaced by \(ar\) for any \(r \in \mathbb{R}\), we conclude that \(b-T(b) = 0\). Thus, \(T\) is the identity on \(V\), contradicting Lemma 5.3.4(iv); and we have eliminated the possibility that \(I\), and hence \(\mathbb{R}\), is not commutative. \(\square\)

Note that in a prime ring, any non-zero two-sided ideal is an essential right ideal. Thus we have:

**Corollary 5.3.3.** Let \(\mathbb{R}\) be a prime ring and \(I\) a non-zero two-sided ideal in \(\mathbb{R}\). If \(\mathbb{R}\) admits a non-identity endomorphism which is strong commutativity-preserving on \(I\), then \(\mathbb{R}\) is commutative.

Further, in the year 1996, Deng and Ashraf [42, Theorem 2] generalized the above result as follows:

**Theorem 5.3.3.** Let \(\mathbb{R}\) be a prime ring with characteristic different from two, and let \(T\) be any endomorphism of \(\mathbb{R}\). Let \(I\) be a subring of \(\mathbb{R}\). If for all \(x, y \in I\), \([T(x), T(y)] - [x, y] \in Z(\mathbb{R})\), then \(T\) is strong commutativity-preserving on \(I\).

**Proof.** For a fixed \(x \in I\), let \(\lambda_x = [T(x), T(y)] - [x, y] \in Z(\mathbb{R})\), and let \(I_a\) be the inner derivation defined by \(I_a(x) = [a, x]\). From
Tl_x(y^2) - l_x(y^2) \in Z(R) \) and \( T l_x(y) = l_x(y) + \lambda_y \), we find that \( T(y)Tl_x(y) + Tl_x(y)T(y) - y l_x(y) - l_x(y)y \in Z(R) \), and \( 2\lambda_y T(y) + (T(y) - y) l_x(y) + l_x(y)(T(y) - y) \in Z(R) \). If we substitute \( l_x(y) \) for \( y \), then \( 2\lambda l_x(y)Tl_x(y) + 2\lambda l_x(y) \in Z(R) \). Combining this with \( Tl_x(y) = l_x(y) + \lambda_y \), we have \( (\lambda_x + \lambda l_x)(y) l_x(y) \in Z(R) \) and hence either \( l_x(y) \in Z(R) \) or \( \lambda_x + \lambda l_x(y) = 0 \). Let \( A_x = \{ y \in l \mid \lambda_x + \lambda l_x(y) = 0 \} \) and \( B_x = \{ y \in l \mid l_x(y) \text{ is in } Z(R) \} \). Obviously \( A_x \) and \( B_x \) are additive subgroup of \( l \). Thus, either \( A_x = l \) or \( B_x = l \).

If \( B_x = l \), then \( l_x(xy) = x l_x(y) \in Z(R) \). i.e. either \( x \in Z(R) \) or \( l_x(y) = 0 \). Hence, \( l_x(y) = 0 \) for all \( y \in l \), and \( \lambda_y = 0 \).

If \( B_x \neq l \), it is easy to see that \( B_x = B_x \neq l \), then \( A_x = A_x = l \). We have,

\[
[T(x), T(y)] - [x, y] = [x, [x, y]] - T([x, [x, y]]) \quad \text{and} \\
T(-x, y) = [-x, [x, y]] - T([-x, [x, y]]), \quad \text{for all } y \in l.
\]

Thus, we get \( 2([T(x), T(y)] - [x, y]) = 0 \) i.e. \( 2\lambda_y = 0 \), and hence \( \lambda_y = 0 \) again. Therefore in every case we have \( [T(x), T(y)] = [x, y] \), for all \( x, y \in l \).

In view of Theorem 5.3.2 the above yields the following:

**Theorem 5.3.4.** Let \( R \) be a prime ring with characteristic different from two, \( T \) a non-identical endomorphism of \( R \), and \( l \) be an essential right ideal of \( R \). Suppose that \( [T(x), T(y)] - [x, y] \in Z(R) \), for all \( x, y \in l \). Then \( R \) is commutative.

**Theorem 5.3.5 ([42. Theorem 4]).** Let \( R \) be a semi-prime ring, and \( l \) a non-zero ideal of \( R \). If \( R \) admits a mapping \( F \), and an endomorphism \( T \) such that \( [F(x), T(y)] = [x, y] \) for all \( x, y \in l \), then \( [x, T(x)] = 0 \). Moreover, if \( T \) is non-identical on \( l \cap T^{-1}(l) \), then \( R \) contains a non-zero central ideal.
Proof. The equation \([F(x), T(y^2)] = [x, y^2]\) gives that 
\((T(y) - y)I_y(x) + I_y(x) (T(y) - y) = 0\), for all \(x, y \in I\). Now replace \(x\) by \(ux\), to get 
\((T(y) - y)I_y(u)x + (T(y) - y)uI_y(x) + I_y(u)x(T(y) - y) + uI_y(x) (T(y) - y) = 0\). Since, \((T(y) - y)I_y(x) = -I_y(x) (T(y) - y)\), the last equation reduce to 
\(I_y(u)I_{T(y)-y}(x) - I_{T(y)-y}(u)I_y(x) = 0\), for all \(x, y, u \in I\). (5.3.3)

For any \(r \in R\), substituting \(ru\) for \(u\) in (5.3.3) and using (5.3.3), we have \(I_y(u)I_{T(y)-y}(x) - I_{T(y)-y}(r)I_y(x) = 0\). Taking \(y = x\) and \(r = T(x)\), we obtain \([x, T(x)]I_{T(x)-x} = 0\) i.e. \([x, T(x)]I[x, T(x)] = 0\), and the semiprimness of \(I\) yields that \([x, T(x)] = 0\).

Moreover, if \(T\) is not identical on \(I \cap T^{-1}(I)\), then \([x, yx] = [F(x), T(yx)]\) implies \([y, x](x-T(x)) = 0\). Replacing \(y\) by \(yu\), this gives that \([x, y]u(x-T(x)) = 0\). For a prime ideal \(P_\alpha \in \Omega(\Omega\), being same as in theorem 5.3.1), since \([x, y]\{x - T(x)\} = 0 \in P_\alpha\), and \(([x, I]IR(x-T(x)) \subseteq P_\alpha\), we get either \([x, I]I \subseteq P_\alpha\) or \(T(x) - x \in P_\alpha\).

Notice that \(\{x \in I | [x, I]I \subseteq P_\alpha\}\) and \(\{x \in I | T(x) - x \in P_\alpha\}\) are additive subgroups of \(I\), we have either \(T(x) - x \in P_\alpha\) for all \(x \in I\) or \([x, I]I \subseteq P_\alpha\), for all \(x \in I\). The later case implies that \([I, I]IR[I, I] \subseteq P_\alpha\) and \([I, I]I \subseteq P_\alpha\). In both the cases, we have \([x, y](T(z) - z) \in P_\alpha\) and \((T(z) - z)[x, y] \in P_\alpha\). and hence \([x, y](T(z) - z) = (T(z) - z)[x, y] = 0\) \(\cap P_\alpha\) for all \(x, y, z \in I\), that is, \((T(z) - z)\) centralizes \([I, I]\). Thus, by Lemma 5.2.2., \((T(z) - z)\) centralizes \(I\).

For all \(x \in W = I \cap T^{-1}(I)\), since \(T(x) - x \in I\) and \(T(x) - x\) centralizes \(I\), we have \(T(x) - x \in Z(I) \subseteq Z(R)\), and \([x, T(x)] = [x, T(x) - x] = 0\). The hypothesis \(T\) being not identical on \(W\) gives \(T(x_0) - x_0 \neq 0\) for
some $x_0 \in W$. Let $I' = I(T(x_0) - x_0)$. Then $I'$ is an ideal of $R$, and $T(x_0) - x_0 \in Z(R)$ implies that $0 \neq (T(x_0) - x_0)^2 \in I'$. Thus the equation $[x(T(x_0) - x_0), y(T(x_0) - x_0)] = [x, y] (T(x_0) - x_0)^2 = 0$, for all $x, y \in I$ shows that $I'$ is a non-zero central ideal.

\[\square\]

§ 5.4

Suppose that $R$ is a prime ring having a non-zero right ideal $I$. If $d$ is a derivation on $R$ such that $d(x)d(y) + d(xy) = d(y)d(x) + d(yx)$, for all $x, y \in I$, we say that $d$ is a $I'$-derivation and if $d(x)d(y) + d(yx) = d(y)d(x) + d(xy)$, for all $x, y \in I$, then $d$ is a $I''$-derivation.

In the year 1995, Bell and Daif [9] studied the commutativity of rings admitting $I'$ and $I''$-derivations and proved the following:

Theorem 5.4.1 ([9, Theorem 1]). Let $R$ be a prime ring and let $I$ be a non-zero right ideal in $R$. If $R$ admits a non-zero $I'$-derivation $d$, then either $R$ is commutative or $d^2(I) = d(I)d(I) = \{0\}$.

In order to prove the above Theorem we use the following known results which are given in the form of Lemmas.

Lemma 5.4.1 (i) ([13, Theorem 4]). Let $R$ be a prime ring and $I$ a non-zero right ideal. If $R$ admits a non-zero derivation $d$ such that $[x, d(x)]$ is central for all $x \in I$, then $R$ is commutative.

(ii) ([13, Lemma 3]). Let $I$ be a non-zero left ideal of a prime ring $R$. If $d$ is a non-zero derivation of $R$, then $d$ is a non-zero on $I$.

(iii) ([16, Lemma 2]). Let $I$ be a subring of a ring $R$, and let $d$ be a derivation of $R$ such that $d(xy) = d(x)d(y)$ for all $x, y \in I$, then $d(x)(y-d(y)) = 0$, for all $x, y \in I$.
(iv) If $R$ is a prime ring, the centralizer of any one-sided ideal is equal to the center of $R$.

Proof of (i) and (ii) can be looked in [13, Theorem 4] and [13, Lemma 3] respectively, whereas (iii) is proved in [16, Lemma 2].

Proof of Theorem 5.4.1. Since $d$ is a $I^*$-derivation, we have
\[ [d(x), d(y)] = [d(y), x] + [y, d(x)], \text{ for all } x, y \in I. \] (5.4.1)
Substituting $xy$ for $y$, we get
\[ d(x)[y, x] = [d(x), x]d(y) + d(x)[d(x), y], \text{ for all } x, y \in I. \] (5.4.2)
Replacing $y$ by $yx$ and using (5.4.2), we have
\[ [d(x), x]yd(x) + d(x)y[d(x), x] = 0, \text{ for all } x, y \in I. \] (5.4.3)
In (5.4.2) we substitute $yd(x)$ for $y$, since $I$ is right ideal, to get
\[ d(x)y[d(x), x] - [d(x), x]yd(x) = 0, \text{ for all } x, y \in I. \] (5.4.4)
From (5.4.3) and (5.4.4), we obtain
\[ [d(x), x]y[d(x) + d^2(x)] = 0, \text{ for all } x, y \in I. \] (5.4.5)
Thus (5.4.5) yields that
\[ [d(x), x]IR(d(x) + d^2(x)) = \{0\}, \text{ for all } x, y \in I. \] (5.4.6)
But $R$ is prime, hence for each $x \in I$, we have either $[d(x), x]I = \{0\}$ or $d(x) + d^2(x) = 0$. If $[d(x), x]I = \{0\}$, then (5.4.4) shows that $d(x)y[d(x), x] = 0$, for all $y \in I$, so that $d(x)IR[d(x), x] = \{0\}$. Therefore, either $d(x)I = \{0\}$ or $[d(x), x] = 0$.
On the other hand, suppose $d(x) + d^2(x) = 0$. In (5.4.1), put $y = yd(x)$ to get
\[ y[d(x), d^2(x)] + [d(x), y]d^2(x) = d(y)[d(x), x] + y[d^2(x), x] + [y, x]d^2(x), \text{ for all } y \in I. \] (5.4.7)
But $d(x) = -d^2(x)$. Hence (5.4.7) implies
\[ d(y)[d(x), x] - [y, x]d(x) + [d(x), y]d(x) = y[d(x), x], \text{ for all } y \in I. \] (5.4.8)
If in (5.4.1) we put $y = yx$, we get
Thus, from (5.4.9) and (5.4.8), we get $y[d(x), x] = 0$, for all $y \in I$, that is

$$I[d(x), x] = \{0\}. \quad (5.4.10)$$

But $I$ is a right ideal, hence $[d(x), x] = 0$. Thus, in any event, for each $x \in I$, either $[d(x), x] = 0$ or $d(x)I = \{0\}$.

Suppose that $[d(x), x] = 0$. Then by (5.4.2), we have

$$d(x)[y, x] = d(x)[d(x), y], \quad \text{for all } y \in I. \quad (5.4.11)$$

Replacing $y$ by $yz$ in (5.4.11) and using (5.4.11), we get

$$d(x)y[z, x] = d(x)y[d(x), z], \quad \text{for all } y \in I, z \in R \text{ i.e. } d(x)y[z, x + d(x)] = 0 \text{ for all } y \in I, z \in R. \quad \text{Thus, } d(x)yR[z, x + d(x)] = \{0\}, \quad \text{for all } y \in I, z \in R. \quad \text{Hence we have either } d(x)I = \{0\} \text{ or } x + d(x) \in Z(R).$$

The sets of all elements $x$ for which these conditions hold are additive subgroups of $I$ with union equal to $I$. Hence either $d(I)I = \{0\}$ or $x + d(x) \in Z(R)$, for all $x \in I$. In the latter case, $R$ is commutative by Lemma 5.4.1 (i); therefore we assume that $d(I)I = \{0\}$.

Under this assumption, the condition that $[d(x), d(yz)] = [d(yz), x] + [yz, d(x)]$ for all $x, y, z \in I$ becomes $[d(x), yd(z)] = [yd(z), x] + [yz, d(x)]$, or $y[d(x), d(z)] + [d(x), y]d(z) = y[d(z), x] + [y, x]d(z) + y[z, d(x)] + [y, d(x)]z$. Using (5.4.1) to eliminate the terms with first factor $y$, and noting that the last summand on the right is zero, we get

$$yd(x)d(z) = [x, y]d(z), \quad \text{for all } x, y, z \in I; \quad (5.4.12)$$

hence,

$$yd(z)d(x) = [z, y]d(x), \quad \text{for all } x, y, z \in I. \quad (5.4.13)$$
Thus (5.4.12) and (5.4.13) gives
\[ y[d(x), d(z)] = [x, y]d(z) - [z, y]d(x), \]
for all \( x, y, z \in I \). Using (5.4.1), we reduce this to
\[ xyd(z) - zyd(x) = 0, \text{ for all } x, y, z \in I. \]  
(5.4.14)
Replacing \( x \) by \( xt \) in (5.4.14) and using (5.4.1), we obtain
\[ [x, yz]d(t) = 0, \text{ for all } x, y, z, t \in I. \]  
(5.4.15)
From (5.4.12), we have
\[ [x, zy]d(t) = zyd(x)d(t). \]
Substituting in (5.4.15), we get
\[ zyd(x)d(t) = 0, \text{ for all } x, y, z, t \in I. \]  
(5.4.16)
Since \( zyR d(x)d(t) = \{0\} \) for all \( x, y, z, t \in I \) and since \( I^2 \neq \{0\} \), we conclude that \( d(x)d(t) = 0 \) for all \( x, t \in I \), which is the desired conclusion that \( d(I) d(I) = \{0\} \). In particular,
\[ [d(x), d(t)] = 0, \text{ for all } x, t \in I. \]  
(5.4.17)
Using (5.4.1), (5.4.17) and \( d(I)l = \{0\} \), we have
\[ yd(x) = xd(y), \text{ for all } x, y \in I. \]  
(5.4.18)
Replacing \( y \) by \( yr \) for arbitrary \( r \in R \), we get
\[ xyd(r) = yr(d(x) - xd(y)r), \]
and substituting \( yd(x) \) for \( xd(y) \) now yields
\[ xyd(r) = y[r, d(x)], \text{ for all } x, y \in I, r \in R. \]  
(5.4.19)
Again substituting \( d(z) \) for \( r \), we obtain
\[ xyd^2(z) = y[d(z), d(x)] \]
for all \( x, y, z \in I \), and application of (5.4.17), gives
\[ xyd^2(z) = 0, \text{ for all } x, y, z \in I. \]
Since \( I^2 \neq \{0\} \), we conclude that \( d^2(I) = \{0\} \).

Similarly one can prove the following.

**Theorem 5.4.2.** Let \( R \) be a prime ring and \( I \) a non-zero right ideal. If \( R \) admits a non-zero \( I^* \)-derivation \( d \), then either \( T \) is commutative or \( d^2(I) = d(I)d(I) = \{0\} \).

Followings are the consequences of Theorem 5.4.1 and 5.4.2.
Corollary 5.4.1. Let $R$ be a prime ring and $I$ a non-zero right ideal of $R$. If $R$ admits a non-zero $I^*$- or $I^{**}$-derivation $d$ with $d^2(I) \neq \{0\}$, then $R$ is commutative.

Corollary 5.4.2. Let $R$ be a prime ring and $I$ a non-zero two-sided ideal. If $R$ admits a non-zero $I^*$- or $I^{**}$-derivation $d$, then $R$ is commutative.

§ 5.5.

Long ago Herstein [51] proved that if $R$ is a prime ring of characteristic different from two which admits a non-zero derivation such that $d(x)d(y) = d(y)d(x)$ for all $x,y \in R$, then $R$ is commutative. In view of this result, it seems appropriate to study derivations such that $d(xy) = d(yx)$ for all $x,y$ in some distinguished subset of $R$. Bell and Daif [9] investigated this problem and proved the following:

Theorem 5.5.1. Let $R$ be a prime ring and $I$ a non-zero two-sided ideal of $R$. If $R$ admits a non-zero derivation $d$ such that $d(xy) = d(yx)$ for all $x,y \in I$, then $R$ is commutative.

Proof. Let $c \in I$ be a constant - i.e. an element such that $d(c) = 0$, and let $z$ be an arbitrary element of $I$. The condition that $d(cz) = d(zc)$ yields $cd(z) = d(z)c$. Now for each $x,y \in I$, $[x,y]$ is a constant. Hence,

$$d(z)[x,y] = [x,y]d(z), \text{ for all } x,y,z \in I. \quad (5.5.1)$$

By Lemma 5.2.2 and Lemma 5.4.1(iv), $d(z)$ is central for all $z \in I$, hence $d$ is a $I^*$-derivation and $R$ is therefore commutative by corollary 5.4.2.

The following example justified that in the above theorem, $I$ can not be replaced by a one-sided ideal.
Example 5.5.1. Let \( R \) be a ring of \( 2 \times 2 \) matrices over a field \( F \) and let

\[
R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in F \right\}.
\]

Suppose \( d \) is an inner derivation given by

\[
d(x) = x \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x, \text{ for all } x \in R.
\]

It is readily verified that \( d \) is a \( \lambda \) and \( \Gamma \) -derivation. But \( R \) is not commutative.

However Bell and Daif [9, Theorem 3] further extended the above theorem as follows:

**Theorem 5.5.2.** Let \( R \) be a prime ring of characteristic different from two, and let \( I \) be a non-zero right ideal. If \( d \) is a derivation such that \( d(xy) = d(yx) \) for all \( x, y \in I \), then either \( R \) is commutative, or \( d(I) = \{0\} = d(I)d(I) \).

**Proof.** Writing \( d(xy) = d(yx) \) in the form \( [x, d(y)] = [y, d(x)] \) and replacing \( x \) by \( x^2 \), we get

\[
[y, x]d(x) + d(x)[y, x] = 0, \text{ for all } x, y \in I.
\]

Recalling (5.5.1) and using the fact that characteristic of \( R \) is different from two, we have

\[
[y, x]d(x) = 0 \text{ and } d(x)[y, x] = 0, \text{ for all } x, y \in I. \tag{5.5.2}
\]

In the first of these equalities replace \( y \) by \( yw, w \in I \), we obtain

\[
[y, x]d(x) = \{0\} = [y, x]d(x), \text{ for all } x, y \in I. \text{ Since } d \neq 0, \text{ we can conclude from the usual additive - group argument that}
\]

\[
[y, x]I = \{0\}, \text{ for all } x, y \in I. \tag{5.5.3}
\]

On the other hand, the second equality of (5.5.2) yields that
\[d(x) I[y, x] = \{0\} = d(x)IR[y, x]\] for all \(x, y \in I\). Thus

\[
either x is central or d(x)I = \{0\}, for all x \in I. \quad (5.5.4)
\]

Assume that \(R\) is not commutative, then \(I\) is not central. By (5.5.3) and (5.5.4), we have \([y, x] I = \{0\}\) for all \(x, y \in I\) and \(d(I)I = \{0\}\). These conditions, together with the \(d(xy) = d(yx)\) for all \(x, y \in I\), yield that \(yd(x) = xd(y)\), for all \(x, y \in I\). But this is (5.4.18), and as in the proof of the theorem 5.4.1, we have

\[xyd^2(z) = y[d(z), d(x)], for all x, y, z \in I. \quad (5.5.5)\]

Now by applying \(d\) to the condition \(zd(x) = xd(z)\), we obtain

\[zd^2(x) + d(z)d(x) = xd^2(z)d(x)d(z).\]

Hence \(zd^2(x) + [d(z), d(x)] = xd^2(z)\) and

\[y[d(z), d(x)] = yxd^2(z) - yzd^2(x). \quad (5.5.6)\]

From (5.5.5) and (5.5.6), we obtain

\[yzd^2(x) = [y, x] d^2(z), for all x, y, z \in I. \quad (5.5.7)\]

Since \([y, x]\) is constant, applying \(d\) to (5.5.2) shows that \([y, x]d(I) = \{0\} = [y, x]d^2(I)\) for all \(x, y \in I\), and (5.5.7) yields that \(I^2d^2(I) = \{0\}\). Since \(I^2 \neq \{0\}\) and \(R\) is prime, we conclude that \(d^2(I) = \{0\}\). Finally, since char. of \(R\) is different from two and using the fact that \(d^2(xy) = 0, for all x, y \in I\) we get \(d(I)d(I) = \{0\}\). \(\square\)