4.1 **INTRODUCTION**

In chapter 2, we have considered the effect of perturbations on a KdV equation. There we have the case of nonlinearity and dispersion. In this chapter we consider the effect of dissipation on the KdV equation. Thus in addition to nonlinearity and dispersion, we consider the dissipation term $\frac{1}{3} \epsilon \eta_{xx}$. The resulting equation is a KdVB type equation.

It is to be noted that we are interested in the effect of perturbation on the perturbed KdV type equation which incorporates the dissipation.

While analytical solution exists for the travelling wave solutions of Burgers' and KdV equations, no comparable analytical solution exists for the KdVB equation. Numerical solutions by Grad and Hu (1967) and Johnson (1970) show that when dispersion dominates on dissipation the solution represents an oscillatory shock wave. Studies by Grad and Hu (1967) and Jeffrey and Kakutani (1972) show that when dissipative effect predominates the solution behaves like a
Burgers' shock wave. In this case Jeffrey (1979) has obtained an analytical solution for a KdVB travelling wave.

In our analysis we follow Jeffrey (1979) and Jeffrey and Mohamad (1991). In the case of KdV equation solitary wave solutions exists due to a balancing of nonlinearity and dispersion. Unlike Jeffrey (1979), we are interested in a perturbation of the solitary wave solution due to dissipation.

4.2 ASYMPTOTIC SOLUTION

In this chapter we study the effect of dissipation using three methods. First we consider the asymptotic solution for KdVB travelling wave solution interms of the parameter $\epsilon$.

We consider the equation (1.21) as

$$
\eta_t + \left(1 - \frac{1}{2} \epsilon \right) \eta_x + \left(\frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \eta \eta_x + \frac{1}{3} \epsilon \beta \eta_{xx} + \cdots = 0.
$$

Equation (4.1) can be written as

$$
\eta_t + \left(1 - \frac{1}{2} \epsilon \right) \eta_x + \left(\frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \eta \eta_x + \cdots = 0.
$$

Equation (4.2) has the form

$$
\tau \eta_t + \left(1 - \frac{1}{2} \epsilon \right) \eta_x + \left(\frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \eta \eta_x + \cdots = 0,
$$

where $\tau$ is the characteristic time.

This equation may be interpreted as an equation of motion...
As pointed above by Jeffrey (1979), equation (4.2) has travelling wave solutions. Such solutions have the form

\[ \eta(x,t) = \tilde{\eta}(\zeta), \]  

(4.4)

where \( \zeta = x - \lambda t \).

The boundary conditions at infinity determine the permissible range of values of \( \lambda \). We consider equation (4.2) with the boundary conditions

\[ \tilde{\eta}(-\infty) = \eta_0^- \quad \text{and} \quad \tilde{\eta}(\infty) = \eta_0^+ \]  

(4.7)

Then \( \tilde{\eta} \) must satisfy the equation

\[ - \lambda \frac{d \tilde{\eta}}{d \zeta} + \left( 1 - \frac{1}{2} \varepsilon \right) \frac{d \tilde{\eta}}{d \zeta} + \left( \frac{3}{2} \alpha + \frac{5}{4} \varepsilon \alpha \right) \eta \frac{d \tilde{\eta}}{d \zeta} + \sqrt{\varepsilon} \frac{d^2 \tilde{\eta}}{d \zeta^2} + \mu \frac{d^3 \tilde{\eta}}{d \zeta^3} = 0. \]  

(4.5)

Integrating equation (4.5) with respect to \( \zeta \) we obtain

\[ \mu \frac{d^2 \tilde{\eta}}{d \zeta^2} = - \frac{1}{2} \left( \frac{3}{2} \alpha + \frac{5}{4} \varepsilon \alpha \right) \eta_0^+ - \frac{1}{4} \eta_0^+ - \frac{1}{4} \eta_0^- + \eta_0^- \]  

(4.6)

where \( \lambda \) is the constant of integration.

This equation may be interpreted as an equation of motion.
for a particle or of an anharmonic oscillator under the action of a nonlinear force together with a friction proportional to velocity provided that we regarded \( \zeta \) and \( \eta \) as the time and space co-ordinates respectively.

Using the boundary conditions and the vanishing of derivatives at infinity we get

\[
\lambda = \left( 1 - \frac{1}{2} \epsilon \right) + \frac{1}{2} \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \left( \eta_\infty^+ + \eta_\infty^- \right),
\]

\[
\Lambda = - \frac{1}{2} \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \eta_\infty^+ \eta_\infty^-.
\]

Therefore equation (4.6) becomes

\[
\mu \frac{d^2 \tilde{\eta}}{d\zeta^2} + \sqrt{\frac{d\eta}{d\zeta}} + \frac{1}{2} \left[ \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right] \tilde{\eta}^2 - \left[ \frac{1}{2} \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \left( \eta_\infty^+ + \eta_\infty^- \right) \right] \tilde{\eta} + \frac{1}{2} \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \tilde{\eta}_\infty^+ \tilde{\eta}_\infty^- = 0.
\]

Making the variable changes

\[
\phi = \frac{\tilde{\eta} - \eta_\infty^-}{\eta_\infty^+ - \eta_\infty^-}, \quad \xi = \frac{\left( \eta_\infty^+ - \eta_\infty^- \right) \zeta + \sqrt{2} \eta_\infty^-}{2}, \quad \tau = \frac{\left( \eta_\infty^+ - \eta_\infty^- \right)}{2\sqrt{2}}.
\]

reduces equation (4.9) to

\[
\mu \frac{d^2 \phi}{d\xi^2} + \sqrt{\phi} + \frac{1}{2} \left[ \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right] \phi^2 - \left[ \frac{1}{2} \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \left( \eta_\infty^+ + \eta_\infty^- \right) \right] \phi + \frac{1}{2} \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \phi_\infty^+ \phi_\infty^- = 0.
\]
\[ \tau \frac{d^2 \phi}{d\xi^2} + \frac{d\phi}{d\xi} + \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon\alpha \right) \phi^2 - \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon\alpha \right) \phi = 0, \]  

(4.11)

with the boundary conditions

\[ \phi(-\infty) = 0, \quad \phi(\infty) = 1. \]  

(4.12)

We look for an asymptotic solution of equation (4.11) in the form

\[ \phi(\xi) = \epsilon \phi_1(\xi) + \epsilon^2 \phi_2(\xi) + \epsilon^3 \phi_3(\xi) + \ldots. \]  

(4.13)

We match an asymptotic solution of \( \phi(\xi) \) to the value of \( \phi \) at the point where the curvature of the travelling wave changes sign. Because of the invariant of the equation under an arbitrary fixed translation we can take the origin at this point. Now to determine \( \phi(0) \) we consider the \((\phi, s)\)-phase plane with \( s = d\phi/d\xi \). Then equation (4.11) becomes

\[ \tau \frac{ds}{d\xi} = -s - \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon\alpha \right) \phi^2 + \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon\alpha \right) \phi, \]  

(4.14)

\[ \frac{d\phi}{d\xi} = s. \]  

(4.15)

This system has critical points at the origin \((0,0)\) and at point \((1,0)\), with the origin representing a saddle point and \((1,0)\) a node or focus. These two points corresponds to the boundary conditions to be satisfied by the solution to equation (4.11). From this we conclude that the solution corresponding to the trajectory joining these two critical points must be unique. Thus the point at which \( ds/d\phi = 0 \) will correspond to the point where the curvature of the wave changes sign.

From equation (4.13) we obtain the boundary value problem

\[ \begin{align*}
\tau \frac{d^2 \phi}{d\xi^2} + \frac{d\phi}{d\xi} + \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon\alpha \right) \phi^2 - \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon\alpha \right) \phi &= 0,
\phi(-\infty) &= 0,
\phi(\infty) &= 1.
\end{align*} \]  

(4.19)

This system has critical points at the origin \((0,0)\) and at point \((1,0)\), with the origin representing a saddle point and \((1,0)\) a node or focus. These two points corresponds to the boundary conditions to be satisfied by the solution to equation (4.11). From this we conclude that the solution corresponding to the trajectory joining these two critical points must be unique.
points must be unique. Thus the point at which \( \frac{ds}{d\phi} = 0 \) will correspond to the point where the curvature of the wave changes sign.

Now to find \( \phi(0) \) we seek an expansion of \( s \) in the form

\[
s(\phi) = \epsilon f_1(\phi) + \epsilon^2 f_2(\phi) + \ldots.
\]

From equations (4.14) and (4.15)

\[
\tau s \frac{ds}{d\xi} = -s - \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \phi^2 + \left( \frac{3}{2} \alpha + \frac{5}{4} \epsilon \alpha \right) \phi.
\]

Substituting for \( s \) and equating terms of \( O(\epsilon) \) we get

\[
s(\phi) = \epsilon \frac{5}{4} \alpha (\phi - \phi^2) + \ldots.
\]

Then to the same order it follows that \( \frac{ds}{d\phi} = 0 \) when \( \phi_1(0) = 0 \) and \( \phi_1'(0) = \frac{5\alpha}{16} \). It is to be noted that the equation (4.13) implies that the \( O(1) \) solution corresponds to \( \phi = 0 \) or \( \phi = 1 \). Thus we are considering the solution corresponds to the perturbation only. The same thing follows, if we consider the corresponding terms in equation (4.16) also. From equation (4.11) we thus obtain the boundary value problem

\[
\tau \phi_1''(\xi) + \phi_1'(\xi) - \frac{3}{2} \alpha \phi_1(\xi) = 0,
\]

with \( \phi_1(0) = \frac{1}{2} \) and \( \phi_1'(0) = \frac{5}{16} \alpha \).

Solving equation (4.19) we get
\[ \phi_1(\xi) = \left( 4 + \frac{(5a + 8)t}{\sqrt{6a+1}} \right) \exp \left( \frac{\sqrt{6a+1} - 1}{2t} \right) \xi \]

Equation (4.1) can be written as

\[ \eta_t + \alpha_\xi \eta_x + \beta \eta_x \exp - \frac{\beta t}{\sqrt{6a+1}} - \left( \frac{7}{2} + \frac{(5a + 8)t}{\sqrt{6a+1}} \right) \exp - \frac{\sqrt{6a+1} + 1}{2t} \xi \cdot \]

where \( a, b, c \) and \( d \) are the constant coefficients of \( \eta_x \).

(4.21)

By using equation (4.10) we get

\[ \eta = \eta_\omega + \varepsilon \left( \eta^+ - \eta^- \right) \left\{ 4 + \frac{(5a + 8)t}{\sqrt{6a+1}} \right\} \exp \left( \frac{\sqrt{6a+1} - 1}{2t} \right) \xi - \left( \frac{7}{2} + \frac{(5a + 8)t}{\sqrt{6a+1}} \right) \exp \left( \frac{\sqrt{6a+1} + 1}{2t} \right) \xi + O(\varepsilon^2). \]

(4.22)

Unlike in Jeffrey (1979) \( \tau \) and hence \( \mu \) enters as a nonlinear factor in the solution.

It is to be noted that to the order of \( \varepsilon \) we are not directly making use of the boundary condition at \( \xi = \pm \omega \). This corresponds to the fact that we cannot prescribe such a boundary condition in our physical problem.

In the next two sections we give exact solutions of equation (4.1). The study involves the applications of the methods proposed by Jeffrey and Mohamad (1991) for the general KdVB equation.
4.3 DIRECT METHOD

Equation (4.1) can be written as

$$\eta_t + a\eta_x + b\eta_x + c\eta_{xx} + d\eta_{xxx} = 0. \quad (4.23)$$

where $a$, $b$, $c$ and $d$ are the constant coefficients of $\eta_x$, $\eta_x$, $\eta_{xx}$, and $\eta_{xxx}$ respectively.

We look for a solution of the KdVB equation (4.23) of the form

$$\eta = \eta(\xi), \quad \text{where} \quad \xi = kx - \omega t. \quad (4.24)$$

Here $k$ and $\omega$ are constants to be determined. Substituting in equation (4.23) we get,

$$-\omega \eta' + a\kappa \eta' + b\kappa \eta' + c\kappa^2 \eta'' + d\kappa^3 \eta''' = 0. \quad (4.25)$$

Equation (4.25) can be integrated to get

$$-\omega \eta + a\kappa \eta + \frac{1}{2} b\kappa \eta^2 + c\kappa^2 \eta' + d\kappa^3 \eta''' = C, \quad (4.26)$$

where $C$ is the constant of integration.

The basis of the method is to assume a travelling wave solution of the form

$$\eta = A \sech^m \xi + B \tanh^m \xi + D, \quad (4.27)$$

which is a superposition of solutions of Burgers' equation and KdV equation. In equation (4.27) $A$, $B$ and $D$ are constants to be determined.

There are five undetermined constants and inorder to
obtain a unique solution, it is necessary to find and then solve five independent algebraic equations.

By setting the integration constant $C$ equal to zero and substituting equation (4.27) into equation (4.26) we get,

\[-\frac{\omega + ak}{A \sech^n \xi + B \tanh^m \xi + D} + \frac{1}{2} b k \left(A \sech^n \xi + B \tanh^m \xi + D\right)^2 + c k^2 \left(-A_n \sech^n \xi \tanh \xi + B_m \sech^2 \xi \tanh^{m-1} \xi\right)
+ d k^3 \left(A_n^2 \sech^n \xi \tanh^2 \xi - A_n \sech^{n+2} \xi\right)
+ B_m (m-1) \sech^4 \xi \tanh^{m-2} \xi
- 2 B_m \sech^2 \xi \tanh^m \xi \right) = 0. \quad (4.28)

Then only when $n = 2$ and $m = 1$ in equation (4.28) we can arrive at five simultaneous equations from which to determine the five remaining unknowns. These equations are

\[\frac{1}{2} b k \left(B^2 + D^2\right) - D \left(\omega - ak\right) = 0, \quad (4.29)\]

\[4 d k^3 A + c k^2 B + \frac{1}{2} b k \left(2 AD - B^2\right) - A \left(\omega - ak\right) = 0, \quad (4.30)\]

\[b k D - \left(\omega - ak\right) = 0, \quad (4.31)\]

\[-6 d k^2 + \frac{1}{2} b A = 0, \quad (4.32)\]

Substituting $\xi = \frac{x - c t}{10 d}$ into equation (4.24) and equation (4.27) we get

\[
\eta(x,t) = \frac{3 c^2}{25 d^2} \left\{ \sech^2 (\xi/2) + 2 \tanh (\xi/2) + 2 \right\},
\]

where

These results are in agreement with those obtained by...
\[ \frac{1}{2} bAB - c kA - d k^2 B = 0. \quad (4.33) \]

Equation (4.37) represent a travelling wave solution to the KdVB equation (4.23).

It follows from equations (4.29) to (4.33) that

\[ k = \pm \frac{c}{10d}, \quad (4.34) \]

We also find the travelling wave solution \( \eta \) given by equation (4.37(a)) has the limit 0 as \( \xi \to -\infty \) and the limit \( \omega = \frac{6c^3}{250d^2} \pm \frac{ac}{10d} \), while the travelling wave solution (4.37(b)) has the limit 0 as \( \xi \to -\infty \) and the limit \( A = \frac{3c^2}{25bd} \), \( B = \pm \frac{6c^2}{25bd} \) and \( D = \pm \frac{6c^2}{25bd} \).

Substituting these results into equation (4.24) and equation (4.27) with \( k = \frac{c}{10d} \) we get

\[ \eta(x,t) = \frac{3c^2}{25bd} \left\{ \text{sech}^2(\xi/2) + 2 \tanh(\xi/2) + 2 \right\}, \quad (4.37a) \]

where

\[ \xi = \frac{c}{5d} \left[ x - \left( \frac{6c^2}{25bd} + a \right) t \right]. \quad (4.38) \]

Similarly when \( k = -\frac{c}{10d} \),

\[ \eta(x,t) = \frac{3c^2}{25bd} \left\{ \text{sech}^2(\xi/2) - 2 \tanh(\xi/2) - 2 \right\}, \quad (4.37b) \]

where

\[ \xi = \frac{12(d/b)}{5d} \left[ x + \left( \frac{6c^2}{25bd} - a \right) t \right]. \quad (4.39b) \]

These results are in agreement with those obtained by Jeffrey. Using equation (4.35), the coefficient of
Equation (4.37) represents a travelling wave solution to the KdVB equation (4.23).

We also find the travelling wave solution \( \eta \) given by equation (4.37a) has the limit 0 as \( \xi \to -\infty \) and the limit \( 12c^2/25bd \) as \( \xi \to \infty \), while the travelling wave solution \( \eta \) given by equation (4.37b) has the limit 0 as \( \xi \to -\infty \) and the limit \( -12c^2/25bd \) as \( \xi \to \infty \).

It is also to be noted that the travelling wave solutions of either Burgers' equation or KdV equation cannot be obtained as limiting cases from the solution (4.37).

4.4 SERIES METHOD

We seek a solution of the KdVB equation (4.23) in the form

\[
\eta(x,t) = \sum_{j=0}^{2} \eta_j F^{j-2},
\]

(4.38)

where \( \eta_j \) and \( F \) are functions of \( x \) and \( t \) respectively. Substituting equation (4.38) into equation (4.23) we find that

\[
\eta_0 = -12(d/b) F_x^2,
\]

(4.39a)

\[
\eta_1 = \frac{12}{5} \left( c/b \right) F_x + 12(d/b) F_{xx},
\]

(4.39b)

where \( \eta_0 \) and \( \eta_1 \) are the coefficients of the powers of \( F^{-5} \) and \( F^{-4} \) respectively. Using equation (4.39), the coefficient of
\( F^{-3} \) can be written in the form

\[
F_t + \left( a - \frac{c^2}{25d} \right) F_x + \frac{6}{5} cF_{xx} + 4dF_{xxx}
\]

\[-3dF_x^{-1} F_{xx}^2 + 6F_x \eta_2 = 0. \quad (4.40)
\]

A solution \( \eta(x,t) \) of the KdVB equation (4.23) may then be written as

\[
\eta(x,t) = \frac{\eta_0}{F^2} + \frac{\eta_1}{F} + \eta_2(x,t). \quad (4.41)
\]

Substituting from equations (4.39), equation (4.41) can be written as

\[
F(x,t) = 1 + \exp(kx - \omega t), \quad (4.43)
\]

where \( k \) and \( \omega \) are constants to be determined.
form of equation (4.43) will be a solution of the KdVB equation provided

\[ 6c^2 k^2 + 5acak - 5c\omega = 0, \quad (4.44a) \]

and

\[ c^2 k^2 - 25d^2 k^4 = 0. \quad (4.44b) \]

From equation (4.44) we have,

\[ \omega = \frac{6c^3}{125d^2} \pm \frac{ac}{5d}, \quad (4.45a) \]

and

\[ k = \pm \frac{c}{5d}. \quad (4.45b) \]

Thus corresponding to \( k = \frac{c}{5d} \) and

\[ \xi = \frac{c}{5d} \left[ x - \left( \frac{6c^2}{25bd} + a \right) t \right], \]

we have

\[ \eta(x,t) = 12(d/b) \frac{\partial^2}{\partial x^2} \left\{ \log \left[ 1 + \exp(kx - \omega t) \right] \right\} \]

\[ + \frac{12c}{5d} \frac{\partial}{\partial x} \left\{ \log \left[ 1 + \exp(kx - \omega t) \right] \right\} \]

\[ = \frac{3c^2}{25bd} \left\{ \text{sech}^2(\xi/2) + 2 \tanh(\xi/2) + 2 \right\}. \quad (4.46) \]

Similarly corresponding to \( k = -\frac{c}{5d} \) and

\[ \xi = -\frac{c}{5d} \left[ x + \left( \frac{6c^2}{25bd} + a \right) t \right], \]
we have

\[ \eta(x,t) = \frac{3c^2}{25bd} \left( \text{sech}^2(\xi/2) - 2 \tanh(\xi/2) - 2 \right) \].

(4.47)

These results are identical with those we have obtained using the direct method.

4.5 DISCUSSION

As a further step in the study of equation (1.15) we have considered here equation (1.21) which includes a term \( \frac{1}{3} \epsilon \beta \eta_{xx} \) in addition to nonlinearity and dispersion. Unlike the usual cases considered else where, \( \frac{1}{3} \epsilon \beta > 0 \) in our case.

First, we have obtained an asymptotic solution. Here we see that unlike in Jeffrey (1979), the dispersion coefficient enters nonlinearly in the solution. It has already been pointed out by Jeffrey that KdVB shock solution is sensitive to a perturbation at the origin.

We have also obtained analytic solutions using two different methods. It is found that the travelling wave solutions of either Burgers' equation or KdV equation cannot be obtained as limiting cases from the solution of KdVB equation. The results obtained by series method are identical with those obtained using direct method.