Chapter 4
Multiplication Modules
CHAPTER-IV

Multiplication Modules

1. Introduction

Let $R$ be a commutative ring with identity. In 1974, Mehdi, Fazal [14] introduced the concept of multiplication module as: A left $R$-module $M$ is said to be a multiplication module if for every pair of submodules $L, N$ of $M$, $L \subseteq N$ implies that there exists an ideal $A$ of $R$ such that $L = AN$.

A submodule $N$ of a Module $M$, which is not equal to $M$ over a ring $R$ is said to be a prime submodule of $M$ if $AN_1 \subseteq N$ and $N_1 \subseteq N$ implies that $AM \subseteq N$, where $A$ is an ideal of $R$ and $N_1$ is a submodule of $M$. In 1976, Mehdi, F. [15] defined weak multiplication module as: A module $M$ over $R$ is called a weak multiplication module if $N \subseteq P$, where $N$ is a submodule of $M$ and $P$ is a prime submodule of $M$, implies that there exists an ideal $A$ of $R$ such that $N = AP$.

In the Lemma 4.14 it is proved that the homomorphic image of weak multiplication module is weak multiplication
module. In general weak multiplication module need not be a multiplication module, example 4.19.

In the last section of chapter generalized multiplication modules are studied. $\mathbb{Z}(p^\infty)$ is an example of generalized multiplication module which is not a multiplication module. Finally the structure of torsion free generalized multiplication modules and torsion generalized modules are studied.

2. Results on Multiplication Modules

In this section we study some results on multiplication modules.

4.1 Lemma [14]: For any multiplication module $M$ all its submodules and quotient modules are multiplication modules.

Proof: Trivial.

If $S$ is any multiplicative subset of a ring $R$, then $M_S$ denotes the quotient module of $M$ with respect to $S$. For each $R$–submodule $N$ of $M$, $N^c$ denotes the extension of $N$ and for each $R_S$–submodule $L$ of $M_S$, $L^c$ denotes the contraction of $L$.  

45
4.2 Lemma [14]: If $M$ is a multiplication $R$-module and $S$ is a multiplicative subset of $R$, then the quotient module $M_S$ is a multiplication $R_S$-module.

**Proof:** Consider any two $R_S$-submodules $N$ and $L$ of $M_S$ such that $N \subseteq L$. Then in $M$, $N^c \subseteq L^c$. Since $M$ is a multiplication $R$-module there exists an ideal $A$ of $R$ such that $N^c = AL^c$. Therefore $N^c = (AL^c)^c = A^cL^c$, hence $N = A^cL$. Therefore $M_S$ is a multiplication $R_S$-module. This completes the proof.

4.3 Lemma [14]: A multiplication $R$-module $M$ is Noetherian (Artinian) if $R$ is itself Noetherian (Artinian).

**Proof:** Consider an ascending sequence $M_1 < M_2 < \ldots$ of submodules of $M$. If $A_i = (M_i : M) = \{a_i | a_i \in R$ and $a_iM \subseteq M_i\}$ then $M_i = A_iM$ and $A_1 < A_2 < \ldots$. Since $R$ is a Noetherian ring there exists an integer $n$ such that $A_n = A_{n+1}$. Therefore $M_n = M_{n+1}$ and hence $M$ is Noetherian. Similarly by taking $R$ Artinian it can be shown that $M$ is Artinian. This completes the proof.

In 1981, Barnard, A. [2] defined multiplication module as: An $R$-module $M$ is a multiplication module if every submodule of $M$ is of the form $IM$, for some ideal $I$ of $R$. 
Clearly this definition is equivalent to the definition given by Mehdi, Fazal [14]

4.4 Proposition [2]: Let $M$ be a multiplication module over a ring $R$ and let $I$ be an ideal of $R$ contained in the Jacobson radical of $R$. Then $M=IM$ implies $M=0$.

Proof: Given $x \in M$, there is an ideal $E$ of $R$ such that $Rx=EM$. Thus $Rx=EM=EIM=IEM=Ix$. Therefore, for some $a \in I$, $x=ax$. But $1-a$ is a unit in $R$. Therefore $x=0$ and hence $M=0$. This completes the proof.

4.5 Lemma [2]: Let $R$ be a ring.

(i) Let $S$ be a multiplicatively closed subset of $R$. If an $R$–module $M$ is a multiplication module, then the $S^{-1}R$–module $S^{-1}M$ is a multiplication module.

(ii) A finitely generated $R$–module $M$ is a multiplication module if and only if, the $R_p$–module $M_p$ is a multiplication module for all prime/maximal ideals $P$ of $R$.

4.6 Lemma [2]: Let $R$ be a ring and let $M$ be an $R$–module whose annihilator is contained in only finitely many maximal
ideals $\mathfrak{R}_1, \ldots, \mathfrak{R}_n$ of $R$. If $M_{\mathfrak{R}_1}$ is a cyclic $R_{\mathfrak{R}_1}$-module for $1, \ldots, n$, then $M$ is a cyclic $R$-module.

**Proof:** See [2].

**4.7 Proposition** [2]: Let $R$ be a semi-local ring. Then an $R$-module is a multiplication module if and only if, it is cyclic.

**Proof:** In view of Lemma 4.5 and 4.6, it is enough to show that every multiplication module over a local ring is cyclic.

Let $R$ be a local ring with maximal ideal $\mathfrak{M}$ and let $M$ be a non-zero multiplication module over $R$. By Proposition 4.4, we can choose an element $x \in M - \mathfrak{M}M$. Then $Rx = IM$, where $I$ is an ideal of $R$ and $I \not\subseteq \mathfrak{M}$. Therefore $I = R$ and so $M = Rx$.

**4.8 Proposition** [2]: A finitely generated module is a multiplication module if and only if, it is locally cyclic.

**4.9 Proposition** [2]: Every finitely generated Artinian multiplication module is cyclic.

**Proof:** Let $M$ be a finitely generated Artinian module over a ring $R$, then $R/\text{Ann}(M)$ is isomorphic to a submodule of the direct sum of finitely many copies of $M$, and is therefore an Artinian ring. Thus $\text{Ann}(M)$ is contained in only finitely
many maximal ideals of $R$. Hence the result follows from Lemma 4.6 and Proposition 4.8.

The following Lemma is instrumental to the further development of the subject.

4.10 Lemma [14]: If $M$ and $N$ are $R$–modules, where $N$ is finitely generated, then $M \oplus N$ is a multiplication $R$–module if and only if $M$ and $N$ are multiplication $R$–modules and $\text{ann}(M) + \text{ann}(N) = R$.

4.11 Theorem [14]: If $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ is a direct sum of finite number of finitely generated modules over a ring $R$, then $M$ is a multiplication $R$–module if and only if each $M_i$ is a multiplication $R$–module and

$$\text{ann}(M_i) + \bigcap_{j \neq i} \text{ann}(M_j) = R.$$  

Proof: Suppose $M$ is multiplication $R$–module then $M_i = A_i M = A_i (M_1 \oplus M_2 \oplus \ldots \oplus M_n)$ for some ideal $A_i$ of $R$. Then $A_i M_j = (0)$ for every $j \neq i$ and $M_i = A_i M_i$. Therefore $A_i \subseteq \bigcap_{j \neq i} \text{ann}(M_j)$ and $A_i + \text{ann}(M_i) = R$. Therefore $\text{ann}(M_i) + \bigcap_{j \neq i} \text{ann}(M_j) = R$. Conversely, suppose that

$$\text{ann}(M_i) + \bigcap_{j \neq i} \text{ann}(M_j) = R.$$
If $M' = M_1 \oplus M_2 \oplus \ldots \oplus M_{i-1} \oplus M_{i+1} \oplus \ldots \oplus M_n$ then $M = M_i \oplus M'$ and $R = \text{ann}(M_i) + \text{ann}(M')$ because $\text{ann}(M') = \bigcap_{j \neq i} \text{ann}(M_j)$. Therefore by Lemma 4.10 $M$ is a multiplication $R$-module.

**4.12 Corollary ([14]):** A torsion free multiplication module over an integral domain $D$ is uniform. In particular a non-zero vector space $V$ over a filed $F$, which is multiplication module is of dimension one.

**Proof:** Let $M$ be a torsion free, multiplication module over an integral domain $D$. Suppose that $M$ is not uniform. Then it has two non-zero submodules $N_1$ and $N_2$ with $N_1 \cap N_2 = (0)$. Then $N = N_1 \oplus N_2 \subseteq M$, is a multiplication module. Hence by Theorem 4.11 $\text{ann}(N_1) + \text{ann}(N_2) = D$. This is not possible as $N_1$ and $N_2$ are both torsion free. The second part follows from Theorem 4.11.

This corollary shows that the direct sum of even two copies of a multiplication module need not be a multiplication module, which is evident by the following example.

**4.13 Example:** Any vector space of dimension two is a direct sum of two multiplication modules, however it is not a multiplication module.
3. Weak multiplication modules

4.14 Lemma ([16], p.50): The homomorphic image of a weak multiplication module is a weak multiplication module.

**Proof:** Suppose \( f: M \rightarrow N \) is an epimorphism from a weak multiplication module \( M \) to a module \( N \). If \( P \) is a prime submodule of \( N \) then \( f^{-1}(P) \) is a prime submodule of \( M \). Therefore if \( N_1 \subseteq P \), then \( f^{-1}(N_1) \subseteq f^{-1}(P) \) and hence \( f^{-1}(N_1) = A f^{-1}(P) \) for some ideal \( A \) of \( R \). Therefore \( N_1 = f^{-1}(A)P \). This proves that \( N \) is a weak multiplication module.

4.15 Proposition [15]: If \( M \) is a weak multiplication \( R \)-module and \( S \) is a multiplicatively closed subset of \( R \), then the quotient module \( M_S \) is weak multiplication \( R_S \)-module

**Proof:** See [15].

4.16 Proposition [15]: If \( M \) is a weak multiplication module and \( P, P' \) are prime submodules of \( M \) such that \( P \) is strictly contained in \( P' \), then \( P = (P: M)P' \).

**Proof:** By definition of weak multiplication module \( P = A P' \) for some ideal \( A \) of \( R \). Now \( P' \not\subseteq P \) implies that \( AM \subseteq P \), that is \( A \subseteq (P: M) \). Now \( P = AP' \subseteq (P: M) P' \subseteq P \). Therefore \( P = (P: M) P' \). This completes the proof.
4.17 Proposition ([16], p.53): Let $M$ be a weak multiplication module, $P$ be a prime submodule of $M$. Then there is no submodule strictly between $P$ and $AP$ for any maximal ideal $A$ of $R$.

**Proof:** Assume that there is a submodule $N$ of $M$ such that $AP < N < P$. Then there exist an ideal $C$ of $R$ such that $N = CP$. If $C \subseteq A$, then $CP \subseteq AP$, that is $N \subseteq AP$. If $C \not\subseteq A$, take $c \in C \setminus A$. Since $(c) \subseteq C$ then $(c)P \subseteq CP = N$. Now $((c)+A)P = (c)P + AP \subseteq N$. But $A$ is maximal therefore $(c)+A = R$ that is $P \subseteq N$. Therefore there is no submodule strictly between $P$ and $AP$. This completes the proof.

4.18 Proposition [15]: Let $M$ be a weak multiplication module over a quasi-local ring $R$, then any prime submodule $N$ of $M$ is cyclic.

**Proof:** Let $P$ be a maximal ideal of $R$. Suppose that $N = PN$. Consider any $0 \neq x \in N$. Then $Rx = AN$ for some ideal $A$ of $R$. Then $P(x) = PAN = AN = Rx$. This implies that $x = px$ for some $p \in P$. Thus $x = 0$, as $(1-p)$ is unit. This is a contradiction, hence $N \neq PN$. Choose $x \in N \setminus PN$ then $Rx = AN$. Now either $A = R$ or $A \subseteq P$. If $A \subseteq P$, then $Rx = AN \subseteq PN$, a contradiction. Therefore $Rx = RN = N$. Hence $N$ is cyclic $R$-module.
Following are examples of weak multiplication modules, which are not multiplication modules.

4.19 Examples [15]:

1. Q is a weak multiplication $\mathbb{Z}$-module but not a multiplication module.

2. Let $R$ be a local discrete valuation ring of rank one with maximal ideal $M$. Consider $N = R/M \oplus R$ as $R$-module. Then $N$ is a weak multiplication ideal but not multiplication ideal.

Proof: See [15].

4. Generalized multiplication modules

4.20 Definition [18]: A module $M_R$ is said to be a generalized multiplication module if for every pair of proper submodules $K$ and $N$ of $M$, $K \subset N$ implies $K = NA$ for some ideal $A$ of $R$.

It is clear from the definition of generalized multiplication module that all the proper submodules of a generalized multiplication module are multiplication modules.
4.21 Example [18]: The quasi-cyclic group $\mathbb{Z}(p^\infty)$ is a generalized multiplication module which is not a multiplication module.

Lemma 4.3 shows that a multiplication module over a Noetherian ring is always Noetherian. Such is not the case with generalized multiplication modules (example 4.21). But if $M$ is a generalized multiplication module over a Noetherian ring $R$ having a maximal submodule, then $M$ is Noetherian.

4.22 Theorem ([16], p. 62): If $M$ is a generalized multiplication module over a Noetherian ring $R$ having a maximal submodule. Then $M$ is Noetherian.

Proof: Let $M$ be a generalized multiplication module over a Noetherian ring having a maximal submodule $N$. As every proper submodule of $M$ is multiplication module, $N$ is multiplication module, hence Noetherian. Take $0 \neq x \in M \setminus N$. Then $M = N + (x)$ and $M$ is Noetherian. This completes the proof.

Now we study Torsion free generalized multiplication modules and Torsion generalized multiplication modules.

4.23 Lemma [18]: Let $M$ be a faithful generalized multiplication module over a Noetherian ring $R$. Then—
(i) Either \( M \) is finitely generated or every proper submodule of \( M \) is finitely generated and small in \( M \).

(ii) If \( R = R_1 \oplus R_2 \), then \( M \) is finitely generated.

4.24 Lemma [18]: If \( M \) is a generalized multiplication module over a domain \( D \), such that \( M \) is not a torsion free module, then \( M \) is a torsion module.

Proof: Let \( N \) be a torsion submodule of \( M \). Now \( N \neq 0 \) and \( M/N \) is a torsion free module. So if \( M/N \neq 0 \), we can find a proper submodule \( T/N \) of \( M/N \). Then \( N = TA \) for some non–zero ideal \( A \) of \( D \). This implies that \( T \) is a torsion submodule of \( M \) and hence \( N = T \). This is a contradiction. This proves that \( M \) is a torsion module.

4.25 Lemma [18]: If \( M \) is torsion free generalized multiplication module over a domain \( D \), then \( D \) is a Dedekind domain and \( M \) is a uniform \( D \)-module.

Proof: As \( M \) is torsion free, \( D \) is embeddable in \( M \). So \( D \) is a multiplication module and hence \( D \) is a Dedekind domain. For the other part, suppose \( M \) is not uniform. Then we can find two non–zero submodules \( A \) and \( B \) of \( M \) such that \( A \cap B = 0 \) and \( A \oplus B < M \). Then for some ideal \( C \) of \( D \),
A=(A+B)C, which is not possible. Hence M is a uniform D-module. This completes the proof.

4.26 Theorem [18]: If M is torsion free generalized multiplication module over a domain D which is not a field, then either M is a multiplication module isomorphic to an ideal of D, or M is isomorphic to the total quotient field Q of D and D is a discrete valuation ring of rank one.

Proof: If M is torsion free generalized multiplication module over a domain D, then D is a Dedekind domain and M is a uniform D-module (By Lemma 4.25). Thus if M is finitely generated then M is isomorphic to an ideal of D, and M is multiplication module. So let M not be finitely generated. We can regard D⊂M⊂Q.

Let M≠Q. Then M is not divisible as D-Module, so far some a≠0, Ma≠M. This gives Ma is finitely generated. Then M≥Ma implies that M is finitely generated. This is a contradiction. Hence M=Q. Suppose, D is not a discrete valuation ring. Consider any prime ideal P≠0 of D, then D<P⊂M=Q. This gives D_P is a finite D-Module; this is a contradiction. Hence D is a discrete valuation ring. This completes the proof.
Gilmer and Mott [4] have proved that any indecomposable multiplication ring is either a Dedekind domain or a special primary ring. Its consequence is:

4.27 Lemma [18]: Any Noetherian multiplication ring is a direct sum of Dedekind domains and special primary rings.

4.28 Definition [18]: A module is said to be uniserial if it has a unique composition series.

4.29 Lemma [18]: Any module over an Artinian principal ideal ring is a direct sum of unisserial modules.

4.30 Lemma [18]: Any multiplication module over an Artinian ring is a direct sum of finitely many uniserial modules. Further if M is a faithful multiplication module over a quasi-local ring R, and if R is not a domain, then M is uniserial and injective.

Thus any finite length multiplication module over a quasi-local ring, is quasi-injective.

4.31 Theorem [18]: Let M be a faithful torsion generalized multiplication module over a Noetherian domain R. Then M has an infinite properly ascending chain of submodules

\[ 0 = x_0 R < x_1 R < \ldots < x_n R \ldots < M \]
module over \( P/P^n \) for some \( n \). Then by Lemma 4.30 \( xR+yR \) is uniserial. Therefore either \( xR \subseteq yR \) or \( yR \subseteq xR \) and each \( xR \) is of finite length. Further if \( xR+yR=zR=R/A \) for some ideal \( A \), then \( R/A \) is a special primary ring with maximal ideal \( P/A \), hence all composition factor of \( xR+yR \) are isomorphic to \( R/P \).

This proves the first part.

Now consider \( E=E_R(M) \). Then by Matlis ([12], Theorem (3.6)) \( E=E_R(R/P) \) is an \( \hat{R}_P \) module, where \( \hat{R}_P \) is the \( P \)-adic completion of \( R_P \). Further by Matlis ([12], Theorem (3.7))

\[
\hat{R}_P = \text{Hom}_R(E,E).
\]

Since each \( x_nR \) is quasi injective by Lemma 4.30, using Johnson an Wong [8] we get that each \( x_nR \) is an \( \hat{R}_P \) - sub module of \( E \). Hence \( M \) itself is an \( \hat{R}_P \) - sub module of \( E \). Hence by Johnson and Wong [8], \( M \) is a quasi injective \( \hat{R}_P \) - module. Consider an annihilator \( A \) of \( M \) in \( \hat{R}_P \). Then \( S=\hat{R}_P/A \) is a complete local ring and \( R \) is embeddable in \( S \).

Further \( M \) is a quasi-injective uniform \( S \)-module; each \( x_nR \) is an \( S \)-module. For each \( n \geq 1 \), let

\[
A_n = \{ s \in S : x_ns = 0 \}.
\]
Then $x_nA_n = 0$. The maximal ideal $N$ of $S$ is $\hat{P}\hat{R}/A$. By ([23], Chapter–VIII, Theorem (13)), $A_n \subseteq N^2$ for some $n$. However, by (4.30) $S/A_n$ is a special primary ring. Thus $S/N^2$ is special primary ring and hence $N/N^2$ is a simple $S$–module. This implies $N$ is principal, and $S$ is a complete discrete valuation ring. However every infinite length torsion, uniform, module over a Dedekind domain is always injective, we get $N$ is injective as an $S$–module. This completes the proof.

It follows from the above proof that if $R$ is a complete local domain, admitting a faithful, torsion generalized multiplication module $M$, then $R$ is a discrete valuation ring and $M$ is an injective $R$–module.