CHAPTER – 4

DESIGN OF SPECIAL BI NOTCH FILTERS *

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* The results of this Chapter have been published in the following papers:

- R. Deshpande, B. Kumar, S. B. Jain, “On the design of multi notch filters”
  (Appeared on line in International Journal of Circuit Theory and Applications,
  Wiley Intersciences) DOI: 10.1002 / cta.725.

- R. Deshpande, B. Kumar, S. B. Jain, “Design simplifications for bi notch filters”
  in International Journal of Analog Integrated Circuits and Signal Processing,
  DOI: 10.1007 / s10470-010-9533-1
A methodology for designing FIR bi notch filters derived from second order prototype IIR notch filters is suggested. Rejection bandwidth for the designed filter can be controlled by suitable choice of ‘r’, the pole radius of the IIR prototype notch filter. The suggested bi notch filter can also be adapted to eliminate second, third and fourth order harmonics of periodic noise besides the fundamental noise frequency component. Such filters find application in switched mode power regulators meant to supply ripple free dc for sensitive electronic gadgets and also in a host of communication circuits. A special case when two notch frequencies \( \omega_1 \) and \( \omega_2 \) are such that \((\cos \omega_1)(\cos \omega_2) = -1/2\) has also been discussed. The IIR bi notch filter design for this special case results reduction in the number of multipliers without affecting the response of the desired notch filter. For the above mentioned condition, the required design weights of FIR bi notch filter reduce to almost half in number resulting in reduced computations. The number of zero weights further reduces with increase in ‘r’ value. Also, the frequency response becomes better with reduced ripples in the pass bands, when ‘r’ is increased and the length ‘L’ of the FIR notch filter is chosen appropriately.

4.1 INTRODUCTION

In many practical applications, the noise is periodic in nature and the harmonic components of such noise are of major concern besides its fundamental frequency component. Elimination of second, third and fourth harmonic contents of periodic noise are desirable as these are the serious sources of disturbance. For the purpose of removing harmonics along with the fundamental noise frequency, a bi notch filter has been suggested. Such a bi notch filter can be used to eliminate second and third order harmonic components to overcome the harmonic distortion in applications such as active-RC inverting lossy integrators having all nonlinear components reported in [48]. Standard design techniques to determine the weights of the filters, such as Fourier Series method, Windowing method or Frequency Sampling method may be used. However, these methods become inefficient for
design of multi notch filters. Designing an analog / digital filter from a prototype filter is a well known design approach. Many such designs can be found in the literature [36, 49]. A number of design approaches for notch / multi notch filters have also been suggested in literature [18, 31-39]. These methods are analytical [31-34, 36 – 39, 18] and a few among them are also recursive [32, 33]. In some of these methods the formulas for computing the design weights were arrived at after involved mathematical manipulations. Moreover, those formulas take considerable time to compute the design weights. Novel analytical designs for maximally flat (MF) and equiripple (ER) FIR notch filters have been proposed in [31-33], based on Zolotarev polynomials and Jacobs’s elliptical functions. Zahradnik and Vlcek [33] have suggested recursive algorithms for computation of impulse response of notch filters, which have been obtained by using highly involved mathematical manipulations using many results proposed by them in [35]. These notch filter designs are optimal and very useful ones though the calculations of the impulse response h(n) requires computations of a number of parameters.

In many practical situations fast computation of design weights h(n) is necessary such as for on-line operations, direction finding, parameter estimation, ARMA control systems or use of radar in chaff clutter conditions, to mention a few. In such cases, we may accept suboptimal but quick design of a filter if the calculation of design parameters could be instant. Taking all these factors into account an alternate design approach for notch filter is proposed. The suggested design may not be optimal in the sense of [33], but the novelty and simplicity of the proposed approach is the major advantage. In this work, the methodology suggested by Dutta Roy et al. [36] has been extended due to its better control of rejection bandwidth, besides straight forward formulas for design weights. Also, this method avoids FFT algorithm [39] required in the analytical design of narrow band FIR filters. In the proposed design, the transfer function N(z) for FIR notch filter is evolved from the transfer function H(z) of second order IIR prototype notch filter. Two such FIR filters with different notch frequencies are then
cascaded to obtain proposed FIR bi notch filter of length ‘2L’. The resulting FIR bi notch filter effectively eliminates noise at chosen notch frequencies. Such notch filters may be exploited for removing the major ripple contents in regulated power supply sources and even for communication circuits where the phase response may not necessarily be linear. Special case of design of a notch filter with two notch frequencies $\omega_1$ and $\omega_2$ such that:

$$(\cos \omega_1)(\cos \omega_2) = -1/2$$

has also been considered. The afore-mentioned condition, if used in the design of IIR bi notch filter, ensures reduction in the number of multipliers without affecting the frequency response of the notch filter. It has been shown that an FIR bi notch filter designed with the condition (4.1) for $r = 0.86$ has nearly ‘L’ design weights instead of (2L+1) needed for conventional bi notch filter obtained by cascading two FIR notch filters each of length ‘L’. With increase in ‘r’ value the number of zero weights reduces but the frequency response improves due to suppression of ripples in the pass bands of the bi notch filter.

4.2 DESIGN

A straightforward method for designing a bi notch filter using IIR notch filter as prototype is suggested here. Consider an IIR notch filter characterized by transfer function [5].

$$H_0(z) = K_0 \frac{1 - 2 \cos \omega_0 z^{-1} + z^{-2}}{1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}}$$

(4.2)
This filter is versatile as it can be designed for any required notch frequency \( \omega_0 \).
This is our prototype from which the required FIR filter will be evolved. In the above equation \( K_0 \) is a scaling factor and ‘\( r \)’ is the pole radius. The value of ‘\( r \)’ is chosen to be positive and less than unity, \( \omega_0 \) is the normalized frequency expressed in radians [i.e. \( 2\pi \times (\text{actual frequency in Hz}) / \text{sampling frequency in Hz} \)]. A pair of complex conjugate zeros are placed at \( \exp(\pm j\omega_0) \) for the notch effect at \( \omega_0 \). Frequency response in the pass bands is made close to unity by placing the conjugate poles very close to the conjugate zeros.

Equation (4.2) can be written in the form

\[
H_0(z) = K_0 H_{01}(z) H_{02}(z) \tag{4.3a}
\]

where,

\[
H_{01}(z) = 1 - 2 \cos \alpha_0 z^{-1} + z^{-2} \tag{4.3b}
\]

\[
H_{02}(z) = 1 / (1 - az^{-1} + bz^{-2}) \tag{4.3c}
\]

with,

\[
a = 2r \cos \omega_0 \quad \text{and} \quad b = r^2 \tag{4.3d}
\]

It is easy to show that \( H_{01} \left( e^{j\omega_0} \right) = 0 \), which ensures a notch at \( \omega_0 \) for \( H_0(z) \). By actual (long) division of unity by the factor \( (1 - az^{-1} + bz^{-2}) \), \( H_{02}(z) \) given by (4.3c) can be written as

\[
H_{02}(z) = 1 + az^{-1} + (a^2 - b)z^{-2} + (a^3 - 2ab)z^{-3} + (a^4 - 3a^2b + b^2)z^{-4} \\
+ (a^5 - 4a^3b + 3ab^2)z^{-5} + (a^6 - 5a^4b + 6a^2b^2 - b^3)z^{-6} \\
+ (a^7 - 6a^5b + 10a^3b^2 - 4ab^3)z^{-7} + \cdots \cdots \cdots \tag{4.4a}
\]
After simple algebraic manipulation the above equation can be put in the following elegant form [36]:

$$H_{02}(z) = \sum_{i=0}^{\infty} h_2(i)z^{-i}$$ (4.4b)

with

$$h_2(i) = \sum_{m=0}^{\left\lfloor \frac{i}{2} \right\rfloor} (-1)^{i-m} \binom{i-m}{m} a^{(i-2m)} b^m, \quad i = 0, 1, 2, \ldots,$$ (4.4c)

where the notation $\left\lfloor x \right\rfloor$ stands for the integral part of $x$. The coefficients $h_2(i)$’s are functions of $r$ and $\omega_0$ only. Now the overall transfer function $N_0(z)$, of the resulting FIR notch filter is given by

$$N_0(z) = K_0 H_{01}(z) H_{02}(z)$$ (4.5)

i.e.

$$N_0(z) = K_0 \left(1 - 2 \cos \omega_0 z^{-1} + z^{-2}\right) \sum_{i=0}^{\infty} h_2(i)z^{-i}$$ (4.6)

Equation (4.6) can be written in modified form

$$N_0(z) = K_0 \sum_{i=0}^{\infty} D_0(i)z^{-i},$$ (4.7a)

In order to arrive at an FIR structure, we truncate the series of equation (4.7a) at $i = L$. Also, we shall see that the error between the responses of $N_0(z)$ and $H_0(z)$ can be decreased by appropriate choice of length $L$ of the designed FIR filter $N_0(z)$. The resulting FIR filter will have the transfer function:

$$N_0(z) = K_0 \sum_{i=0}^{L} D_0(i)z^{-i},$$ (4.7b)
where $K_0$ is chosen such that dc response of $N_0(z)$ is unity and weights $D_0(i)$’s are computed by recursive formula.

$$D_0(i) = h_2(i) - 2\cos\omega_0 h_2(i-1) + h_2(i-2), \quad i = 2, 3, \ldots, L$$  \hspace{1cm} (4.8a)

with $D_0(0) = h_2(0)$, and $D_0(1) = (-2\cos\omega_0)h_2(0) + h_2(1)$  \hspace{1cm} (4.8b)

All the design weights $D_0(i)$’s for the FIR notch filter $N_0(z)$ can thus be computed from (4.4c), (4.8a) and (4.8b). Note that the filter $N_0(z)$ has a notch at $\omega_0$. As expected, the implementation complexity of FIR notch filter $N_0(z)$ in (4.7b) is higher than that of IIR notch filter $H_0(z)$ in (4.2) due to increase in the number of multiplications, additions and delays. The comparison of (4.2) and (4.7b) in terms of required multipliers, adders and delays is shown in Table 4.1.

The increase of implementation complexity, however, is not of much concern due to the tremendous advances in DSP and field programmable gate array (FPGA) technology. The decisive advantages of FIR filters are their constant group delay and superior time response [33, 35]. FIR notch filters such as $N_0(z)$ derived from IIR notch filter $H_0(z)$ also have the advantage of unconditional stability and are desirable in many practical applications. From the proposed method, it is also possible to evolve an analytical method for tuning the rejection bandwidth of an FIR notch filter using pole length ‘r’. This idea is due to [31] where an analytical procedure for notch frequency tuning of FIR notch filter based on the differential equation of the transformed Chebyshev polynomial is proposed.
**TABLE - 4.1**

Comparison of implementation complexity of designed FIR notch filter of length ‘L’ and prototype IIR notch filter

<table>
<thead>
<tr>
<th>Filter types</th>
<th>No. of multipliers</th>
<th>No. of Adders</th>
<th>No. of delays</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prototype IIR notch filter (Eq 4.2)</td>
<td>5</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Designed FIR notch filter of length ‘L’ (Eq. 4.7b)</td>
<td>L+1</td>
<td>L</td>
<td>L</td>
</tr>
</tbody>
</table>
The ideal frequency response of the designed FIR notch filter $N_0(z)$ very closely follows that of prototype IIR notch filter $H_0(z)$. However, for all practical purposes, response of $N_0(z)$ which is least deviated from that of $H_0(z)$ can be taken as the ideal. One such response of the designed FIR notch filter for a notch frequency of 1.8 radians and a length of 170 with ‘$r$’ value equal to 0.85 is shown in Figure 4.1. This response is overlapping the response of IIR filter $H_0(z)$. The responses of filters $N_0(z)$ and $H_0(z)$ are indistinguishably close to each other over the frequency range $0 \leq \omega \leq \pi$. The error between this ideal response of the designed FIR notch filter and that of the prototype IIR notch filter over the entire frequency range has been shown in Figure 4.2. Note that the error is very small in the stop band and it does not exceed $-1 \times 10^{-10}$ ($\leq -200$ dB) in the pass band. The maximum error increases to $\pm 6.5 \times 10^{-10}$ ($< -180$ dB) when length is reduced to 130. The maximum error also increases to $-3.8 \times 10^{-6}$ ($< -108$ dB) for increase in ‘$r$’ value from 0.85 to 0.91. This low error is indeed insignificant for practical notch filters.

Notch filter with very narrow rejection bandwidth with pole length ‘$r$’ = 0.99 was also investigated. For any given length of the designed FIR notch filter when ‘$r$’ value was increased beyond 0.97, ripples were observed in the pass bands. Also, response at the notch was found to be non zero. With increase in the length of the FIR notch filter non zero response improved. However, ripples though reduced in amplitude, continued to exist in the pass bands. The degradation in the
(a) Ideal frequency response of the designed FIR notch filter (Response of $N_0(z)$ overlaps the response of $H_0(z)$) at $\omega_d = 1.8$ rad. $L=170$, $r = 0.85$
Maximum error = $-1 \times 10^{-10}$

(b) Frequency response of the designed FIR notch filter at $\omega_d = 1.8$ rad. $L=170$, $r = 0.91$
Maximum error increases from $-1 \times 10^{-10}$ to $-3.8 \times 10^{-6}$

Figure 4.1 Ideal frequency response of the designed FIR notch filter

(a) The frequency response of the designed FIR notch filter $N_0(z)$ is indistinguishably same as that of $H_0(z)$ at $\omega_d = 1.8$ rad. $L=170$ and $r = 0.85$. Maximum error in the entire frequency band is the least and is equal to $-1 \times 10^{-10}$

(b) Maximum error increases from $-1 \times 10^{-10}$ to $-3.8 \times 10^{-6}$ on increasing $r$ value from 0.85 to 0.91 for the same length and notch frequency.
Figure 4.2  Error $E(\omega) = |N_0(\omega) - H_0(\omega)|$ between the ideal frequency response of the designed FIR notch filter $N_0(z)$ and the response of the prototype IIR notch filter $H_0(z)$ at notch frequency of 1.8 radian for $L = 170$ and $r = 0.85$ (refer curve (a) in Figure 4.1. Note that the maximum error in the frequency band $0 \leq \omega \leq \pi$ is $-1 \times 10^{-10}$ (i.e. $\leq -200$ dB).
performance for ‘r’ > 0.97 seems to be due to quantization effect. The pole for the case ‘r’ = 0.99 is almost on the unit circle. It is also important to analyze the performance of the designed FIR notch filter with respect to phase response, for confirming design effectiveness. Phase response of the designed FIR notch filter of length 150 is compared with that of prototype IIR notch filter for ‘r’ value equal to 0.97 in Figure 4.3. The two phase responses are indistinguishably close to each other. The error between them is found to be almost negligible throughout the spectrum. Thus, the performance analysis of the designed FIR notch filter carried out in terms of both magnitude response and phase response shows that performance of the proposed FIR notch filter is indistinguishably similar to that of IIR notch filter.

4.2.1 Design of FIR bi-notch filter

Now consider that we require a filter with two notches, say, at \( \omega_1 \) and \( \omega_2 \); then the prototype IIR filters are chosen as

\[
G_1(z) = K_1 \frac{1 - 2 \cos \omega_1 z^{-1} + z^{-2}}{1 - 2r \cos \omega_1 z^{-1} + r^2 z^{-2}} \tag{4.9a}
\]

\[
G_2(z) = K_2 \frac{1 - 2 \cos \omega_2 z^{-1} + z^{-2}}{1 - 2r \cos \omega_2 z^{-1} + r^2 z^{-2}} \tag{4.9b}
\]

Following the steps as for FIR notch filter \( N_0(z) \), we obtain the following two notch filters both of order L:
(a) Phase response of the designed FIR notch filter and the prototype IIR notch filter. (Response of $N_0(z)$ overlaps the response of $H_0(z)$) at $\omega_d = 1.8$ rad., $L=150$, $r = 0.97$.

(b) Error between the two phase responses.

Figure 4.3 Phase responses of the designed FIR notch filter and the prototype IIR notch filter.

(a) The phase response of the designed FIR notch filter $N_0(z)$ is indistinguishably the same as that of $H_0(z)$ at $\omega_d = 1.8$ rad., $L=150$ and $r = 0.97$.

(b) Error between the phase response of the designed FIR notch filter $N_0(z)$ and the response of the prototype IIR notch filter $H_0(z)$ at notch frequency of 1.8 rad. for $L = 150$ and $r = 0.97$. 
\[ N_1(z) = \sum_{i=0}^{L} D_1(i) z^{-i} \quad \text{(for notch at } \omega_1) \quad (4.10a) \]

\[ N_2(z) = \sum_{i=0}^{L} D_2(i) z^{-i} \quad \text{(for notch at } \omega_2) \quad (4.10b) \]

The weights \( D_1(i) \) and \( D_2(i) \) can readily be computed using methodology as for \( D_0(i) \).

Cascading \( N_1(z) \) and \( N_2(z) \), we obtain

\[ N_3(z) = N_1(z)N_2(z) \quad (4.11a) \]

\[ N_3(z) = \sum_{i=0}^{2L} D_3(i) z^{-i} \quad \text{(with notches at } \omega_1 \text{ and } \omega_2) \quad (4.11b) \]

The composite weights \( D_3(i) \) of the bi-notch filter are, therefore, given by

\[ D_3(i) = \sum_{j=0}^{i} D_1(j) D_2(i-j) \quad , \quad i = 0, 1, \ldots, L \quad (4.12a) \]

\[ D_3(L+i) = \sum_{j=1}^{L} D_1(L+j-i) D_2(j) \quad , \quad i = 1, 2, \ldots, L \quad (4.12b) \]

4.2.2 Design Example

To design an FIR bi notch filter with notches at \( \omega_1 = 0.8 \) radian and \( \omega_2 = 2.4 \) radians, choose \( r = 0.8 \) and individual FIR notch filter with order \( L = 4 \). Since \( \omega_1 \) and \( r_1 \) are known to be 0.8 radians and 0.8 respectively, \( a_1 = 2 r \cos \omega_1 \) will be 1.11473, \( b_1 = r^2 = 0.64 \). Using (4.4c) first compute \( h_2(i) \) for \( i = 0 \) to 4. Then
compute $D_1(i)$ using (4.8) for $i = 0$ to 4. Proceeding similarly, calculate $D_2(i)$. Values of $D_1(i)$, $D_2(i)$ and the overall weights $D_3(i)$ of the cascaded notch filter (with notches at $\omega_1$ and $\omega_2$) computed by using (4.12), are shown in Table 4.2. With weights $D_3(i)$ the frequency response of the FIR bi notch filter is obtained by using (4.11b) and by replacing $z = e^{j\omega}$.

4.3 PERFORMANCE OF THE FIR BI NOTCH FILTER AS A HARMONICS ELIMINATOR

FIR bi notch filter $N_3(z)$ so designed will have two notches, at $\omega_1$ and $\omega_2$. By choosing $\omega_1$ as the fundamental frequency component of periodic noise and $\omega_2$ equal to $n\omega_1$, the $n^{th}$ harmonic component of the noise where $n$ is 2, 3 or 4, $N_3(z)$ can be used as harmonic eliminator. Frequency responses of the bi notch filters obtained by the design approach given in Section 4.2.1 enable us to investigate efficacy and its performance as a harmonics eliminator.

4.3.1 FIR bi notch filter as a first and second Harmonics Eliminator.

The suggested FIR bi notch filters effectively removes second order harmonics $2\omega_0$ together with the fundamental component $\omega_0$ of any periodic noise.
TABLE - 4.2

The design weights $D_3(i)$ of FIR bi notch filter (composite filter) in terms of constituent weights $D_1(i)$ and $D_2(i)$ for order $L = 4$, $r = 0.8$ and notches at $\omega_1 = 0.8$ rad. & $\omega_2 = 3\omega_1 = 2.4$ rad. (Design Example)

<table>
<thead>
<tr>
<th>i</th>
<th>$D_1(i)$</th>
<th>$D_2(i)$</th>
<th>$D_3(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00000000000000</td>
<td>1.00000000000000</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>1</td>
<td>-0.27868268373887</td>
<td>0.29495748621650</td>
<td>0.01627480247763</td>
</tr>
<tr>
<td>2</td>
<td>0.04934384713641</td>
<td>0.01200032529938</td>
<td>-0.02085537141189</td>
</tr>
<tr>
<td>3</td>
<td>0.23336202057678</td>
<td>-0.20293113431490</td>
<td>0.04164094051332</td>
</tr>
<tr>
<td>4</td>
<td>0.22855575454094</td>
<td>0.23174402081874</td>
<td>0.58627718566961</td>
</tr>
<tr>
<td>5</td>
<td>-------</td>
<td>-------</td>
<td>-0.00438179755390</td>
</tr>
<tr>
<td>6</td>
<td>-------</td>
<td>-------</td>
<td>-0.03317853460007</td>
</tr>
<tr>
<td>7</td>
<td>-------</td>
<td>-------</td>
<td>0.00769917443166</td>
</tr>
<tr>
<td>8</td>
<td>-------</td>
<td>-------</td>
<td>0.05296642953858</td>
</tr>
</tbody>
</table>
with $\omega_0$ in the frequency range 0.6 radians to 1.3 rad. Frequency response of one such noise eliminator designed for the fundamental frequency $\omega_1 = 1$ rad. and its second harmonic ($2\omega_1$), for filter length of 100, at selected values of $r$ is shown in Figure 4.4. From the figure, it is evident that the frequency response has two notches, one exactly at the first harmonic frequency $\omega_1 = 1$ rad. and the second at $\omega_2 = 2\omega_1$ i.e. at 2 rad., as desired. For the filter length of 100, the responses are drawn for selected values of $r = 0.85$, 0.88 and 0.91. A number of such eliminators were designed and performance was studied over the frequency range 0.6 to 1.3 rad. It is seen that such noise eliminator works effectively i.e. without ripples up to a maximum length of 100 and ‘$r$’ equaling 0.91. With this value of ‘$r$’, a rejection bandwidth of $9.16^\circ$ (0.18 rad.) is achieved. When the length of the filter is decreased to 96, the maximum value of ‘$r$’ has to be reduced to 0.88 to avoid ripples in the pass bands. Further reduction in filter length to 56 results in reduction of ‘$r$’ value to 0.85 and rejection bandwidth increases to $17.19^\circ$ (0.3rad.). Rejection bandwidth narrows down with increase in the length of the filter in general. Performance results are shown in Table 4.3.

4.3.2 **FIR bi notch filter as a first and third Harmonics Eliminator.**

The suggested FIR bi notch filter can be also designed to work as a first and third harmonic eliminator if the second notch frequency $\omega_2$ is set equal to $n\omega_1$ where $n$ is 3 and $\omega_1$ is set to first harmonic frequency of any periodic noise.
Figure 4.4 FIR bi notch filter as a first and second harmonic eliminator. The frequency responses are drawn for filter length equal to 100 and for selected values of ‘r’ equaling 0.91, 0.88 and 0.85; $\omega_1 = 1 \text{ rad.}$, $\omega_2 = 2 \text{ rad.}$
Performance analysis shows that this filter effectively removes first and third harmonics of periodic noise for the fundamental frequencies of 0.7 to 0.8 radians. Such an eliminator response for $\omega_1 = 0.8$ radians, $\omega_2 = 3 \omega_1$ i.e. 2.4 radians and filter length of 100 is shown in Figure 4.5. The three curves in Figure 4.5 are for the selected values of ‘r’ equaling 0.91, 0.88, and 0.85. The eliminator performance is excellent up to a maximum filter length of 100 and maximum ‘r’ value of 0.91 which corresponds to the rejection bandwidth of $9.16^0$ (0.18 rad.). When the filter length is decreased to 76 the value of ‘r’ has to be reduced to 0.88 to avoid ripple and it further reduces to 0.85 for the filter length of 64. The performance parameters are as shown in Table 4.3.

4.3.3 FIR bi notch filter as a first and fourth Harmonic Eliminator.

It is also possible to have two notches one say at 0.6 radians and another at 2.4 radians for the aforementioned FIR bi notch filter. The designed notch filter can remove first harmonic component (i.e. 0.6 radians) and its fourth harmonic component (i.e. 2.4 radians). The response of such a harmonic eliminator for a filter length of 88 is shown in the Figure 4.6. This can work satisfactorily up to the length of 88 and ‘r’ value up to 0.91 with tolerable ripple in the pass bands. If the filter length is reduced to 68, maximum ‘r’ value reduces to 0.88 and it further reduces to 0.85 for a length of 52. In general, high value of ‘r’ results in narrower notch filters. Additional performance details are also given in Table 4.3.
Figure 4.5  FIR bi notch filter as a first and third harmonics eliminator. The frequency responses are drawn for filter length equal to 100 and for selected values of ‘r’ equaling 0.91, 0.88 and 0.85. \( \omega_1 = 0.8 \text{rad}; \omega_2 = 2.4 \text{rad.} \)
Figure 4.6 FIR bi notch filter as a first and fourth harmonics eliminator. The frequency responses are drawn for filter length equal to 88 and selected values of 'r' equal to 0.91, 0.88 and 0.85. $\omega_1 = 0.6\text{rad}$, $\omega_2 = 2.4\text{rad}$. 

$r = 0.91$
RBW = 9.16 degree

$r = 0.88$
RBW = 13.75 degree

$r = 0.85$
RBW = 17.19 degree

NORMALISED MAGNITUDE RESPONSE

FREQUENCY IN RADIANS

Figure 4.6 FIR bi notch filter as a first and fourth harmonics eliminator. The frequency responses are drawn for filter length equal to 88 and selected values of 'r' equal to 0.91, 0.88 and 0.85. $\omega_1 = 0.6\text{rad}$, $\omega_2 = 2.4\text{rad}$. 
# TABLE - 4.3

Performance parameters of Harmonics Eliminator

<table>
<thead>
<tr>
<th>Types of Eliminator</th>
<th>Length of the Filter (L)</th>
<th>Maximum ‘r’ value (r)</th>
<th>Range of $\omega_1$ in radians</th>
<th>Rejection bandwidth (RBW) for Maximum length (*) and selected values of ‘r’</th>
</tr>
</thead>
<tbody>
<tr>
<td>Second Harmonic</td>
<td>100 *</td>
<td>0.91</td>
<td>0.7 - 1.2</td>
<td>0.91 $9.16^0$</td>
</tr>
<tr>
<td>Eliminator</td>
<td>96</td>
<td>0.88</td>
<td>0.6 - 1.3</td>
<td>0.88 $13.75^0$</td>
</tr>
<tr>
<td>$\omega_1$ = 1 rad, $\omega_2$ = 2 rad</td>
<td>56</td>
<td>0.85</td>
<td>0.8 - 1.1</td>
<td>0.85 $17.19^0$</td>
</tr>
<tr>
<td>Third Harmonic</td>
<td>100 *</td>
<td>0.91</td>
<td>0.7 - 0.8</td>
<td>0.91 $9.16^0$</td>
</tr>
<tr>
<td>Eliminator</td>
<td>76</td>
<td>0.88</td>
<td>0.7 - 0.8</td>
<td>0.88 $13.75^0$</td>
</tr>
<tr>
<td>$\omega_1$ = 0.8 rad, $\omega_2$ = 2.4 rad</td>
<td>64</td>
<td>0.85</td>
<td>0.7 - 0.8</td>
<td>0.85 $17.19^0$</td>
</tr>
<tr>
<td>Fourth Harmonic</td>
<td>88 *</td>
<td>0.91</td>
<td>0.4 – 0.6</td>
<td>0.91 $9.16^0$</td>
</tr>
<tr>
<td>Eliminator</td>
<td>68</td>
<td>0.88</td>
<td>0.5 – 0.6</td>
<td>0.88 $13.75^0$</td>
</tr>
<tr>
<td>$\omega_1$ = 0.6 rad, $\omega_2$ = 2.4 rad</td>
<td>52</td>
<td>0.85</td>
<td>0.55 – 0.65</td>
<td>0.85 $17.19^0$</td>
</tr>
</tbody>
</table>

* Maximum filter length up to which the Eliminator performance is free of ripples.
4.4 SPECIAL TYPES OF IIR AND FIR BI NOTCH FILTERS

In this section, we shall consider some interesting results for bi notch filters designed with condition: \( \cos \omega_1 \cos \omega_2 = -1/2 \). First the design of special condition IIR bi notch filter is considered.

We aim at realizing a notch at \( \omega = \omega_1 \). Therefore, the IIR prototype of such a filter is

\[
N_1(z) = \frac{1 - 2 \cos \omega_1 z^{-1} + z^{-2}}{1 - 2r \cos \omega_1 z^{-1} + r^2 z^{-2}} \tag{4.13}
\]

Let us choose another notch at \( \omega = \omega_2 \) such that

\[
\cos \omega_1 \cos \omega_2 = -1/2 \tag{4.14}
\]

If \( 0 \leq \omega_1 \leq \pi/3 \), then \( \omega_2 \) lies in the range \( 2\pi/3 \leq \omega_2 \leq \pi \). An IIR notch filter for notch at \( \omega = \omega_2 \) is

\[
N_2(z) = \frac{1 - 2 \cos \omega_2 z^{-1} + z^{-2}}{1 - 2r \cos \omega_2 z^{-1} + r^2 z^{-2}} \tag{4.15}
\]

We have chosen the pole length ‘\( r \)’ in \( N_1(z) \) and \( N_2(z) \) equal (with \( r < 1 \)) to simplify the design. Cascading \( N_1(z) \) and \( N_2(z) \), we have

\[
N_3(z) = N_1(z)N_2(z) \tag{4.16}
\]

i.e.

\[
N_3(z) = \left( \frac{1 - 2 \cos \omega_1 z^{-1} + z^{-2}}{1 - 2r \cos \omega_1 z^{-1} + r^2 z^{-2}} \right) \left( \frac{1 - 2 \cos \omega_2 z^{-1} + z^{-2}}{1 - 2r \cos \omega_2 z^{-1} + r^2 z^{-2}} \right) \tag{4.17}
\]
Using (4.14) in (4.17) and simplifying, we obtain

\[ N_3(z) = \frac{1 - (2Cz^{-1}) - (2z^{-3}) + z^{-4}}{1 - (2Crz^{-1}) - (2Cr^3z^{-3}) + r^4z^{-4}} \]  \hspace{1cm} (4.18)

where

\[ \cos \omega_1 + \cos \omega_2 = C \]  \hspace{1cm} (4.19)

For analyzing the performance of so designed IIR bi notch filter with special condition, consider an example:

Let \( \omega_1 = \pi/4 \), using the condition (4.14), we have \( \omega_2 = 3\pi/4 \). Also, by using (4.19), we have \( C = 0 \). Hence (4.18) reduces to

\[ N_3(z)\big|_{C=0} = N_3(z) = \frac{1 + z^{-4}}{1 + r^4z^{-4}} = N_3(z)\big|_{\omega_1=\pi/4} \quad \text{(with notches at } \pi/4, 3\pi/4) \]  \hspace{1cm} (4.20)

If we put \( \omega_1 = \pi/4 \) in (4.13), we have

\[ N_1(z)\big|_{\omega_1=\pi/4} = \frac{1 - \sqrt{2}z^{-1} + z^{-2}}{1 - \sqrt{2}rz^{-1} + z^{-2}} \quad \text{(with notch at } \pi/4) \]  \hspace{1cm} (4.21)

Frequency responses of \( N_1(z)\big|_{\omega_1=\pi/4} \) and \( N_3(z)\big|_{\omega_1=\pi/4} \) are shown in Figure 4.7. It may be seen that \( N_3(z)\big|_{\omega_1=\pi/4} \) has only one multiplier (viz. \( r^4 \)) and gives two notches (at \( \omega = \pi/4, 3\pi/4 \)) where as \( N_1(z)\big|_{\omega_1=\pi/4} \) has two multipliers (viz. \( \sqrt{2} \) and \( \sqrt{2}r \)) and gives only one notch (at \( \omega = \pi/4 \)). Thus we obtain two notches at \( \omega_1 \) and at \( \omega_2 \) that require lesser multipliers for IIR notch filter, for the condition: \( \cos \omega_1, \cos \omega_2 = -1/2 \). Also, the two notches of \( N_3(z) \) (at \( \omega = \pi/4, 3\pi/4 \)) have the same rejection bandwidths as that of \( N_1(z) \).
Figure 4.7  

(a) The frequency response of IIR Notch filter \( N(z) \) evaluated at \( \omega = \pi/4 \) for \( r = 0.91 \). This IIR filter requires two multipliers, and gives only one notch at \( \pi/4 \).

(b) Frequency response of special IIR bi notch filter \( N(z) \) evaluated at \( \omega = \pi/4 \) designed with the condition \( \cos\omega_1 \cos\omega_2 = -1/2 \) for \( r = 0.91 \). This condition reduces the number of multipliers to one while maintaining the response sharpness. It gives two notches at \( \pi/4 \) and \( 3\pi/4 \).
Next, we bring out some interesting results for the design of FIR bi notch filter with special condition \( \cos \omega_1 \cos \omega_2 = -1/2 \).

Let \( \omega_1 = \cos^{-1}(2/3) = 0.8410 \text{ rad.} \) and \( \cos \omega_1 \cos \omega_2 = -1/2 \),

Now, \( \omega_2 = \cos^{-1} \left( -\frac{1/2}{\cos \omega_1} \right) = 2.4188 \text{ rad.} \)

The values of \( D_1(i) \), \( D_2(i) \) and \( D_3(i) \) are computed by procedure explained in Section 4.2. Weights \( D_3(i) \) calculated for \( r = 0.86 \) are given in Table 4.4. The frequency response of FIR bi notch filter \( N_3(z) \bigg|_{\omega_1=\cos^{-1}(2/3)} \) is shown in Figure 4.8, for \( L = 100 \). The impulse response has weights \( D_3(i) \) zero for \( i = 51 \) to 101 which is almost half of \( L \). The rejection bandwidth for the notches \( \omega_1 \) and \( \omega_2 \) of FIR bi notch filter with 50 zero weights (and 51 non zero weights) is exactly the same as that of single notch FIR filter of order 50. In other words, an FIR bi notch filter, with two notches at \( \omega_1 \) and \( \omega_2 \) has been realized with the length of a single notch filter with \( \omega_1 \) as its notch frequency. Also, important observations to be made for the design with “special condition” are:

(a) The rejection bandwidth of special FIR bi notch filter is exactly the same as that of single notch FIR filter for frequency \( \omega_1 \).

(a) The extra notch \( \omega_2 = 2.418858406 \text{ rad.} \) is almost around third harmonic component of \( \omega_1 = 0.84106867 \text{ rad.} \).
The weights $D_3(i)$ of special FIR bi-notch filter for the case $\cos \omega_1 \cdot \cos \omega_2 = -1/2$, length $L=100$, $r = 0.86$ notches at $\omega_1 = 0.8410$ rad. and $\omega_2 = 2.4188$ rad.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$D_3(i)$ for ‘$r$’ = 0.86</th>
<th>$i$</th>
<th>$D_3(i)$ for ‘$r$’ = 0.86</th>
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<tr>
<td>0</td>
<td>1.00000000000000</td>
<td>26</td>
<td>-0.00007829490377</td>
</tr>
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<td>1</td>
<td>0.023333333214501</td>
<td>27</td>
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<td>51 to 100</td>
<td>All zeros</td>
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</table>
Figure 4.8  

(a) The frequency response of single FIR Notch filter $N_1(z)$ with 51 weights for a filter order of 50, $r = 0.86$ and notch at $\omega_1 = 0.84106867$ radians.

(b) Frequency response of special FIR bi notch filter $N_3(z)_{\alpha_1=\cos^3(2/3)}$ designed with the condition $\cos\omega_1 \cos\omega_2 = -1/2$ for order 100 and $r = 0.86$. (This condition generates 51 non zero weights and 50 zero weights). Rejection bandwidth is same as for a single notch FIR filter of length 50 at $\omega_1 = 0.84106867$ radians.
For the same FIR bi notch filter if ‘r’ is increased from 0.86 to 0.88 keeping the length same as 100 and for the same notches $\omega_1$ and $\omega_2$, the number of zero weights reduce from 50 to 36. However, in this case, frequency response of FIR bi notch filter at $\omega_1$ is sharper and without ripple when compared to equivalent single notch filter of the same length. This feature of special FIR bi notch filter with special condition is shown in Figure 4.9.

4.5 CONCLUSION

1. A novel design technique for designing FIR bi notch filters from two prototype IIR filters is proposed. The design weights of the proposed filters are computed from the exact mathematical formulas obtained in this chapter. Designed FIR bi notch filter can be an effective eliminator of harmonic contents of periodic noise or unwanted signals.

2. The rejection bandwidth of the bi notch FIR filter can be controlled from 17.19° to 9.16° by varying the pole length of prototype IIR filters used for the derivation of FIR filters. It is observed that ‘r’ cannot be increased beyond 0.91 and the filter length is limited to 100, if we aim at ripple free performance of the designed FIR filter.

3. Design simplification for the special condition: $\cos \omega_1 \cdot \cos \omega_2 = -1/2$ has been
Figure 4.9  
(a) The frequency response of single FIR Notch filter $N_1(z)$ with 65 weights for a filter length of 64 and $r = 0.88$. Note the ripples in the pass band.

(b) Frequency response of special FIR bi notch filter $N_3(z)$ designed with the condition $\cos \omega_1 \cos \omega_2 = -1/2$ for a length 100 and $r = 0.88$. This condition generates 36 zero weights. Rejection band width at $\omega_1 = 0.84106867$ rad. is the same as for a single notch FIR filter of length 64 and also the two notches are now free from ripples in the pass bands.
brought out. A special IIR bi notch filter designed with this condition requires less number of multipliers. An FIR bi notch filter designed with the special condition reduces half the number of weights to zero without compromising with the rejection band width for the desired response. Also, with increase in ‘r’ value the number of zero weights reduces. However, the quality of the desired response improves due to elimination of ripples in the pass bands for the desired rejection bandwidth.